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THE UNILATERAL CONTACT BETWEEN A RIGID CIRCULAR PUNCH ON A HALFSPACE AND A MINDLIN FORCE

A.P.S. Selvadurai
Department of Civil Engineering,
Carleton University, Ottawa, Ontario, Canada K1S 5B6

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ABSTRACT

This paper presents exact closed form solutions and relevant numerical results to the axisymmetric problem of the unbonded contact between an internally loaded elastic halfspace and a rigid circular punch which is subjected to a central force.

INTRODUCTION

The axisymmetric interaction between a rigid circular punch on a isotropic elastic halfspace and a Mindlin force was examined by Selvadurai [1]. In this treatment, the interface between the rigid circular punch and the elastic halfspace was considered to be smooth. It was also assumed that the relative magnitudes of the force on the rigid punch (P) and the internal Mindlin force (P_M), located a point along the axis of symmetry and directed along the axis, were such that there was no separation at the rigid punch-elastic halfspace interface. In general, for this assumption to be realized either the external force on the punch must be significantly greater than the internal Mindlin force or the internal force should be remotely located from the plane surface of the halfspace region. In situations where these conditions are not satisfied, separation will occur at the smooth interface between the rigid punch and the elastic halfspace. In this paper, we present an analysis of the unilateral contact problem which results from the occurrence of separation at the contact between the punch and the smooth surface of the halfspace (Figure 1). The numerical results presented in the paper illustrate the manner in which the dimensions of the contact region and the resulting displacement of the punch are influenced by the relative magnitudes of the external load and the internal Mindlin force.

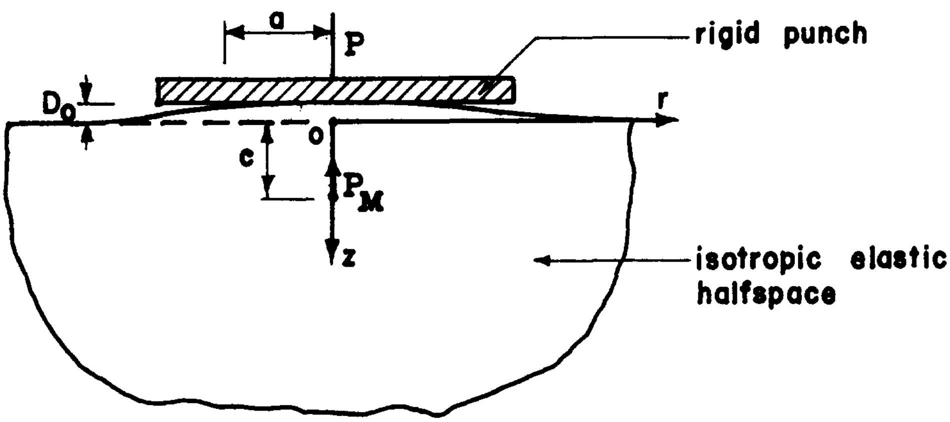

Figure 1. Geometry of the unilateral contact problem.

ANALYSIS

The axisymmetric problem associated with the unilateral contact problem can be formulated by appeal to Hankel transforms and the theory of dual integral equations. The mixed boundary conditions associated with the unilateral contact problem are

$$u_z(r,0) = -D + f(r) \quad ; \quad 0 \leq r \leq a \tag{1}$$

$$\sigma_{zz}(r,0) = 0 \quad ; \quad a < r < \infty \tag{2}$$

where u_z is the axial displacement, σ_{zz} is the axial stress and a is the radius of the contact region. In (1), the function $f(r)$ is prescribed in such a way that $f(0) = 0$ and D is a constant. In addition to (1) and (2) it is assumed that σ_{rz} is zero over the entire surface of the halfspace region. Following Sneddon [2], we can show that the mixed boundary conditions (1) and (2) yield the following system of dual integral equations for an arbitrary function $\psi(\xi)$; i.e.

$$H_0\left[\xi^{-1}\psi(\xi); \xi \to r\right] = D - f(r) \quad ; \quad 0 \leq r \leq a \tag{3}$$

$$H_0\left[\psi(\xi); \xi \to r\right] = 0 \quad ; \quad a < r < \infty \tag{4}$$

where H_0 is the Hankel transform of order zero which is defined by

$$H_0\left[\Omega(\xi, z); \xi \to r\right] = \int_0^\infty \xi\Omega(\xi, z)J_0(\xi r)d\xi \quad . \tag{5}$$

The solution of the dual system (3) and (4) is straightforward; representing $\psi(\xi)$ by a result of the form

$$\psi(\xi) = \int_0^a \chi(t)\cos(\xi t)dt \tag{6}$$

the boundary condition (4) is identically satisfied and the equation (3) reduces to the Abel integral equation. The details of the solution procedure are given in [2] and will not be pursued here. It is sufficient to note that the function $\chi(t)$ is related to the displacements D and $f(r)$ by relationship

$$\chi(t) = -\frac{2D}{\pi} + \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{rf(r)dr}{(t^2 - r^2)^{\frac{1}{2}}} \tag{7}$$

In order to develop the results relevant to the unilateral smooth contact problem, we assume that at the boundary contact region $r = a$ the stress σ_{zz} is finite. Consequently, the condition $\chi(a) = 0$ gives the result for the displacement D; i.e.

$$D = \int_0^a \left(\frac{df}{dr}\right) \frac{dr}{(a^2 - r^2)^{\frac{1}{2}}} \tag{8}$$

Also it can be shown that the total load on the punch is given by

$$P = \frac{4G}{(1-\nu)} \int_0^a \left(\frac{df}{dr}\right) \frac{r^2 dr}{(a^2 - r^2)^{\frac{1}{2}}} \tag{9}$$

THE UNILATERAL CONTACT PROBLEM

We apply the procedures and results described in the previous section to examine the unilateral contact problem. Considering the displacements of the elastic halfspace due to the Mindlin force we have

$$f(r) = \frac{P_M(1-\nu)}{2\pi G}\left[\frac{1}{(r^2+c^2)^{\frac{1}{2}}} + \frac{c^2}{2(1-\nu)(r^2+c^2)^{\frac{3}{2}}} - \frac{(3-2\nu)}{2(1-\nu)c}\right] \tag{10}$$

and the constant D is given by

$$D = D_0 - \frac{P_M(3-2\nu)}{4\pi G c} \tag{11}$$

where D_0 is the net displacement of the indenting punch due to the combined action of P and P_M (Figure 1). Considering (8), (10) and (11) it can be shown that for a particular radius of contact a, the displacement of the contact region D_0 is given by

$$\frac{D_0}{W_1(0)} = 1 - \frac{2(1-\nu)a^2}{(3-2\nu)(a^2+c^2)}\left[1 + \frac{(a^2+3c^2)}{2(1-\nu)(a^2+c^2)}\right] \tag{12}$$

where

$$W_1(0) = \frac{P_M(3-2\nu)}{4\pi G c} \quad . \tag{13}$$

Considering (9), (10) and (11), the external load on the punch P required to initiate the contact region is given by

$$\frac{P}{P_M} = \frac{2}{\pi}\left[\frac{ac}{(a^2+c^2)} - \tan^{-1}\left(\frac{a}{c}\right) - \frac{a^3 c}{(1-\nu)(a^2+c^2)^2}\right] \quad . \tag{14}$$

The results (12) and (13) can be combined to evaluate a relationship between the displacement of the contact region D_0 and the relative magnitudes of the Mindlin force and the external force.

The normal stress within the contact region can be evaluated by making use of the result

$$\sigma_{zz}(r,0) = \frac{G}{r(1-\nu)} \frac{d}{dr} \int_r^a \frac{t\chi(t)dt}{(t^2 - r^2)^{\frac{1}{2}}} \tag{15}$$

Considering (7) and (10) it can be shown that

$$\begin{aligned} \frac{\sigma_{zz}}{P/\pi a^2} &= -\left(\frac{P_M}{P}\right) \frac{1}{\pi(1-\nu)} \left\{ \left[\frac{(1-2\nu)a^2 c}{2(r^2+c^2)^{\frac{3}{2}}} + \frac{3a^2c^3}{2(r^2+c^2)^{\frac{5}{2}}} \right] \times \right. \\ &\times \left. \left[\tan^{-1}\left(\sqrt{\frac{a^2-r^2}{r^2+c^2}}\right) + \frac{[(a^2-r^2)(r^2+c^2)]^{\frac{1}{2}}}{(a^2+c^2)} \right] + \frac{a^2c^3(a^2-r^2)^{1/2}}{(r^2+c^2)(a^2+c^2)^2} \right\} \end{aligned} \tag{16}$$

NUMERICAL RESULTS

Exact closed form relationships have been developed for the problem related to the unilateral contact between a loaded rigid circular punch and an internally loaded isotropic elastic halfspace. The loadings are assumed to be such that the state of deformation is axisymmetric. The Figure 2 illustrates the manner in which the displacement of the rigid punch is influenced by the Poisson's ratio of the elastic medium and the relative magnitudes of the external load on the punch and the internal Mindlin force. These results are normalized with respect to the maximum surface displacement which occurs at the surface of the halfspace due to Mindlin force, in the absence of the rigid punch. It is evident that a greater part of the influence of P_M occurs when $(P_M/P) \epsilon (0, 10^2)$. The Figure 3 illustrates the extent to which the radius of the contact region is influenced by the load ratio P_M/P. These results are consistent with the limiting conditions $a/c \to \infty$ as $P_M/P \to 0$ and $a/c \to 0$ as $P_M/P \to \infty$. The Figures 4 and 5 illustrate the manner in which the stress within the contact zone is influenced by radius of the contact zone and consequently, through Figure 3, the force ratio P_M/P.

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REFERENCES

[1] A.P.S. Selvadurai, The interaction between a rigid circular punch on an elastic half-space and a Mindlin force, *Mechanics Research Communications*, Vol.5, pp.57-64 (1978).

[2] I.N. Sneddon, The relation between load and penetration in the axisymmetric Boussinesq problem for a punch of arbitrary profile, *International Journal of Engineering Science*, Vol.3, pp.47-57 (1965).

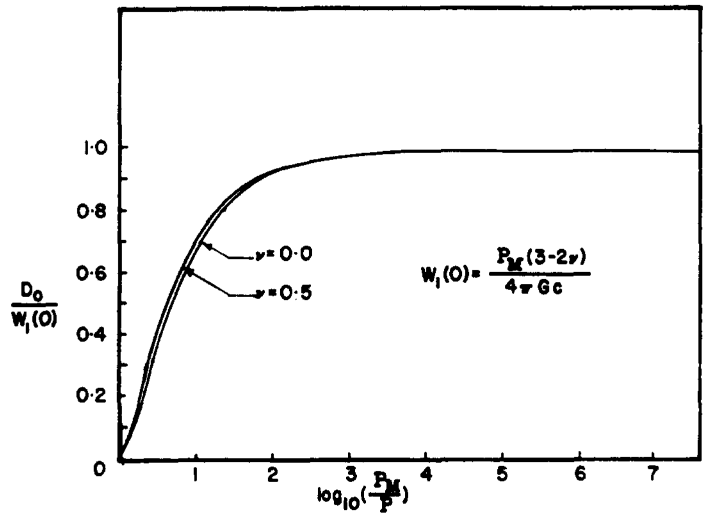

Figure 2. The displacement of the rigid punch.

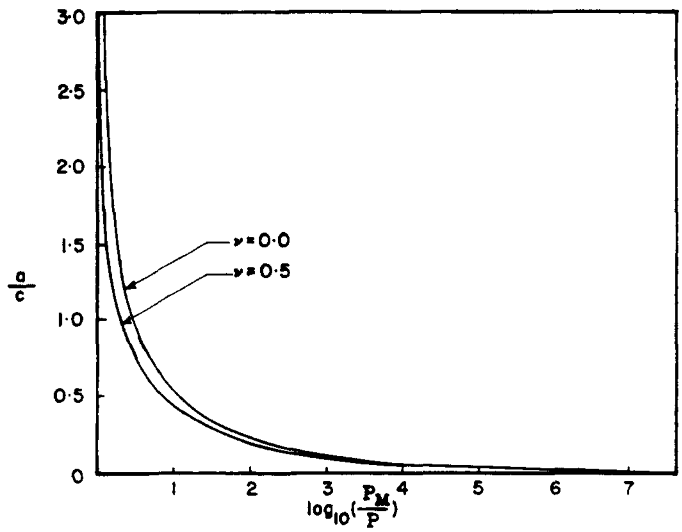

Figure 3. The variation in the radius of the contact zone.

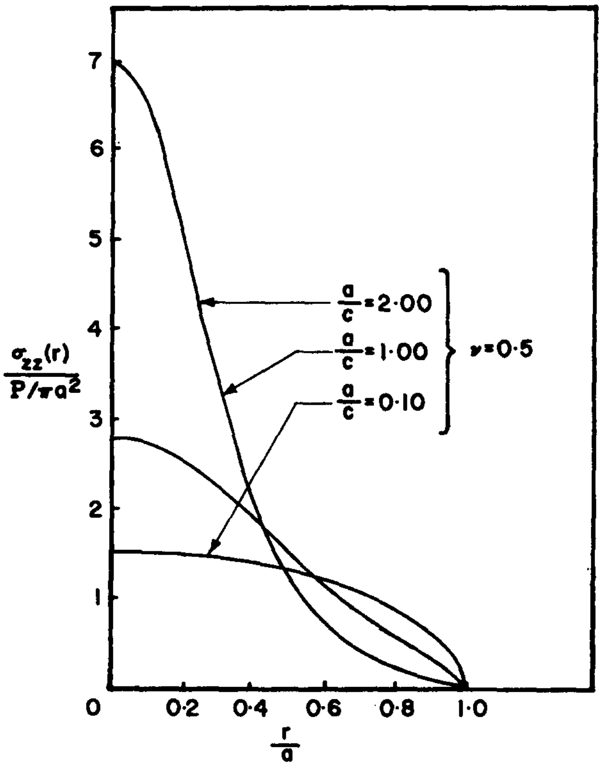

Figure 4. The normal stresses at the contact zone.

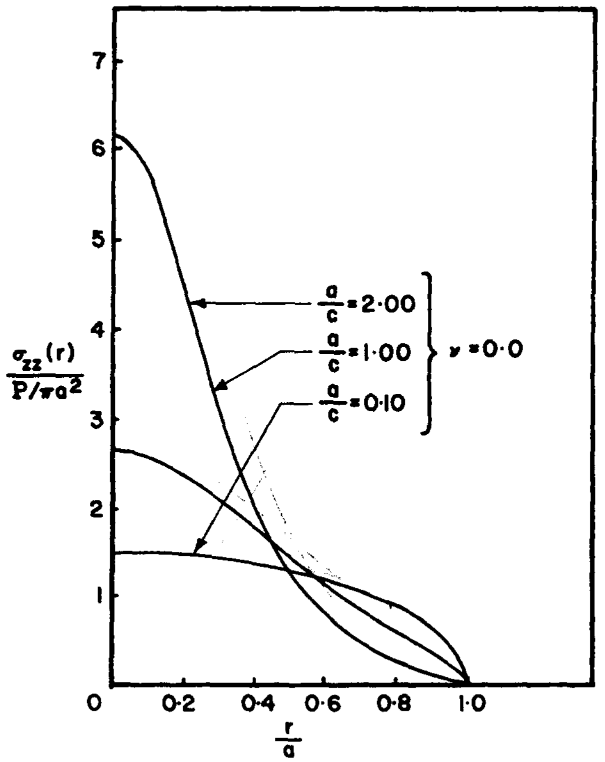

Figure 5. The normal stresses at the contact zone.