UPPER BOUNDS FOR AMERICAN FUTURES OPTIONS: A NOTE

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Under standard perfect market assumptions, the cost-of-carry formula can be applied to calculate the value of a pure (futures-style margining) European futures option at time t with maturity at T [Duffie (1989, p. 285); Lieu (1990, p. 332)]:

$$\text{Call: PEC}(t) = \text{EC}(t) \exp[r(T - t)] \tag{1}$$

$$\text{Put: PEP}(t) = \text{EP}(t) \exp[r(T - t)] \tag{2}$$

where EC and EP are conventional European futures option prices for call and put, respectively, and are given by Black (1976) for lognormal futures prices. Pricing equations (1) and (2) apply to pure American call (PAC) and pure American put (PAP) options as well [Lieu (1990)] even if the interest rate is stochastic [Chen and Scott (1992)], since the pure European option price never falls below the intrinsic value of the option. Building on these pure options results and using rational pricing arguments [Merton (1973)], new upper bounds for conventional American options are proposed in this note.

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Proposition A. *The price of a pure European call on futures is an upper bound for the price of a conventional American call on futures:*

Proof. Consider the strategy of rolling over short positions in a conventional call on futures and long positions in a pure call on futures, both American:[1]

Day t:	At the beginning of the day, short $\exp[-r(T - t)]$ units of conventional call and long $\exp[-r(T - t)]$ units of pure call; at the end of the day, close out the positions and invest (borrow) the gain (loss) till time T.
Day $t + 1$:	At the beginning of the day, short again $\exp[-r(T - t - 1)]$ conventional calls and long again $\exp[-r(T - t - 1)]$ pure call; at the end of the day, close out the previous positions and invest (borrow) the gain (loss) till time T.
....	... ...
Day $T - 1$:	At the beginning of the day, short again $\exp(-r)$ conventional calls and long again $\exp(-r)$ pure calls; at the end of the day, close out the positions and invest (borrow) the gain (loss) till time T.
Day T:	At the beginning of the day, short again one conventional call and long again one pure call; at the end of the day, close out the positions. The terminal value of the rollover short positions in the conventional call is: $\mathrm{AC}(t) - \mathrm{AC}(T)$, where AC denotes the price of a conventional American call; the terminal value of the rollover long positions in the pure call is: $\mathrm{PAC}(T) - \mathrm{PAC}(t)$.

Since at maturity a conventional call should be valued the same as a pure call, both equal to the intrinsic value, terminal value of the rollover portfolio is thus guaranteed to be $\mathrm{AC}(t) - \mathrm{PAC}(t)$ if premature exercise of the conventional call does not take place.

Suppose premature exercise takes place at end of day $(T - s) > t$. After investing (borrowing) till time T the day's gain (loss) on both long and short positions, no new position is initiated. The terminal value at time T of the positions would be:

Short conventional calls:	$\mathrm{AC}(t) - \mathrm{AC}(T - s)$
Long pure calls:	$\mathrm{PAC}(T - s) - \mathrm{PAC}(t)$
Overall:	$[\mathrm{AC}(t) - \mathrm{PAC}(t)] + [\mathrm{PAC}(T - s) - \mathrm{AC}(T - s)]$

[1]Please refer to Lieu (1990, Appendix A) for details about the rollover portfolio approach. Lieu used this approach to establish the put–call parity of pure futures options. Further, it can be shown that the rollover trading strategy is self-financing.

Note that $AC(T - s) = F(T - s) - X$ due to premature exercise, where $F(T - s)$ and X denote the futures price and the striking price, respectively. Since the value of a pure call is always above the intrinsic value prior to maturity, the second bracketed term in the overall terminal value expression is positive.[2] If $AC(t) > PAC(t)$, the rollover portfolio which needed no initial capital will have guaranteed positive terminal value at T whether the conventional call has been prematurely exercised or not. If $AC(t) = PAC(t)$, then the terminal value is guaranteed to be nonnegative with some probability of positive terminal value (when the conventional call has been prematurely exercised).

Thus, if $AC(t)$ is either greater than or equal to $PAC(t)$, the rollover portfolio will dominate the riskless investment of $\exp[-r(T - t)][AC(t) - PAC(t)]$ according to the rational pricing arguments of Merton (1983, p. 143). In perfect markets, to prevent arbitrage, it is necessary that $AC(t)$ is less than $PAC(t)$. The latter is of course equal to $PEC(t)$. Thus, the pure European futures call price should be an upper bound for the conventional American futures call price. Q.E.D.

This result also applies when the optioned asset is a stock with no dividends prior to maturity. The proof uses the relevant pure European put–call parity which can be derived by either following Lieu (1990, Appendix A) or, using the put–call parity for conventional European stock options and eqs. (1) and (2) of this note.

Proposition B. *The price of a pure European put on futures is an upper bound for the price of a conventional American put on futures.*

Proof: The proof is identical to that of Proposition A, except that now the conventional and pure American put options on futures are rolled over. The result that pure American futures put price is always above the intrinsic value can then be used. Q.E.D.

Unlike Proposition A, Proposition B does not hold for stock options since it can be shown that the pure European put price can be below the intrinsic value without violating the relevant pure European put–call parity.

Proposition C. *Propositions A and B hold for stochastic interest rates as well.*

Proof: This proposition can be proved, following the same portfolio strategies as in the proofs of Propositions A and B, but with the number of options to short or long in the rollover portfolio

[2]Based on the results in Chen and Scott (1992), this also holds for stochastic interest rates.

changed to the price of a default-free discount bond with \$1 face value payable at time T. The compounding factor on the daily gains and losses should accordingly change to the reciprocal of the discount bond price. Also needed is the result of Chen and Scott (1992) that the pure American futures (call or put) option price is always above the intrinsic value under a stochastic interest rate regime. Q.E.D.

Note that the upper bounds in Propositions A and B are tighter than the ones suggested in Ramaswamy and Sundaresan (1985) and Ball and Torous (1986); the pure call price is always lower than the futures price and the pure put price is always lower than the striking price.

TABLE I
Upper Bounds for Conventional American Call Options on Futures [Striking Price (X) = 100]

r	*Sigma*	$T - t$	F	*EC*	*AC*	*Upper Bound*	*Dollar Difference*	*Percent Difference*
0.08	0.2	0.25	80	0.04	0.04	0.041	0.001	2.020
0.08	0.2	0.25	90	0.7	0.7	0.714	0.014	2.020
0.08	0.2	0.25	100	3.91	3.92	3.989	0.069	1.760
0.08	0.2	0.25	110	10.74	10.82	10.957	0.137	1.266
0.08	0.2	0.25	120	19.75	20.03	20.149	0.119	0.594
0.12	0.2	0.25	80	0.04	0.04	0.041	0.001	3.045
0.12	0.2	0.25	90	0.69	0.69	0.711	0.021	3.045
0.12	0.2	0.25	100	3.87	3.89	3.988	0.098	2.516
0.12	0.2	0.25	110	10.63	10.76	10.954	0.194	1.800
0.12	0.2	0.25	120	19.55	20.01	20.145	0.135	0.677
0.08	0.4	0.25	80	1.16	1.16	1.183	0.023	2.020
0.08	0.4	0.25	90	3.52	3.53	3.591	0.061	1.731
0.08	0.4	0.25	100	7.81	7.83	7.968	0.138	1.760
0.08	0.4	0.25	110	14.01	14.08	14.293	0.213	1.513
0.08	0.4	0.25	120	21.71	21.87	22.149	0.279	1.274
0.08	0.2	0.5	80	0.3	0.3	0.312	0.012	4.081
0.08	0.2	0.5	90	1.7	1.71	1.769	0.059	3.472
0.08	0.2	0.5	100	5.42	5.46	5.641	0.181	3.319
0.08	0.2	0.5	110	11.73	11.9	12.209	0.309	2.594
0.08	0.2	0.5	120	19.91	20.36	20.723	0.363	1.781
						Mean	0.121	2.114
						Minimum	0.001	0.594
						Maximum	0.363	4.081

r: riskfree rate, T: maturity time, t: current time.
Sigma: futures price volatility, F: current futures price.
EC: Black's (1976) European call price [from Barone-Adesi and Whaley (1987)].
AC: Finite difference American call price [from Barone-Adesi and Whaley (1987)].

TABLE II
Upper Bounds for Conventional American Put Options on Futures [Striking Price $(X) = 100$]

r	*Sigma*	$T - t$	F	*EP*	*AP*	*Upper Bound*	*Dollar Difference*	*Percent Difference*
0.08	0.2	0.25	80	19.64	20	20.037	0.037	0.184
0.08	0.2	0.25	90	10.5	10.59	10.712	0.122	1.153
0.08	0.2	0.25	100	3.91	3.92	3.989	0.069	1.760
0.08	0.2	0.25	110	0.94	0.94	0.959	0.019	2.020
0.08	0.2	0.25	120	0.14	0.14	0.143	0.003	2.020
0.12	0.2	0.25	80	19.45	20	20.042	0.042	0.212
0.12	0.2	0.25	90	10.4	10.53	10.717	0.187	1.773
0.12	0.2	0.25	100	3.87	3.89	3.988	0.098	2.516
0.12	0.2	0.25	110	0.94	0.93	0.969	0.039	4.153
0.12	0.2	0.25	120	0.14	0.14	0.144	0.004	3.045
0.08	0.4	0.25	80	20.77	20.94	21.190	0.250	1.192
0.08	0.4	0.25	90	13.32	13.39	13.589	0.199	1.487
0.08	0.4	0.25	100	7.81	7.83	7.968	0.138	1.760
0.08	0.4	0.25	110	4.21	4.22	4.295	0.075	1.778
0.08	0.4	0.25	120	2.1	2.11	2.142	0.032	1.537
0.08	0.2	0.5	80	19.51	20.06	20.306	0.246	1.227
0.08	0.2	0.5	90	11.31	11.48	11.772	0.292	2.540
0.08	0.2	0.5	100	5.42	5.46	5.641	0.181	3.319
0.08	0.2	0.5	110	2.12	2.14	2.207	0.067	3.108
0.08	0.2	0.5	120	0.69	0.69	0.718	0.028	4.081
						Mean	0.106	2.043
						Minimum	0.003	0.184
						Maximum	0.292	4.153

r: riskfree rate, T: maturity time, t: current time.
Sigma: futures price volatility, F: current futures price.
EP: Black's (1976) European put price [from Barone-Adesi and Whaley (1987)].
AP: Finite difference American put price [from Barone-Adesi and Whaley (1987)].

Tables I and II show the upper bounds for the cases considered by Barone-Adesi and Whaley (1987) in their Table III for lognormal futures prices. First, as expected, the upper bounds are always higher than the conventional American prices (finite difference method). Second, the dollar differences and the percentage differences are modest in most cases.

The upper bounds should be of interest to practitioners as well as researchers as a guide to the valuation of American futures options and tests of market efficiency.[3]

[3]An anonymous referee indicated that the pure options prices are often used as estimates of conventional options prices by many traders as well as the Chicago Mercantile Exchange.

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