

Correlated Noise and Memory Effects in Neural Firing Statistics

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Abstract. This paper discusses two problems at the forefront of neurobiology and of noise research. They arise from non-renewal firing processes in nerve cells, due to various forms of memory. The combination of short and long-term correlations between firing intervals has been shown to enhance information transfer, namely by causing a minimal variability in the spike count distribution at a specific counting time. The first problem concerns first passage time calculations in a model that combines these two forms of correlations. It is a two-dimensional leaky integrate-and-fire (LIF) model in which the threshold is also a dynamical variable. The second problem concerns the effect of long-range correlations on neuron firing statistics. We show new results on the interspike interval densities as well as the spike count Fano factor for the perfect integrate-and-fire (PIF) model forced by a slow (long-correlation time) Ornstein-Uhlenbeck process, which is a simplification of the previous model. These theoretical results are obtained using a quasi-static noise approximation. There remain, however, many exciting challenges in relating correlations with signal detection in neurobiological systems, some of which are highlighted in our paper.

INTRODUCTION

The long time statistics of neural output is important when studying topics such as signal detection and neural encoding. In order to assess how much information a neuron can transmit in response to certain stimuli we must know the variability of its output spike train under different conditions. Long range correlations are present in many natural stimuli such as natural images [1] as well as music [2], so it is of interest to study neuron models with implicit long range correlation.

It has also recently been shown [3] that the resting discharges of a certain class of electroreceptors have long range correlations. In particular, the ratio of the spike count variance to the spike count mean, also known as the Fano factor, goes through a minimum as a function of the duration of the counting window. This is useful from the point of view of signal detection, since this minimal Fano factor will produce a high signal-to-noise ratio for determining the presence of small amplitude low-frequency stimuli (such as *Daphnia* on which the weakly electric fish feed). The origin of this effect in a dynamical model of an electroreceptor has been explained using a combination of theory and numerics [4]. The effect relies on a combination of positive and negative correlations in firing time intervals. We have further found that correlations produce a counter-intuitive increase in mutual information for higher frequency signals. A general theory for this effect is presently lacking, but we are making progress, as we will show below.

In fact, there are few (but a growing number of) theoretical results on the statistics of

correlated firing. Most formalisms, such as first passage time to threshold calculations, have been developed for renewal processes in which there are no correlations between successive first passage times. Also, there is little known about first passage time problems in non-Markovian systems, such as those involving delayed feedback (i.e. where the dynamical equations are formulated as stochastic delay-differential equations); this is true even for white noise forcing, i.e. regardless of whether there are correlations between successive passage times or not. In this paper we report recent progress on our analysis of correlated firing in simple neuron models without delays.

CORRELATED FIRING IN THE LIF MODEL

The model in which we have discovered that correlated firing can improve signal detection is the standard leaky integrate-and-fire model (LIF), with two sources of noise and a memory that persists beyond the firing of a spike [4, 5]. One noise is quasi-white and of moderate strength, and accounts for the noisy phase locking properties of the receptors' firings to the carrier wave emitted by these fish (≈ 1000 Hz); it can be seen as a mixture of conductance noise and synaptic noise. The memory factor is accounted for by a dynamic threshold, which is raised every time the cell fires, and decays between firings. This dynamic threshold produces negative correlations between successive firing intervals, and is responsible for decreasing the Fano factor below its value of one in the Poisson limit. So what accounts for the minimum of the Fano factor? It has to be a mechanism that counteracts this decrease, i.e. which increases the Fano factor at longer counting times.

In our earlier work, we have found that an increase in the Fano factor can occur via the introduction of a small, slow Ornstein-Uhlenbeck process in the model; this is the second noise source in the model, which can be associated with e.g. slower fluctuations in vesicular release rates at synapses. For spike counting times less than the correlation time of this OU noise, the intervals are positively correlated on average. It is as though the noise appears as non-stationary over the counting time. Positive correlations increase the Fano factor. The combination of the decrease (due to memory) and increase (due to weak slow noise) sets the position of the minimum of the Fano factor, and thus the time scale of integration of information over which signal detection is optimal.

Such increases in Fano factors have been seen in various data from different neurons, and in statistical models of neural firing in the form of modulated Poisson processes. Such behavior had not been shown in dynamical models of excitability until our recent study [5]. As mentioned earlier, first passage time to threshold calculations usually assume the "renewal" property, which the interspike interval (ISI) data we are modeling clearly do not have. From the point of view of our model, the "non-renewability" is due to the slow OU process with respect to the counting window, and also to the memory effects carried by the threshold (this is a form of adaptation).

The threshold memory is also difficult to deal with, because the escape time process is then in a two-dimensional space. We have made some progress on calculating the mean first passage time for the case of a simple leaky integrate-and-fire model with Gaussian white noise forcing and dynamical threshold. The idea is to solve the problem

self-consistently. The firings act as delta-function forcing on the first-order threshold dynamics. In the asymptotic state, the mean firing rate produces a mean threshold value. This value can be put in the expression for the mean first passage time of the leaky integrate and fire dynamics - this is basically the classic Ornstein-Uhlenbeck process, and one can use the standard formula for its mean first passage time(MFPT). Since this mean rate (the inverse of the MFPT) affects the threshold, and vice-versa, one can find the steady state value for both the mean rate and threshold. The result (not shown) agrees surprisingly well with numerical simulations for a range of integration time constants. This is so even though, in this description, the correlations between firing intervals are not taken into account. It is not clear however how to extend this formalism in a way that will allow the calculation of the correlations.

CORRELATIONS IN THE PIF MODEL

Model

The first passage time with slow OU forcing in the previous model is currently under investigation using a phase representation of the receptors' voltage. The receptor is forced by a 1000 Hz carrier wave; the system is designed to detect modulations of this carrier. Our model predicts that without noise, the receptor simply fires periodically at a high rate, with leakiness playing a secondary role. Thus, its noisy periodic firing dynamics can be approximated by such a phase (rotations of 2π count as firings). The model then amounts to the perfect integrate and fire (PIF) neuron forced by the slow OU process. We will see that many features of the Fano factor and other firing statistics can be understood from this simple model with only one noise source (the slow one) and no dynamical threshold.

The type of input noise we are using is an Ornstein-Uhlenbeck (OU) process; the statistics we are calculating are the interspike interval probability density, its first and second moments, and the Fano-factor which is a measure of spike count variability.

The dynamical equations defining our system are

$$\begin{aligned}\frac{dv(t)}{dt} &= \mu + \eta(t) \\ \frac{d\eta(t)}{dt} &= -\frac{\eta(t)}{\tau} + \sqrt{\frac{D}{\tau}}\xi(t)\end{aligned}\tag{1}$$

where $v(t)$ is the membrane voltage, μ is the average rate of increase of the voltage, and τ and D are, respectively, the time constant and variance of the OU process. $\xi(t)$ is a Gaussian white noise process with zero mean and unity variance.

The voltage increases due to the mean rate μ and the OU process and is reset to zero when it reaches a threshold level, v_{th} ; the noise is not reset. In this model the times at which the voltage is reset represent the spike times of the neuron, or the times at which an action potential is initiated.

Quasi-Static Approximation

Since we are looking at the effect of long time correlated noise, we will use a quasi-static approximation of the noise. If τ is known to be much larger than the average ISI, then on that time scale we can approximate η as being constant. In this way we can associate each ISI with a corresponding, unique value of the OU process

$$I_k = \frac{v_{th}}{\mu + \eta_k} \quad (2)$$

where k denotes the index in a sequence of ISIs.

This approximation not only allows us to write down a conditional probability density function (PDF) between I and η , but it also allows us to reduce this conditional PDF to a delta function due to the unique one-to-one correspondence.

$$P(I_k|\eta_k) = \delta\left(I_k - \frac{v_{th}}{\mu + \eta_k}\right) \quad (3)$$

Here the process η_k is a sequence of the values of the OU at times when the voltage process reaches threshold. As will be shown the statistics between $\eta(t)$ and η_k differ slightly due to biased sampling.

Stationary Probability Density Function

The first quantity of interest is the stationary PDF of ISIs. In order to obtain the stationary ISI PDF we can average the conditional PDF between I and η over all values of η

$$P_{st}(I) = \int_{-\infty}^{\infty} d\eta P(I|\eta_k) P_{st}(\eta_k) \quad (4)$$

The statistics of the values η_k is not the same as for the continuous OU process. Imagine we measure the noise value at the beginning of each interval of a long spike train. Then a positive value leads to a shorter interval and hence to more intervals than a negative value of η . This problem is known as biased sampling of a stochastic variable [6] and is resolved by a corrective factor given by the inverse interspike interval (see also [7].) Normalization yields

$$P_{st}(\eta_k) = \frac{e^{-\eta_k^2/2D}}{\sqrt{2\pi D}} \left(1 + \frac{\eta_k}{\mu}\right). \quad (5)$$

The normalization as well as any integration in the remainder of the paper is performed with respect to the full range of noise values (including $\eta < -\mu$) for simplicity. Inserting

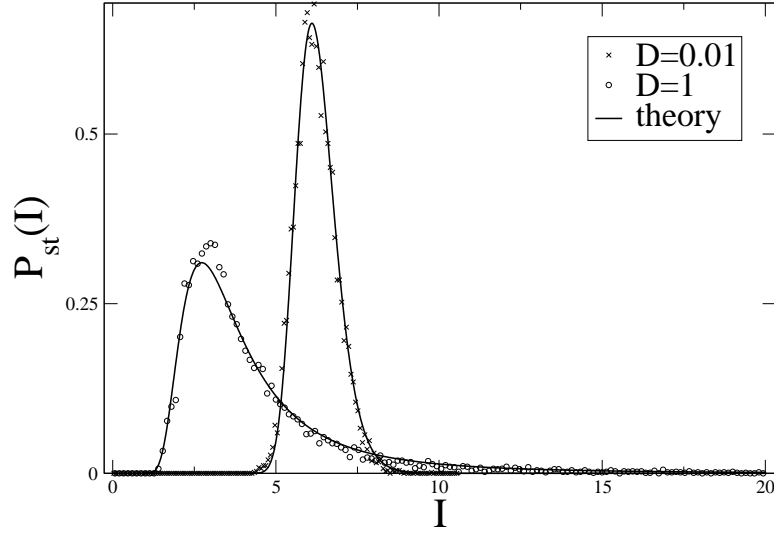


FIGURE 1. Stationary ISI probability density functions. Numerical simulations results for fixed τ and different values of variance, D . The solid lines show the theoretical densities.

equation (5) into (4).yields the PDF for the interspike interval density.

$$\begin{aligned}
 P_{st}(I_k) &= \int_{-\infty}^{\infty} d\eta_k \delta\left(I_k - \frac{v_{th}}{\mu + \eta_k}\right) \frac{e^{-\eta_k^2/2D}}{\sqrt{2\pi D}} \left(1 + \frac{\eta_k}{\mu}\right) \\
 &= \int_{-\infty}^{\infty} d\eta_k \delta\left(\eta_k - \frac{v_{th}}{I_k} + \mu\right) \frac{e^{-\eta_k^2/2D}}{\sqrt{2\pi D} v_{th} \mu} (\mu + \eta_k)^3 \\
 &= \frac{v_{th}^2}{\sqrt{2\pi D} \mu} \frac{e^{-(\frac{v_{th}}{I_k} - \mu)^2/2D}}{I_k^3}
 \end{aligned} \tag{6}$$

Figure 1 shows the stationary PDF for fixed τ and several values of D from both numerical simulation and the corresponding theoretical curves (in black) using (6). The mean of the sampled OU process is given by

$$\langle \eta_k \rangle = \int_{-\infty}^{\infty} d\eta_k \frac{e^{-\eta_k^2/2D}}{\sqrt{2\pi D}} \left(\eta_k + \frac{\eta_k^2}{\mu}\right) = \frac{D}{\mu}. \tag{7}$$

Fano Factor

The Fano factor which is the variance to mean ratio of a counting process, $n(t)$, for a given counting window is useful for determining on which time scales the process is most regular. A simple expression for the Fano-factor for our model neuron can

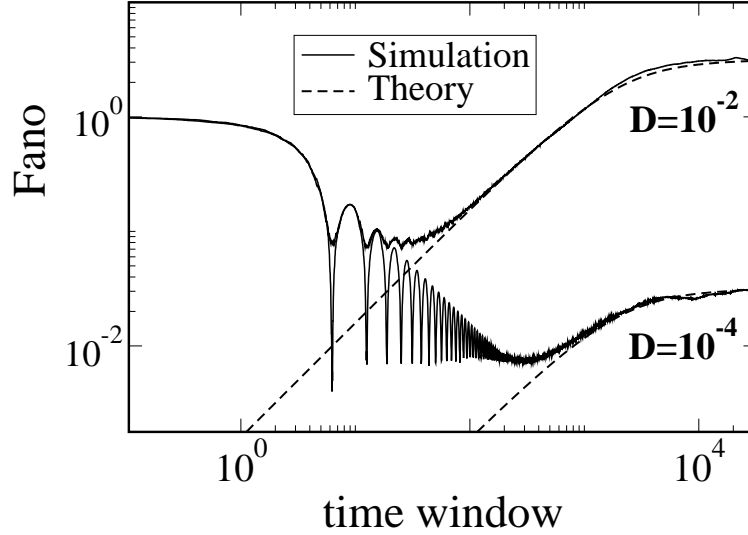


FIGURE 2. The numerical results (symbols) of the Fano-factor for different noise intensities with $\tau = 1000$. The dashed lines are the theoretical curves which are valid in the large counting time limit.

be obtained considering that on a scale much larger than v_{th} fluctuations in the actual membrane voltage, $v(t)$ and fluctuations in the discrete process $n(t)$ times v_{th} become indistinguishable. This is valid provided that there is some nonnegative bias on the dynamics of the voltage. With this in mind we can see that the dynamical equations of our system can be mapped to those of biased Brownian motion. The Fano factor at time t is then equivalent to the variance of the corresponding Brownian motion [8] divided by the mean drift of the Brownian motion

$$F(t) = \frac{2D\tau}{v_{th}\mu} \left(1 - \frac{\tau}{t} \left(1 - e^{-t/\tau} \right) \right) \quad (8)$$

The Fano factor can also be obtained with the SCC and using the relation [6]

$$Var(I_k) = Var(I_1) \left[k + 2 \sum_{l=1}^{k-1} (k-l)\rho_l \right] \quad (9)$$

and the equality

$$\lim_{t \rightarrow \infty} F(t) = \lim_{k \rightarrow \infty} \frac{Var(I_k)}{\langle I \rangle^2 k} \quad (10)$$

This relation holds very well even when the limits are taken to finite values. The approximate equality holds for the times $t = \langle I \rangle k$.

Figure 2 shows $F(t)$ for different variances of numerically generated OU processes with $\tau = 1000$ (solid line). Note that the various simulation values converge with the theoretical Fano curves (8) (dashed lines) at different times depending on the noise variance.

CONCLUSIONS

In this paper we have discussed outstanding unsolved problems on correlated firing in neurons. We have obtained analytic expressions for certain statistical measures for the perfect integrate-and-fire model with additive correlated noise. This model can be seen as a limiting case of the standard leaky integrate-and-fire neuron with vanishing leakiness, or with a strong drive to threshold (i.e. when it is periodically firing at a moderate rate even in the absence of stochastic input). We have derived approximate expressions for the stationary ISI probability density function and for the mid-to-long time behavior of the Fano-factor of the spike count over a given time period.

One interesting consequence from the numerical simulations shown in Figure 2 is that, for a range of noise variances, there is an apparent minimum in the Fano factor. Previously it was thought that a minimum in the Fano factor was only possible in a leaky integrate and fire neuron (LIF) with dynamic threshold [4, 5]. This minimum sets a time scale for the optimal discriminability of two distinct stimuli. This implies that the perfect integrate and fire neuron could also make a good signal detector for a selected range of noise variances if the counting time is chosen to correspond to a minimum of the Fano factor. Short-lived negative correlations will further influence the position of the minimum and are likely to deepen the minimum, leading to higher detectability.

Future work will consider expressions for the full Fano factor curve which will allow us to relate the minimum to specific biophysical factors. We will also consider problems in which there are dynamical memory effects in the form of delayed feedback, in which the spikes propagate back to the cell after a delay time and act e.g. as a simple input current. The presence of this delay clearly renders the whole problem non-Markovian. We have made some progress on understanding escape times in potentials with delayed dynamics [9] for small delays. For larger delays, a new formalism is needed. Here again we can hope that a self-consistent approach can work to some extent, but that might neglect the effects that the delay can induce, such as oscillations in the firing rate. We have made some recent progress here by assuming that the "neuron" is a simple threshold crossing detector (forthcoming work from A. Longtin and R. Morse). It is our hope that such simplified dynamical models with memory will yield some insight into the dynamics of real neural loops with delays. Such circuits are ubiquitous in neurobiology, in particular in the pyramidal cell circuits of the weakly electric fish which receive input from the aforementioned electroreceptors.

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