Simulating a Standard Agency Model

Michel A. Robe

School of Business
University of Miami
5250 University Drive
Coral Gables, Florida 33124
mrobe@homer.bus.miami.edu
(305) 284-1884

Faculty of Management*
McGill University
1001 Sherbrooke Street W.
Montreal, Canada H3A 1G5
robe@management.mcgill.ca
(514) 398-5204

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ABSTRACT

For typical parametrizations of the standard Holmström (1979) agency model, we show how the three first-order conditions that characterize the optimal contract can be reduced to a single equation. This simplification is feasible both when the uninformed principal maximizes her expected income and when the informed agent designs the contract to maximize his expected utility. In numerical simulations of the model, the reduced-form equation greatly facilitates the choice of initial conditions for the solution algorithms. What is more important, it eliminates the sensitivity of the solution algorithms to the choice of initial conditions.

JEL-Classification: D82, C50, C61, C63

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1. Introduction.

The principal-agent model of Holmström (1979) has been used to tackle a wide range of issues in economics and finance.[1] Preference and technology restrictions that keep the contract mathematically tractable have been studied in detail (Rogerson, 1985; Jewitt, 1988; Faynzilberg and Kumar, 1997) and comparative statics can readily be performed in that model. In contrast, it usually remains impossible to explicitly solve the first-order conditions that define the decision variables' values in the optimal incentive-compatible -- or "second-best" -- contract. Indeed, the history of the model is replete with frustrations in obtaining not only exact, closed-form, but also numerical, solutions. Yet, quantitative applications of the principal-agent model require numerical simulations. A natural question, therefore, is whether the model can be made computationally manageable without further loss of much generality.

In this paper, we show how the three first-order conditions of the second-best problem can be reduced to a single equation. For typical parametrizations of the technology and of the agent's preference over consumption, this simplification is feasible not only in the original framework (in which the uninformed principal maximizes her expected income) but also in the "dual" problem (in which the informed agent designs the contract to maximize his expected utility).

Our reduced-form equation can be graphically depicted in two dimensions, which greatly facilitates the choice of initial conditions. What is more important, employing one equation eliminates the sensitivity of numerical search algorithms to initial conditions. This differs sharply from the fine-tuning required when all three first-order equations are used to simulate the model over a range of parameter values. We illustrate our method's usefulness by solving numerically a simple model of investment financing under moral hazard.

The key to our result resides in positing that the agent has power disutility from effort. Our choice entails no loss of generality *per se*. As long as the chosen function is convex, its particulars do not affect the functional form of the second-best contract. This specific functional form,

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[1] Overviews of the related literature include Holmström and Hart (1987) and Sappington (1991).

however, makes it possible to reduce the second-best problem to one equation while retaining enough free parameters for numerical simulations.

Our paper is most closely related to Kaplan and Mukherji (1993). These authors develop a package for solving Kuhn-Tucker problems and apply it to contract-design in the Maskin and Riley (1984) and Sappington (1983) frameworks. Our goal is different: we identify conditions that yield computational tractability across search algorithms. In addition, Kaplan and Mukherji tackle risk-neutral agents with hidden information. The agent here is risk-averse and takes hidden actions.

Section 2 summarizes the setup. Section 3 gives, for a simple parametrization, the reduced-form equation that characterizes the second-best contract. Section 4 does the same for the "dual" problem. Section 5 illustrates the computational advantages of the reduced-form equation. Section 6 discusses robustness to technology and preference assumptions. Section 7 concludes.

2. The setup.

Consider the usual one-period principal-agent model. At time 0, a risk-neutral principal hires a risk-averse agent to exploit a stochastic production process F. The technology's uncertain time-1 return, $\tilde{y}$, belongs to the principal and is publicly observable. The output y depends on the agent's unobservable effort level, a, and can be viewed as a random variable with distribution $F(y,a)$ parametrized by a. $F(.)$ has a density function $f(y,a)$. The support of y, denoted $Y \quad R^{+}$, is independent of a.

The principal designs the agent's payment schedule $s(y)$. The agent's utility is a separable function of his income, c, and time-0 effort, a: $U(c,a) \equiv u(c) - v(a)$. He has a reservation utility level $\bar{U}$. The time-1 payment $s(y)$ constitutes the only modeled source of income for the agent: $c = s(y)$. The functions $u(.)$: R^{+}->R and $v(.)$: R^{+}->R^{+} are thrice continuously differentiable, with $u(.)$ strictly increasing concave and $v(.)$ increasing convex. For simplicity, the discount rate is set equal to 0. The statistical properties of y, $F(.)$ and the preferences of all players are common knowledge.

The principal chooses a and $s(y)$ to maximize her expected income. She faces two constraints. One is participation: the agent will not sign the contract unless his expected utility is at least as high as his reservation utility level $\overline{U}$. The second constraint is incentive compatibility. Given the agent's effort level *a* is unobservable, the contract must ensure that he finds it optimal to work as promised given the payment schedule *s*(*y*). We impose the weaker but mathematically more tractable requirement that he choose an effort level such that his utility is at a stationary point.

Following Jewitt (1988), some restrictions must be imposed on *F*(.) and *u*(.) so that the optimal contract can be characterized through this "first-order approach." One restriction on *F*(.) is that it must have a concave, monotone likelihood ratio. Jewitt (1988) shows that the two other restrictions on *F*(.) are weaker than assuming a random production function with decreasing marginal returns in each state of nature. The Gamma, Poisson and Chi-squared distributions, among others, satisfy these three conditions. Finally, Jewitt (1988) argues that the restrictions on preferences are met by any constant absolute risk-averse (CARA) utility and by any non-decreasing relative risk averse utility with coefficient of relative risk aversion strictly greater than one-half.

Formally, the principal's problem is to:

$$\max_{\substack{a,\, s(.),\\ \lambda,\, \mu}} \left(\int [y-s(y)]\, f(y,a)\, dy + \lambda \left[\int u(s(y))\, f(y,a)\, dy - v(a) - \overline{U} \right] + \mu \left[\int u(s(y))\, f_a(y,a)\, dy - v'(a) \right] \right) \quad \textbf{(1)}$$

where integrals are taken over *Y*, and where λ and μ are the Lagrange multipliers for the agent-participation and incentive-compatibility constraints, respectively.

3. Characterization of the model by a single reduced-form equation.

Simulating the second-best contract would seem to always involve simultaneously solving all three first-order conditions of program (1) with respect to *a*, λ and μ. For typical combinations of the technology and of the agent's preferences over consumption, however, we claim that this system of three non-linear equations in *a*, λ and μ can be reduced to a single equation.

The assumption needed for this result is that the agent has power disutility from effort:

$$v(a) \equiv \frac{a^n}{A}\ , \quad n > 1 \qquad \textbf{(2)}$$

where $A > 0$ is a scaling factor. Varying A and n provides two degrees of freedom for calibrations, which makes parametrization (2) general enough for numerical simulations. Furthermore, as long as *v(.)* is convex, the optimal payment schedule *s(y)* need only satisfy the familiar Euler equation:

$$\frac{1}{u'(s(y))} = \lambda + \mu \frac{f_a(y,a)}{f(y,a)} \tag{3}$$

It is immediate from Equation (3) that the functional form of *s(y)* is independent of that of *v(.)*. In that sense, positing (2) entails no loss of generality *per se* -- see Faynzilberg and Kumar (1997).

In the remainder of this section, we derive the reduced-form equation when the agent has constant absolute risk-averse (CARA) preferences over consumption and output is distributed exponentially. Both functional forms meet the Jewitt (1988) conditions. CARA utility is widely used in the agency literature.[2] The exponential technology shortens the expressions in Equations (4)-(17) below. In Section 6, we generalize our results to some other typical parametrizations.

When $u(c) \equiv -e^{-\gamma c}$, $\gamma > 0$, and $f(y,a) \equiv \dfrac{e^{-y/(a\theta)}}{a\theta}$, $\theta > 0$, the Euler equation (3) becomes:

$$s(y) = \frac{1}{\gamma} Ln\big(K + H y\big) \quad \text{for a.e. } y, \text{ with } K \equiv \gamma\left[\lambda - \frac{\mu}{a}\right] \text{ and } H \equiv \frac{\gamma\mu}{a^2\theta} \tag{4}$$

The optimal values of a, λ and μ in Equation (4) solve the system of first-order conditions of program (1) with respect to these three variables. Let $X \equiv \dfrac{a\lambda}{\mu}$. Then, substituting Equation (4) into these three first-order conditions yields the system of three equations:

$$\frac{\overline{U}}{a} = \frac{g_1(X)}{\gamma\mu} - \frac{a^{n-1}}{A} \tag{5}$$

$$1 + g_2(X) = -\frac{\gamma\mu\, n\, a^{n-1}}{A} \tag{6}$$

$$\frac{a\theta\gamma + g_3(X)}{n-1} = \frac{\gamma\mu\, n\, a^{n-1}}{A} \tag{7}$$

where $g_1(X) \equiv e^{X-1} Ei\big(1-X\big)$; $g_2(X) \equiv X\, g_1(X)$; $g_3(X) \equiv [1+X][1+g_2(X)]$; and *Ei(z)* denotes the

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[2] For instance, exponential utility is used by Schattler and Sung (1997) in a dynamic agency framework and by Bhattacharya and Pfleiderer (1985) and Stoughton (1993) in a portfolio management problem.

exponential integral function $Ei(z) = -\int_{-z}^{\infty} \frac{e^{-t}}{t}\, dt$ (z<0). Adding Equations (6) and (7) gives a as a function of X:

$$a = g_4(X) \equiv - \frac{(n\text{-}1)[1+g_2(X)] + g_3(X)}{\gamma\,\theta} \tag{8}$$

Replacing a by $g_4(X)$ in Equation (6) leaves μ as a function of X only:

$$\mu = g_5(X) \equiv - \frac{A\,[1+g_2(X)]}{\gamma\, n \left[g_4(X)\right]^{n\text{-}1}} \tag{9}$$

Finally, we can introduce $g_4(X)$ and $g_5(X)$ into (5) to obtain a unique, non-linear, equation in X:

$$g_6(X) \equiv \overline{U}\,\gamma\, g_5(X) - g_1(X)\, g_4(X) + \frac{\gamma\, g_5(X) \left[g_4(X)\right]^n}{A} = 0 \tag{10}$$

In Equation (10), X must be solved for numerically. The numerical solution to Equation (10) in turn yields $a = g_4(X)$; $\mu = g_5(X)$; and $\lambda = \frac{X\,\mu}{a}$.

<u>4. Characterization of the "dual" problem by a single reduced-form equation.</u>

In most agency setups, the principal writes the contract to maximize her expected income. In some cases, it is appropriate to posit instead that the agent designs the contract. Solutions to the original model and this "dual" problem characterize points on the same utility-possibility frontier and are therefore qualitatively similar -- see Innes (1990). Consequently, the "first-order approach" remains valid under the Jewitt (1988) restrictions. The contract's reduction to a single equation, however, proceeds somewhat differently.

For example, in the model of investment financing under moral hazard of Robe (1995), a company run by an entrepreneur (the "agent") needs a fixed amount I of outside capital to fund a new project. The entrepreneur designs the financing package to maximize his total expected utility. The risk-neutral investors (the "principal") must earn at least a competitive expected rate of return on I, their investment. Without loss of generality, this rate is set equal to zero. The contract must also be incentive compatible. Let $t(y)$ denote the entrepreneur's monetary payoff as a function of the project's return, y. Given $t(y)$ is his sole source of income, his problem can be written:

$$\max_{a,\, t(.),\, \lambda,\, \mu} \left(\int u(t(y))\, f(y,a)\, dy - v(a) + \lambda \left[\int [\, y - t(y)\,]\, f(y,a)\, dy - I \right] + \mu \left[\int u(t(y))\, f_a(y,a)\, dy - v'(a) \right] \right) \quad \mathbf{(11)}$$

where λ and μ now stand for the Lagrange multipliers of outside investors' participation constraint and of the entrepreneur's incentive-compatibility constraint, respectively.

With $u(c) \equiv -e^{-\gamma c}$, $\gamma>0$, and $f(y,a) \equiv \frac{e^{-y/(a\theta)}}{a\theta}$, $\theta>0$, the Euler equation for (11) yields:

$$t(y) = \frac{1}{\gamma} Ln\left(K_1 + H_1\, y\right) \quad \text{for a.e. } y, \text{ with } K_1 \equiv \frac{\gamma}{\lambda}\left[1 - \frac{\mu}{a}\right] \text{ and } H_1 \equiv \frac{\gamma\,\mu}{\lambda\, a^2\, \theta} \quad \mathbf{(12)}$$

When $v(a) \equiv \frac{a^n}{A}$ ($n>1$, $A>0$), the system of first-order conditions of program (11) with respect to a, λ and μ can again be reduced to a single equation -- in $X \equiv \frac{a}{\mu}$. This system can be written:

$$a\,\theta = h_1(X) - \frac{1}{\gamma} Log(\lambda) \quad \mathbf{(13)}$$

$$\lambda = \frac{\gamma\, n\, a^n}{A\, X\, h_2(X)} \quad \mathbf{(14)}$$

$$\lambda = \frac{n\,(n\text{-}1)\, a^n}{A\, X \left[a\,\theta - \frac{h_3(X)}{\gamma} \right]} \quad \mathbf{(15)}$$

where $h_1(X) \equiv I + \frac{1}{\gamma}\left[Ln\left(\gamma \,.\, \frac{X-1}{X}\right) - e^{X-1} Ei\left(1\text{-}X\right) \right]$; $h_2(X) \equiv -\left[1 + X\, e^{X-1} Ei\left(1\text{-}X\right)\right]$; and $h_3(X) \equiv [1+X]\, h_2(X)$. From (14) and (15), we get a as a function of X: $a = h_4(X) \equiv \frac{(n\text{-}1)\, h_2(X) + h_3(X)}{\gamma\,\theta}$.

Replacing a by $h_4(X)$ in Equation (14) then yields λ as a function of X only:

$$\lambda = h_5(X) \equiv \frac{\gamma\, n \left[h_4(X)\right]^n}{A\, X\, h_2(X)} \quad \mathbf{(16)}$$

Finally, we can introduce $h_4(X)$ and $h_5(X)$ into (13) to obtain a unique, non-linear, equation in X:

$$h_6(X) \equiv \theta\, h_4(X) - h_1(X) + \frac{1}{\gamma} Ln\left(h_5(X)\right) = 0 \quad \mathbf{(17)}$$

Solving Equation (17) numerically for X yields $a = h_4(X)$; $\mu = \frac{a}{X}$; and $\lambda = h_5(X)$.

5. Computational benefits: An example.

The dual framework of section 4 is used by Robe (1995) to quantify the deadweight costs that arise from using equity, debt and warrants in situations where combinations of these financing instruments are suboptimal. His analysis relies on numerical simulations. This makes the setup a natural choice to illustrate the reduced-form equation's computational advantages.

Solving Equation (17) instead of the system of Equations (13)-(14)-(15) obviously reduces computing time. The less patent but much more relevant benefits of employing one equation arise from the fact that only one initial condition is needed for the numerical solution algorithm.

Consider, for example, the parametrization of Section 4. The free parameters are I, γ, n, A, θ and the impact on output of the entrepreneur's effort. Figure 1 shows a representative plot of Equation (17), using the parameter values chosen by Robe (1995) to simulate program (11). These values, listed in the first column of Table I, are consistent with the calibrations of Haubrich (1994) and Boyd and Smith (1994). They have the following implications. (i) The marginal productivity of the entrepreneur at the optimum lies between 1% and 10% of the expected output $E[y]$. This range seems reasonable given that the entrepreneur's effort is, by assumption, crucial to the project. (ii) The entrepreneur's relative risk aversion at his expected consumption is well within the range (0,10] deemed acceptable by Mehra and Prescott (1985). (iii) The expected rate of return on the investment I varies from 5 to 55%.

<Insert Table I and Figure 1>

The ability to draw a two-dimensional plot of Equation (17) greatly facilitates the choice of initial conditions. That $X > 1$ is already clear from Equation (17) and the definition of $Ei(1-X)$. It is immediate from Figure 1 that Equation (17) has a single root X very close to 1, i.e., that a and μ are almost equal. Taking $X = 1+10^{-8}$ as the initial choice for numerical search algorithms, we readily find the solution $\{a=50.6402; \mu=50.64; \lambda=0.0095221\}$. By comparison, the fact that X is very close to 1 is of little use when considering the full system of Equations (13)-(14)-(15). As a result, it is very difficult to simultaneously venture initial values for all three unknowns a, μ and λ.

Strikingly, the same starting point $X=1+10^{-8}$ works for all the parameter values for which the problem has an interior solution $0<a<\infty$ (last column, Table I). In contrast, for the system of Equations (13)-(14)-(15), sets of initial conditions that appear reasonable may not allow either *Mathematica* or a *Fortran*-based algorithm to solve the system. Suppose that we wish to assess the impact on the contract of high intrinsic project productivity and that we set $\theta=1.5$ instead of $\theta=1.121$ -- the other parameters remain unchanged. A natural set of initial conditions for this sensitivity analysis would be the solution found with our reduced-form equation when $\theta=1.121$. Yet, when $\theta=1.5$, no solution is obtained from this starting point. Indeed, even if we keep the firm size I constant, starting from $\{a=50.6402;\ \mu=50.64;\ \lambda=0.0095221\}$ leads to the correct solution for fewer than 75% of the parameter combinations in Table I.[3] The alternative starting point: $\{a=I;\ \mu=I\text{-}10^{-8};\ \lambda=0.01\}$ looks sensible, and in fact yields the correct solution in the case at hand. Over the range of parameter values considered, though, employing the constant set of initial conditions: $\{a=I;\ \mu=I\text{-}10^{-8};\ \lambda=0.01\}$ fails 7% of the time when the firm size I is fixed and almost 15% of the time when the firm size is modified.

6. Robustness.

We have so far focused on exponential technology. Other technologies that meet the Jewitt (1988) conditions may likewise allow us to easily solve the second-best problem by solving a single equation. Take, for instance, the alternative distribution of output:

$$f(y,a) \equiv \frac{y^{\tau-1}\, e^{\frac{-y}{a\theta}}}{(a\theta)^{\tau}\,(\tau-1)!}\ ,\ \theta>0,\ \tau>1 \qquad \textbf{(18)}$$

In contrast to the exponential, the probability density of output (18) is not monotone decreasing in the level of output and most of the probability mass is not concentrated on very low output levels.

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[3] Percentages were averaged by *Mathematica* over 2772 combinations of parameter values (3 for I; 11 for n; 7 for γ; and 12 for θ). The *Fortran*-based algorithm is even more sensitive to initial conditions. When a "solution" is found that differs from the correct one, the Lagrange multipliers are negative in a few cases and imaginary otherwise.

With $\tau = 2$, a reduced-form equation in the same variable X as in Sections 3 or 4, respectively, can readily be derived in the Holmström (1979) model or its dual.

All our results thus far have also relied on exponential utility. With CARA preferences, the agent's second-best payoffs are concave in the output y. It is natural to also consider situations in which they are convex in y. One way to do so is to assume that he has constant relative risk-averse (CRRA) preferences over consumption and is not very risk averse:

$$u(c) \equiv \frac{c^{1-\sigma}-1}{1-\sigma} \ , \ 0 < \sigma < 1 \tag{19}$$

CRRA preferences are common in financial economics. Together with an exponential distribution or the Gamma distribution (18), preferences (19) ensure that all the integrals that define the second-best contract have closed forms as long as $\sigma=1/j$ (j |N). These closed forms allow us to derive a single reduced-form equation whose solution yields the optimal values of a, λ and μ. With exponential technology, for example, the equation is in $X = 2\mu/\lambda a$ ($X = a/\mu$) if $\sigma = 1/2$ ($\sigma = 1/3$). In contrast to the CARA case, there is a small price to pay for this simplification. There is no guarantee that the first-order approach remains valid when $\sigma \leq 0.5$. In simulations, one therefore has to assume validity and to verify numerically that the second-best contract does implement the promised effort level.

7. Conclusions.

For typical parametrizations, the three first-order conditions that characterize the second-best contract in a standard agency model can be reduced to a single equation. This reduction is feasible both when the uninformed principal maximizes her expected income and when the informed agent designs the contract to maximize his expected utility. Employing this equation allows for easy numerical simulation of the model. It greatly facilitates the choice of initial conditions for the solution algorithms and, what is more important, eliminates the sensitivity of the latter to the choice of initial conditions.

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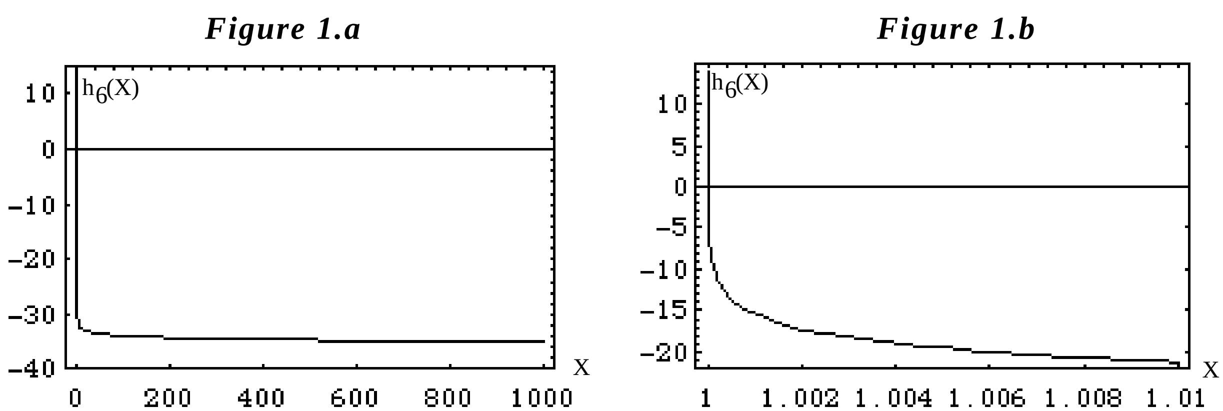
Table I: Parameter Choices for Section 5.

parameter	symbol	central value	range
agent's absolute risk aversion	γ	0.5	0.1 -> 0.6
agent's disutility from effort	n	1.5	1.1 -> 1.6
intrinsic asset productivity *(rate of return on assets, in %)*	θ *(θ-1)*	1.121 *(12.1%)*	1.05 -> 1.6 *(5% -> 60 %)*
marginal productivity of agent's effort	1/I	2%	1% -> 10%

The scaling factor A is set equal to I^2. This ensures that levels of disutility from effort are in the same range as the agent's expected utility from consumption. The central value n=1.5 is chosen so that the rate of return on the investment I be equal to 12.1% when $\gamma = 0.5$, $\theta = 1.121$ and $I = 50$.

Figure 1: Representative Plot of Equation (17).



Figures 1.a and 1.b both plot the value taken by $h_6(X)$, as given by Equation (17), for the central parameter choices in Table I. Figure 1.a illustrates the fact that Equation (17) has only one real solution. Figure 1.b shows that this solution is very close to 1.