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ON THE STRIKING PRICE BIAS OF THE BLACK–SCHOLES FORMULA WITH AN ESTIMATED VARIANCE RATE

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The use of an estimated variance rate in the Black–Scholes formula produces biased estimates of the theoretical price. This study analytically examines the bias with respect to moneyness of the option.

1. Introduction

In their 1973 seminal paper, Black and Scholes derived the following valuation formula for a European call option:

$$h(v^2; x, c, r, T) = x\left[\phi(d_1) - \frac{1}{g}\phi(d_2)\right], \tag{1}$$

where v^2 is the constant variance rate of lognormal stock return; x is the current stock price; c is the call's striking price; r is the riskless rate of return; T is the time to maturity of the call; $g = x/[c\exp(-rT)]$, $d_1 = [\ln(g)/\sqrt{v^2T}] + [\sqrt{(v^2T)}/2]$, $d_2 = d_1 - (\sqrt{v^2T})$; and ϕ is the standard normal distribution function operator.

In (1), all the parameters are empirically observable except v^2. Practitioners have instead used some estimate, s^2, of the variance rate in the formula to compute Black–Scholes (B/S) prices.

The price estimates given by $h(s^2)$ are, however, biased even if we assume s^2 to be an unbiased estimate of v^2 [Boyle and Ananthanarayanan (1977), Ingersoll (1976), Merton (1976), and Thorp (1976)]. This is because $h(.)$ is a non-linear function and unbiasedness is not preserved under non-linear transformation. The average magnitude of this non-linearity bias is about 2.2% of the model price [Butler and Schachter (1986)], which is close enough to that found empirically [Whaley (1982) and Rubinstein (1985)]. More interestingly, the non-linearity bias shows systematic tendencies with respect to variables such as moneyness and time to maturity of option, and stock return variance rate. In principle, such systematic biases contributed to the ones found empirically, among which the striking price bias (systematic tendency with respect to moneyness) has invoked most interests. [1]

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The striking price bias arising solely due to the estimation problem has been studied previously by Boyle and Ananthanarayanan (1977) and Butler and Schachter (1986). The former authors' numerical integration study showed that, given the striking price, as the stock price increases, a small positive bias reaches a maximum, then declines to negative values, reaches a minimum and then increases towards another positive maximum before declining through small positive values. However, the authors did not entertain any discussion as to how the above behaviour is generated.

Butler and Schachter (1986) advanced an analytical explanation based upon second-order Taylor series approximation to the non-linearity bias. [2] The bias in this form is equal to half of the product of the variance of s^2 and $h''(v^2)$, the second-order derivative of $h(v^2)$ with respect to v^2. It is $h''(v^2)$ that determines the sign of the bias and also incorporates the effect of the striking price. The sign and magnitude of $h''(v^2)$, in turn, are related to $\partial^2\phi(d_i)/\partial(v^2)^2$, $i = 1, 2$. After recognizing this, Butler and Schachter, however, carried on their discussion in terms of the general second-order derivative's $(\partial^2\phi(d)/\partial d^2)$ behaviour. This leads to serious errors in capturing the direction and movement of the bias over wide ranges of moneyness. For an example, we compared the behaviour of $\partial^2\phi(d_1)/\partial d_1{}^2)$ with that of the non-linearity bias in Boyle and Ananthanarayanan's (1977) table 2. We found Butler and Schachter's analysis to mispredict the sign of the bias over two of the four ranges of moneyness, [3] predict only two finite extreme value points instead of three and even then mispredicting their locations, As the following expression for $h''(v^2)$ demonstrates, these errors result from oversimplification of the analysis:

$$h''(v^2) = x\left[\phi'(d_1)\frac{\partial^2 d_1}{\partial(v^2)^2} + \left(\frac{\partial d_1}{\partial v^2}\right)^2 \phi''(d_1) - \frac{1}{g}\left\{\phi'(d_2)\frac{\partial^2 d_2}{\partial(v^2)^2} + \left(\frac{\partial d_2}{\partial v^2}\right)^2 \phi''(d_2)\right\}\right]. \tag{2}$$

2. Reexamination of the striking price bias

The expression for $h''(v^2)$ in (2) can be reduced to a simpler form:

$$h''(v^2) = \frac{x\sqrt{T}}{4v_3}\left[(d_1 d_2 - 1)\phi'(d_1)\right]. \tag{3}$$

Using (3), the in the money and out of the money zero bias points can be located as $g = \exp(w)$ and $g = \exp(-w)$, where $w = (v^2T/2)\sqrt{(1 + 4/v^2T)}$. For the hypothetical situation, $c = 50$, $r = 0.015$, $T = 1$, $v^2 = 0.025$, the zero bias stock prices are thus predicted to be 42.03 and 57.72. These predictions compare remarkably well against Boyle and Ananthanarayanan's (1977) numerical integration results of 42.17 and 57.53 for the same situation. Our analysis also correctly predicts positive (negative) bias beyond (between) the zero bias points.

Our analysis suggests that extreme values of the bias function occur at $d_1 = \pm\infty$ and at the roots of the following cubic equation in d_1:

$$d_1^3 - d_1^2 v\sqrt{T} - 3d_1 + v\sqrt{T} = 0. \tag{4}$$

[1] See Galai (1983) for a survey of reported B/S pricing biases. Geske and Roll (1984) and Rubinstein (1985) are two of the more recent studies.

[2] We have found that the second-order approximation closely traces the behaviour of the non-linearity bias. In terms of magnitude, it captures more than 90% on the average.

[3] The moneyness ranges are given by d_1 varying from $-\infty$ to -1, -1 to 0, 0 to $+1$ and $+1$ to $+\infty$.

Table 1
Analytical predictions vis-à-vis Monte Carlo results about the stock prices where the bias is either zero or has an extreme value. [a]

v^2	T	In the money zero bias stock price	Out of the money zero bias stock price	In the money relative maximum bias stock price	Out of the money relative maximum bias stock price	Around the money minimum bias stock price
0.010	0.5	53.26	46.24	55.95	43.79	49.50
		53.00	46.00	56.00	44.00	50.00
0.025	0.5	55.51	44.37	59.86	40.63	49.32
		55.00	43.00	61.00	40.50	50.00
0.045	0.5	57.68	42.70	63.65	37.83	49.07
		58.00	41.50	65.00	38.00	50.00
0.010	1.0	54.44	44.56	58.28	41.21	49.01
		55.00	44.00	58.50	41.00	50.00
0.025	1.0	57.72	42.03	63.99	36.98	48.64
		58.00	41.00	65.00	37.50	50.00
0.045	1.0	60.96	39.79	69.61	33.32	48.16
		62.00	39.00	71.00	34.00	50.00
0.010	1.5	55.27	43.24	60.00	39.24	48.52
		56.00	42.50	62.00	40.00	50.00
0.025	1.5	59.39	40.24	67.15	34.28	47.98
		60.00	40.00	68.50	35.50	50.00
0.045	1.5	63.53	37.62	74.25	30.09	47.27
		65.00	37.50	79.00	30.00	50.00

[a] For a given combination of v^2 and T, the two rows indicate analytical predictions of this paper and Chaudhury's (1985) Monte Carlo results respectively. In the above computations, $r = 0.015$ and $c = 50$ were assumed.

Using the solution method in Spiegel (1968, p. 32), the roots are found to be all real. Like the zero bias points, these roots are functions of v^2 and T only. Continuing with our previous example, the roots are calculated as -1.73, 0.0 and 1.73. The corresponding stock prices 37, 49 and 64 compare well to Boyle and Ananthanarayanan's (1977) values around 40, 50 and 65. Second-order derivative test of the bias function correctly suggests maxima at 37 and 64, and minimum at 49.

In order to consider the goodness of our predictions over a wider range of parameter variations, a comparison with the Monte Carlo results of Chaudhury (1985) is presented in table 1. As can be seen from the table, the analysis of this paper yields critical stock prices within a dollar for almost all cases considered. Given the magnitude of stock prices, these errors are quite negligible.

3. Conclusions

Our analytical examination confirms the finding of Butler and Schachter's (1986) simulation study: as the variance rate and/or time to maturity increases, the moneyness range over which underpricing by the conventional B/S price estimator takes place expands. Consequently, whether an option with a given degree of moneyness would be underpriced or overpriced depends upon the variance rate and time to maturity. Empirical studies with inadequate control for these variables would thus have reported diluted striking price biases.

Another implication of our analysis is that unless the sample options share similar stock return variance rates and time to maturity, no one single cut-off value can be meaningfully used to classify options into near in the money vs. deep away from the money. This raises such interesting questions as: were the deep away from the money options of Black (1975) truly deep away?

The analytical construct presented in this paper may be utilized in trying to answer such questions and help formulate a better test of the B/S model. For example, using the time to maturity of an in the money option and a variance rate estimate of the underlying stock's return, one can tentatively identify the in the money zero bias point. If the option is placed to the left (right), it may be termed near in the money (deeper in the money). For a sample of options one can repeat this process to classify them into groups and then calculate the average statistical bias for each group. If the B/S model is valid, the average for near in the money (deeper away from the money) options should be significantly less (greater) than zero. Further, a trader can possibly do a better job utilizing our analysis in selecting temporarily under- or over-priced options if the B/S model prices fairly represent the market prices.

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