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AN APPROXIMATELY UNBIASED ESTIMATOR FOR THE THEORETICAL BLACK-SCHOLES EUROPEAN CALL VALUATION

*Mohammed M. Chaudhury**

ABSTRACT

Black–Scholes price estimate of a call is commonly formed by using an estimate of the stock return variance rate in the formula. This, however, produces systematic bias with respect to the model's value with the true variance rate. This paper proposes a new procedure to form Black–Scholes price estimates using Taylor series approximation. Our Monte Carlo results favour the new procedure over the common and the recently proposed Butler–Schachter approaches when bias magnitude and any systematic pattern thereof are the relevant concerns.

I. INTRODUCTION

One of the cornerstones of modern finance theory is the European call valuation formula of Black and Scholes (1973):

$$CB(V; S, X, r, T) = S[\phi(d_1) - \phi(d_2)/g] \tag{1}$$

where

V = Constant variance rate of stock return,
S = current stock price,
X = call's striking price,
r = riskless rate of return,
T = time to maturity of the call,

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$g \quad = S/[X\exp(-rT)]$
$d_1 \quad = [\ln(g)/\sqrt{VT}] + [(\sqrt{VT})/2]$
$d_2 \quad = d_1 - (\sqrt{VT})$, and
$\phi(.)$ = the standard normal distribution function.

In (1), all the inputs are empirically observable except V. Thus the Black–Scholes model price, $CB(V)$, would be unknown to the practitioners and must be estimated.

Let s^2 be an unbiased estimate of the stock return variance rate V. The estimator commonly used by practitioners is:

$$CB(s^2) = S[\phi(\hat{d}_1) - \phi(\hat{d}_2)/g] \tag{2}$$

where

$$\hat{d}_1 = [\ln(g)/\sqrt{s^2T}] + [(\sqrt{s^2T})/2] \quad \text{and} \quad \hat{d}_2 = \hat{d}_1 - (\sqrt{s^2T})$$

Recalling the well-known statistical result $E[f(y)] \neq f(E[y])$ for a non-linear function f of a random variable, Boyle and Ananthanarayanan (1977) have shown that $CB(s^2)$ is a biased estimator of $CB(V)$.[1]

The average magnitude of the above non-linearity bias is about 2.2% of the model price (Butler and Schachter (1986, p. 354)), which is close enough to that found empirically (Whaley (1982), Rubinstein (1985)).[2] Perhaps more interestingly, the nonlinearity bias shows systematic tendencies with respect to variables such as moneyness and time to maturity of option, and stock return variance rate. In principle, the nonlinearity bias thus contributes to the magnitude and the systematic tendencies of the reported empirical biases.[3]

Price estimators reducing both the magnitude and systematic nature of the nonlinearity bias would be useful for the purposes of empirical testing, improved portfolio performance and many other decision-makings based upon the Black–Scholes valuation framework. Recently Butler and Schachter (1986) proposed an estimator which attempts to alleviate the nonlinearity bias problems of the usual plug-in estimator. Their approach

[1] This problem has previously been noted in the literature (e.g., Thorp (1976, pp. 249–251), Merton (1976, p. 336), and Ingersoll (1976, pp. 107–108)). Also, notice that the usual estimator is asymptotically unbiased. Hence it may appear that a large enough sample of stock return to estimate V would solve the problem. However, probably non-stationarity (Christie (1982)) of V frustrates such attempts. See Butler and Schachter (1986) for a discussion on the viability of some other solutions.

[2] One may argue that a 2.2% bias magnitude is not economically significant since it is often less than the average bid-ask spread of exchange-traded options. Phillips and Smith (1980, p. 183), however, point out that the average bid-ask spread overstates the trading cost. Also, a large number of options are traded inside the spread. Moreover, Rubinstein (1985, p. 478) argues that whether a bias magnitude of 2% is economically signifiant depends upon the nature of trader. For example, a scalper would find this magnitude important enough.

[3] For a survey of empirical biases of Black–Scholes pricing, see Galai (1983). Also see Geske and Roll (1984a,b), Ball and Torous (1985) and Rubinstein (1985) for more recent findings.

is essentially to derive a representation of the Black–Scholes theoretical value in the form of a power series in the variance rate:[4]

$$CB(V) = 0.5(1 - 1/g) + (1/\sqrt{2\pi})\left[\sum_{j=-\infty}^{\infty} A_j V^{j+0.5}\right] \quad (3)$$

where A_j's are functions of j and g only. For a given j, A_j is the difference of two power series in $\ln(g)$.

Unbiased estimate of V^m can be formed as:

$$(s^2)^m (K/2)^m [\Gamma(K/2)/\Gamma\{(K/2)+m\}] \quad (4)$$

where

$$K = N - 1;$$

N is the stock return sample size from which unbiased estimate of variance rate V is computed as:

$$s^2 = \sum_{i=1}^{N} (q_i - \bar{q})^2/(N-1)$$

where

$$q_i = (S_t - S_{t-1})/S_{t-1}$$

Replacing V^m's in (3) by their unbiased estimates from (4) leads to an unbiased estimate of the series. While operationalizing the estimator, one would truncate the series to achieve desirable accuracy.

The numerical results of Butler and Schachter (1986) show significant reduction in the bias magnitude using their estimator as opposed to the usual estimator. There remain, however, some theoretical and practical problems with their approach. First, a closer look into the power series representation of the Black–Scholes model price reveals some discrepancies in the case of large majority of options, viz., the not-at-the-money options. For these options, the appropriate series representation around $V=0$ would be of the form of a Laurent series which does not have the first term of (3) and also the powers of V differ by 0.5.

Second, except in the case of infrequently traded at-the-money options, the applicability of the Butler–Schachter approach is limited to larger samples. The minimum sample size in Butler and Schachter's study was 60. If the sample size is not large enough, the gamma function in (4) may encounter inadmissible argument values for not-at-the-money options.[5]

[4] This is our derivation based upon Butler and Schachter (1986).

[5] This would be the case if a relatively large number of terms of the power series (3) needs to be used to achieve desirable level of precision of the price estimate. While no formal evidence is available at this time in this regard, informal conversation with one of the authors of Butler and Schachter (1986) revealed that they used more than thirty terms for their numerical study. A satisfactory answer could only be provided by a detailed sensitivity analysis which is beyond the scope of our current study.

However, it is the small sample bias that we are concerned about, the usual estimator being asymptotically unbiased. Further, larger sample sizes covering longer periods may entail problems of nonstationarity in the variance rate.

Third, the Butler–Schachter estimator seems to have failed in alleviating the problem of systematic pattern in biases. It tends to underestimate at-the-money option values while there is some evidence of overestimation for not-at-the-money options.

In what follows, we present a new estimator which is, in spirit, similar to the Butler–Schachter estimator but overcomes its weaknesses. In the process of operationalizing our estimator, we have devised an easy-to-implement recursive algorithm for computing higher order derivatives of the model price with respect to the variance (and volatility) rate.[6] Higher order derivative values can also be used in forming weighted average implied standard deviations (Latane and Rendelman (1976), Chiras and Manaster (1978)), and unbiased estimates of option value's sensitivity to the variance (and volatility) rate (Cox and Rubinstein, (1985, pp. 215–235)).

II. A NEW ESTIMATOR FOR THE BLACK-SCHOLES PRICE

Like Butler and Schachter (1986) we start with a Taylor series expansion and then truncate the series to operationalize the estimator. However, our series applies directly to the price while Butler and Schachter expand the two cumulative normal probabilities (which leads to probable discrepancies in the resultant power series for the prices).

Using Taylor series, let us expand $CB(V)$ around an arbitrary point $V=V_0$, $V_0 \neq 0$:[7]

$$\begin{aligned} CB(V) = {} & CB(V_0) + \{\partial CB(V)/\partial V | V=V_0\}(V-V_0) + \\ & [\{\partial^2 CB(V)/\partial V^2 | V=V_0\}\{(V-V_0)^2/2!\}] + \ldots + \\ & [\{\partial^n CB(V)/\partial V^n | V=V_0\}\{(V-V_0)^n/n!\}] + \ldots \end{aligned} \quad (5)$$

Substituting for $(V-V_0)^m$'s by their binomial expansions leads to a series linear strictly in the positive powers of V as compared to both positive and negative powers of the Butler–Schachter series. As pointed out in the previous section, negative powers would require larger stock return sample if the option is not at-the-money.

Truncating the series in (5) after the term involving the n-th derivative

[6] The algorithm is available on request from the author.

[7] Since the true value of the variance rate is not known, it cannot be guaranteed that the convergence criterion $0 < V < 2V_0$ will be satisfied. Given the positivity of the true variance rate V, choice of a larger V_0 may reduce the possibility of the lack of convergence.

and replacing the powers of V by their unbiased estimates from (4) produces an approximately unbiased estimate of the Black-Scholes model price, $CB(V)$. Both the Butler-Schachter and the new estimators are only approximately unbiased due to the truncation error. However, unlike the Butler-Schacter estimator, the new estimator is usable for all (both at- and not-at-the-money) options irrespective of the stock return sample size. Further, our approach is virtually free of the series representation problems.

III. MONTE CARLO RESULTS

According to the assumptions of Black and Scholes (1973), stock returns q_i's are normally distributed with mean α and variance V (Jarrow and Rudd (1983, pp. 90-91), Butler and Schachter (1986, p. 346)). Since the option value $CB(V)$ is independent of α, we have conveniently assumed α to be zero for our Monte Carlo.

For a given variance rate and a sample size, we generated 500 samples of q_i's using normal distribution random number generator of the Fortran Library IMSL. These samples were in turn used to compute s^2's — the unbiased estimates of the assumed variance rate V. Then the s^2's were used to generate 500 price estimates according to each of Butler-Schachter (BTS), new(NEW) and usual(USL) estimators for a given option described by a combination of g and V.[8]

On the basis of those estimates, we computed the mean mispricing (negative of the bias), the mean percentage (taken out of the Black-Scholes model price) error, the variance, and the mean square error, for each estimator.

In our computations, we truncated the series after the term containing the 31st derivative of the Black-Scholes formula with respect to the variance rate. Thus for the BTS estimator, the highest order of derivative for the standard normal density is 30. For the CC estimator, V_0 was taken to be 50% higher than the true variance rate.[9]

In total, we considered 40 different combinations of moneyness(g), variance rate(V), and sample size(N). Since the order of expansion for the BTS and the NEW estimator was 31, we could compute for the BTS in only 16 of the cases which correspond to sample size of 35.[10]

[8] For the sake of comparability with the existing studies, we would use $S/X\exp(-rT)$ as the indicator of moneyness. The contracts are to buy one share with current price $S = \$1.0$.

[9] These choices were to some extent constrained by the capability of the software used. But the knowledge of the true variance rate V is not required to choose V_0. Any value in the range of convergence would do. For better comparisons across cases, we have chosen the V_0's to be the same percentage distance away from V's.

[10] The sample sizes 31 and 35 were chosen since sizes around 30 are often regarded as small samples. The sizes 11 and 15 were chosen to examine the results at extremely small sample sizes which may be warranted by severe nonstationarity problem. The odd choices are for the ease of valuing Gamma functions.

TABLE 1

The Mean Percentage Errors and Variances of Usual (USL), Proposed (NEW) and Butler–Schachter (BTS) Estimators for the Black–Scholes Model Price

(a) V	*0.03*		*0.04*			*0.03*		*0.04*	
N	*11*		*11*			*15*		*15*	
(b) g	*(1)*	*(2)*	*(1)*	*(2)*		*(1)*	*(2)*	*(1)*	*(2)*
0.80	−7.64	0.60	−0.03	1.02	USL	−4.92	0.39	−2.17	0.83
	1.75	0.56	0.37	1.50	NEW	1.89	0.40	0.99	0.78
0.95	3.53	2.11	3.24	2.75	USL	3.08	1.58	1.88	2.04
	0.28	2.19	0.33	3.04	NEW	0.68	1.62	0.74	2.24
1.00	2.73	2.22	0.03	2.95	USL	2.35	1.66	2.35	2.21
	0.09	2.40	0.03	3.15	NEW·	4.78	1.73	0.41	2.27
	0.13	2.35	0.09	3.14	BTS	0.46	1.73	0.42	2.31
1.05	1.73	1.93	1.74	2.50	USL	1.50	1.44	1.01	1.86
	0.12	2.00	0.16	2.77	NEW	0.32	1.47	0.39	2.04
1.20	−0.16	0.65	−0.25	0.99	USL	−0.08	0.44	−0.13	0.72
	0.04	0.68	0.01	1.12	NEW	0.09	0.46	0.07	0.77

V	*0.03*		*0.04*			*0.03*		*0.04*	
N	*31*		*31*			*35*		*35*	
g	*(1)*	*(2)*	*(1)*	*(2)*		*(1)*	*(2)*	*(1)*	*(2)*
0.80	−2.80	0.20	−1.73	0.34	USL	−2.69	0.17	−1.33	0.34
	−0.85	0.20	−0.52	0.37	NEW	−0.90	0.17	−0.79	0.32
					BTS	−2.01	0.17	−1.77	0.35
0.95	0.88	0.80	1.27	1.05	USL	0.67	0.69	0.66	0.94
	−0.40	0.81	−0.22	1.10	NEW	−0.47	0.70	−0.30	0.95
					BTS	−0.47	0.71	−0.51	0.97
1.00	0.70	0.83	0.70	1.10	USL	0.54	0.72	0.54	0.96
	−0.04	0.80	−0.10	1.05	NEW	−0.14	0.70	−0.20	0.92
	0.02	0.71	0.29	0.94	BTS	0.20	0.66	0.16	0.88
1.05	0.43	0.73	0.68	0.96	USL	0.33	0.63	0.36	0.86
	−0.19	0.73	−0.12	1.00	NEW	−0.23	0.63	−0.16	0.86
					BTS	−0.23	0.65	−0.27	0.88
1.20	−0.06	0.22	−0.05	0.42	USL	−0.06	0.19	−0.03	0.34
	0.00	0.21	−0.06	0.37	NEW	−0.02	0.19	−0.09	0.33
					BTS	−0.09	0.20	−0.12	0.35

(a) *V* is the assumed true value of the stock return variance rate. *N* is the stock return sample size from which *V* is being estimated.
(b) *g* is the ratio of stock price to the present value of the striking price. Option prices are based upon an option to purchase one stock. The current price of the stock is assumed to be $1 and the time to expiration of the option is also assumed to be 1.

(1) Mean percentage error of estimator.
(2) 10000 × variance of estimator.

In general, the ranking of the absolute magnitudes is the same whether we consider the mispricing or the percentage error. Also, the variance of estimators gives identical ranking as the mean square error. Except for the cases of $g=0.8$, all rankings are the same for the two variance rates considered. Hence, we report only the percentage errors and the variance of the price estimates in Table 1.

In terms of percentage error, both BTS and NEW improves upon the usual estimator, but NEW improves most. NEW is lowest in 12 cases, BTS in 3 (all 3 at-the-money) cases, and the USL estimator in the lone case of relatively deeper-in-the-money ($g=1.2$) option with higher variance rate ($V=0.04$) for the stock.[11] In 13 cases NEW performs better than BTS.

If we consider performance in terms of the variance, USL outperforms both BTS and NEW. Out of the 16 common cases, BTS is lowest in 4 (all 4 at-the-money higher sample sizes), NEW is lowest in 3 (2 relatively deeper-in-the-money, 1 relatively deeper-out-of-the-money, and all largest sample size), and the USL in 9 cases.[12] Between BTS and NEW, NEW performs better in 10 out of the 16 cases.

Overall, BTS seems to have some advantage over the other two, for at-the-money option. But at-the-money options are least frequently traded options. For other options, NEW performs better in terms of the mispricing magnitude, while the USL performs better in terms of the variance or the mean square error.

At small sample sizes, the applicability of BTS is limited while the bias reduction due to the use of NEW over the usual estimator is substantial. For example, at sample sizes like 15 or 11, for near-out-of-the-money options, the percentage error of USL is above 3% while that of NEW is below 1%.

Considering the directions of mispricing, the usual estimator, as found by Boyle and Ananthanarayanan (1977), underprices at-the-money and near-the-money options, and overprices relatively deeper-away-from-the-money options. Our Monte Carlo results are in conformity with this pattern. We find BTS to underprice at-the-money options, and overprice not-at-the-money options.[13] For NEW, no such patterns seem to emerge.

In summary, our Monte Carlo results show that for the most frequently traded not-at-the-money options, the new estimator prevails over both the Butler–Schachter and the usual estimators in terms of bias magnitude. Also, unlike the latter estimators, the new estimator does not exhibit systematic deviations. It is to be noted that in terms of variance and mean square error of estimator, the usual estimator prevails with the new estimator standing next best. However, the weight of interest in the literature is towards bias magnitude and systematic deviations. The variance of option

[11] Among all 40 cases, NEW improves upon formula in 38 cases.

[12] Out of all 40 cases, USL wins over NEW in 29 cases.

[13] Butler and Schachter (1986)'s results also indicate underpricing of at-the-money option by BTS, and the cases of overpricing are all not-at-the-money.

price estimates is, in its own right, an interesting problem to which no satisfactory solution seems to be currently available (Butler and Schachter, (1986, pp. 344–345)). Thus there appears to be satisfactory reasons for preferring the estimator proposed in this paper over the usual and the Butler–Schachter estimators.

University of Saskatchewan
Saskatoon, Saskatchewan,
Canada, S7N OWO

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REFERENCES

Ball, C. A. and Torous, W. N. (1985). 'On Jumps in Common Stock Prices and Their Impact on Call Option Pricing', *Journal of Finance,* Vol. 40, pp. 155–73.

Black, F. and Scholes, M. (1973). 'The Pricing of Options and Corporate Liabilities', *Journal of Political Economy,* Vol. 81, pp. 637–59.

Boyle, P. and Ananthanarayanan, A. L. (1977). 'The Impact of Variance Estimation in Option Valuation Models', *Journal of Financial Economics,* Vol. 5, pp. 375–87.

Butler, J. S. and Schachter, B. (1986). 'Unbiased Estimation of the Black–Scholes Formula', *Journal of Financial Economics,* Vol. 15, pp. 341–57.

Chiras, D. P. and Manaster, S. S. (1978). 'The Information Content of Option Prices and a Test of Market Efficiency', *Journal of Financial Economics,* Vol. 6, pp. 213–34.

Christie, A. A. (1982). 'The Stochastic Behaviour of Common Stock-Variances', *Journal of Financial Economics,* Vol. 10, pp. 407–32.

Cox, J. and Rubinstein, M. (1985). *Options Markets,* New Jersey: Prentice-Hall, Inc.

Galai, D. (1983). 'A Survey of Empirical Tests Of Option Pricing Models', in: M. Brenner, ed., *Option Pricing: Theory and Applications,* Lexington, Mas.: D. C. Heath and Co., Lexington Books.

Geske, R. and Roll, R. (1984a). 'On Valuing American Call Options With Black–Scholes European Formula', *Journal of Finance,* Vol. 39, pp. 443–55.

——— (1984b). 'Isolating the Observed Biases in American Call Option Pricing: An Alternative Variance Estimator', Unpublished working paper.

Jarrow, R. and Rudd, A. (1983). *Option Pricing,* Homewood, Illinois: Richard D. Irwin, Inc.

Latane, H. A. and Rendleman, R. R., Jr. (1976). 'Standard Deviation of Stock Price Ratios Implied by Option Premia', *Journal of Finance,* Vol. 31, pp. 369–82.

Merton, R. (1976). 'The Impact on Option Pricing of Specification Error in the Underlying Stock Price Returns', *Journal of Finance,* Vol. 31.

Phillips, S. M. and Smith, C. W. (1980). 'Trading Costs for Listed Options: The Implications for Market Efficiency', *Journal of Financial Economics,* Vol. 8, pp. 179–201.

Rubinstein, M. (1985). 'Nonparametric Tests of Alternative Option Pricing Models Using all Reported Trades and Quotes on the 30 Most Active CBOE Option Classes from August 23, 1976 through August 31, 1978', *Journal of Finance,* Vol. 40, pp. 455–80.

Thorp, E. O. (1976). 'Common Stock Volatilities in Option Formulas', *Proceedings of the Seminar on the Analysis of Security Prices,* Vol. 2, pp. 235–76.

Whaley, R. E. (1982). 'Valuation of American Call Options on Dividend-Paying Stocks: Empirical Tests', *Journal of Financial Economics,* Vol. 10, pp. 29–58.