Harmonic Response of Smoothly Embedded Rigid Sphere

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Abstract: The category of problems that examines the dynamic behavior of inclusions embedded in elastic media is of particular interest to the study of dynamic soil-structure interaction. In a majority of investigations, the interface between the inclusion and the surrounding elastic medium is assumed to be perfectly bonded. This paper examines the elastodynamic problem pertaining to the steady rectilinear oscillations of a spherical rigid inclusion that is embedded in smooth bilateral contact with an isotropic elastic medium of infinite extent. The bilateral contact is maintained by the application of a uniform radial stress field at infinity. The inclusion is subjected to a harmonic oscillation. The analytical solution for the resulting elastodynamic problem is developed in exact closed form. The numerical results presented in the paper illustrate the manner in which the dynamic compliance of the embedded rigid spherical inclusion is influenced by the frictionless boundary conditions at the inclusion-elastic medium interface.

Introduction

The dynamic behavior of rigid inclusions embedded in elastic media serves as a useful model for the examination of the dynamic behavior of structures and foundations embedded in soil media. Although soil media possess complicated stress-strain characteristics, such elastodynamic modeling provides a valid first approximation for the treatment of dynamic soil-structure interaction phenomena associated with embedded foundations. Two types of embedded foundations are encountered in engineering practice. A partially embedded foundation is one in which the soil medium associated with the interaction problem can be modeled as a half-space region. The elastodynamic modeling associated with the half-space model has been applied quite successfully in the analysis of a variety of soil-structure interaction problems. Extensive accounts of these developments are given by Eringen and Suhubi (1974), Dasgupta and Chopra (1979), Selvadurai and Rahman (1981) and Gazetas (1983).

In this paper we shall focus attention on the type of problem in which the foundation is deeply embedded in an isotropic elastic medium of infinite extent. Such structural foundations can range from end-bearing pile foundations to anchor regions that are used to resist uplift loads. The problem of the deeply embedded foundation also serves as a useful model for the examination of the performance of in situ dynamic testing devices such as the screw plate test and the penetration test (Fig. 1). In relation to the analysis of dynamic in situ testing devices, the results developed for the dynamic behavior of foundations resting on a half-space region are of limited relevance. The fully embedded nature of an in situ testing device makes it nec-

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Note. Discussion open until February 1, 1991. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on April 21, 1989. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 116, No. 9, September, 1990. ©ASCE, ISSN 0733-9399/90/0009-1945/$1.00 + $.15 per page. Paper No. 25007.

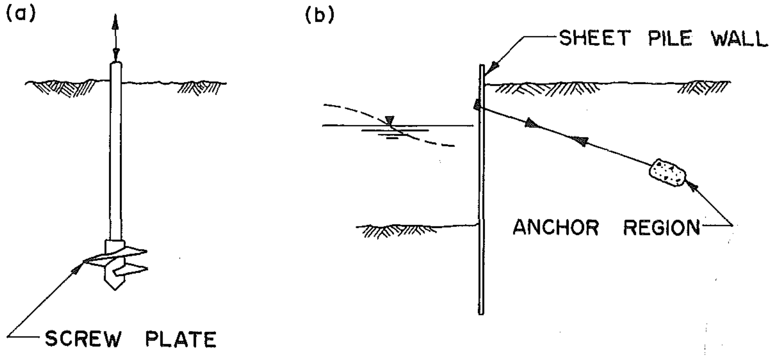

FIG. 1. Dynamic Loading of Embedded Foundations

essary to adopt the analytical treatment in relation to an infinite space rather than a half-space. With reference to the problem of the dynamic behavior of rigid objects embedded in elastic media, Mal et al. (1968), Datta (1970), and Mal (1971) have examined the dynamic interaction between an embedded rigid disc and a normally incident plane harmonic compressional wave. The dynamic problems related to the steady axisymmetric and rotational oscillations of a rigid disc inclusion that is embedded in bonded contact with an isotropic elastic medium of infinite extent has also been examined by Selvadurai (1980, 1981). Kanwal (1965) has presented an approximate solution to the embedded rigid inclusion problem that is based on matched asymptotic expansion techniques. Selvadurai and Rahman (1981) have presented further approximate solutions to the embedded disc inclusion problem in which the contact pressure distributions at the inclusion-elastic medium interface are assigned a priori. Selvadurai (1981) has extended the study of the dynamic loading of a disc inclusion to include viscoelastic behavior of the surrounding medium.

Of specific interest to this paper are the solutions for the steady oscillations of three-dimensional objects such as spheres and ellipsoids embedded in full bonded contact with elastic regions of infinite extent. Such solutions have been developed by Williams (1966), Chadwick and Trowbridge (1967), and Datta and Kanwal (1979). In particular, the solution to the problem of the steady oscillations of a rigid sphere embedded in bonded contact with an elastic medium of infinite extent has been used by Selvadurai (1980) to provide a basis for the theoretical evaluation of the dynamic screw plate test. In view of the uncertainties that are associated with the modeling of such an in situ test, it is necessary to consider a variety of plausible models in which the boundary conditions at the inclusion-elastic medium are subject to variations. Several interface conditions can be utilized in the solution of the embedded rigid spherical inclusion problem. These range from the perfectly bonded interface to the smooth case with Coulomb friction or finite friction occupying an intermediate position. In this paper we focus attention on the smoothly embedded rigid spherical inclusion where a bilateral constraint ensures that there is no separation at the interface. The assumption of a smooth interface with a bilateral contact is a suitable assumption for a deeply embedded foundation since the effects of self-weight stresses ensure that the interface will not experience separation under the action of a steady oscillatory axial loading within the elastic limit.

FUNDAMENTAL EQUATIONS

We consider the problem of an isotropic elastic infinite space that contains a rigid spherical inclusion. The contact at the inclusion-elastic medium interface is considered to be smooth and bilateral. The bilateral contact is maintained by a radially symmetric isotropic stress field σ_0 that is applied at infinity. The rigid spherical inclusion is subjected to steady rectilinear oscillations along the fixed axis z (Fig. 2). The analysis is referred to a system of spherical polar coordinates (R, Θ, Φ), and for axial symmetry, the non-zero components of the displacement vector $\mathbf{u}$ are $(u_R, u_\Theta, 0)$, where u_R and u_Θ = functions of R, Θ, and time t. We assume that the rectilinear oscillations of the smoothly embedded rigid inclusion with bilateral contact imposes the following boundary conditions at the smooth interface $R = a$:

$$u_R(a, \Theta, t) = \zeta(t) \cos \Theta \qquad (1a)$$

$$\sigma_{R\Theta}(a, \Theta, t) = 0 \qquad (1b)$$

where $\zeta(t)$ = the axial displacement; and $\sigma_{R\Theta}$ = the shear stress. Following Chadwick and Trowbridge (1967) and Gurtin (1971), we seek scalar wave function representations of $\mathbf{u}$ of the form

$$\mathbf{u} = \text{grad}\,\mathbf{\Phi} + \text{curl}\,(\mathbf{x} \wedge \text{grad}\,\mathbf{\Psi}) \qquad (2)$$

In order to satisfy the boundary conditions, we assume that ϕ and ψ admit representations of the form

$$\phi(R, \Theta, t) = \frac{\partial}{\partial R}\left[\frac{1}{R} F(t_P)\right] \cos \Theta \qquad (3a)$$

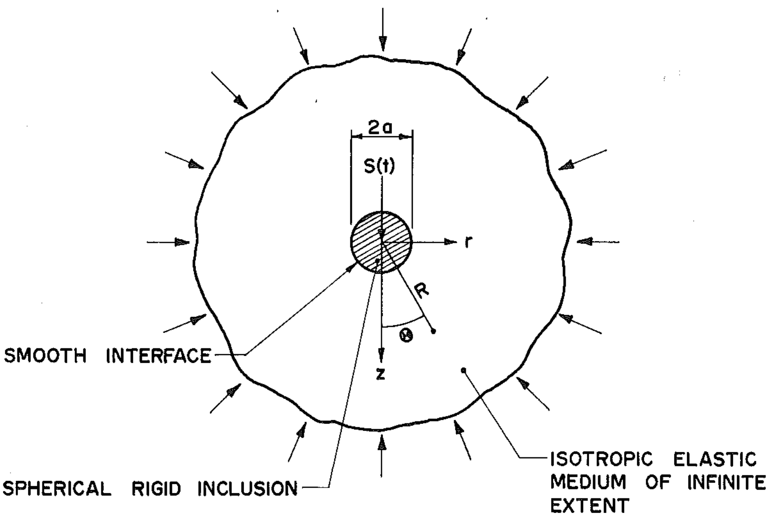

FIG. 2. Geometry of Smoothly Embedded Spherical Rigid Inclusion

$$\psi(R,\Theta,t) = \frac{\partial}{\partial R}\left\{\frac{1}{R}\,G(t_S)\right\}\cos\Theta \quad \text{(3b)}$$

where

$$t_P = t - \frac{(R-a)}{v_P} \quad \text{(4a)}$$

$$t_S = t - \frac{(R-a)}{v_S} \quad \text{(4b)}$$

In Eq. 4, v_P and v_S = the compressional and shear wave speeds, respectively, defined by

$$v_P^2 = \frac{(\lambda + 2\mu)}{\rho_0} \quad \text{(5a)}$$

$$v_S^2 = \frac{\mu}{\rho_0} \quad \text{(5b)}$$

and λ and μ = Lame's constants. The relevant displacement and stress components can be written as

$$u_R = \frac{\partial\phi}{\partial R} + \frac{1}{R\sin\Theta}\frac{\partial}{\partial\Theta}\left(\sin\Theta\,\frac{\partial\psi}{\partial\Theta}\right) \quad \text{(6)}$$

$$\sigma_{RR} = \lambda\nabla^2\phi + 2\mu\left[\frac{\partial^2\phi}{\partial R^2} + \frac{\partial}{\partial R}\left(\frac{1\partial^2\psi}{R\partial\Theta^2} + \frac{\cot\Theta}{R}\frac{\partial\psi}{\partial\Theta}\right)\right] \quad \text{(7)}$$

$$\sigma_{R\Theta} = 2\mu\left[\frac{\partial^2}{\partial R\partial\Theta}\left(\frac{\phi}{R}\right) + \frac{1}{2}\nabla^2\frac{\partial\psi}{\partial\Theta} - \frac{1}{R}\frac{\partial}{\partial R}\left(R\,\frac{\partial^2\psi}{\partial R\partial\Theta}\right) + \frac{1}{2R^2}\frac{\partial}{\partial R}\left(R\,\frac{\partial\psi}{\partial\Theta}\right)\right.$$
$$\left. - \frac{\cot^2\Theta}{2R^2}\frac{\partial\psi}{\partial\Theta}\right] \quad \text{(8)}$$

From the representations (Eq. 3) and the results (Eqs. 6 and 8), we can rewrite the expressions for u_R and $\sigma_{R\Theta}$ in the form

$$u_R = \left[\frac{1}{Rv_P^2}\,F''(t_P) + \frac{2}{R^2v_P}\,F'(t_P) + \frac{2}{R^3}\,F(t_P)\right.$$
$$\left. + \frac{2}{R^2v_S}\,G'(t_S) + \frac{2}{R^3}\,G(t_S)\right]\cos\Theta \quad \text{(9)}$$

$$\sigma_{R\Theta} = -\mu\left[\frac{1}{Rv_S^3}\,G'''(t_S) + \frac{1}{Rv_S^2}\,G''(t_S)\right]\sin\Theta - 2\mu\left[\frac{1}{R^2v_P^2}\,F''(t_P) + \frac{3}{R^3v_P}\,F'(t_P)\right.$$
$$\left. + \frac{3}{R^4}\,F(t_P) + \frac{1}{R^2v_S^2}\,G''(t_S) + \frac{3}{R^3v_S}\,G'(t_S) + \frac{3}{R^4}\,G(t_S)\right]\sin\Theta \quad \text{(10)}$$

In Eqs. 9 and 10, the prime denotes the derivative of the particular function with respect to R.

Steady Oscillations of Embedded Inclusion

Considering the boundary conditions (Eq. 1) associated with the smoothly embedded rigid spherical inclusion problem and the results (Eqs. 9 and 10), we obtain the following set of coupled differential equations for $F(t)$ and $G(t)$, i.e.

$$\frac{1}{av_P^2} F''(t) + \frac{2}{a^2 v_P} F'(t) + \frac{2}{a^3} F(t) + \frac{2}{a^2 v_S} G'(t) + \frac{2}{a^2} G(t) = \zeta(t) \quad (11)$$

$$\frac{2}{av_P^2} F''(t) + \frac{6}{a^2 v_P} F'(t) + \frac{6}{a^3} F(t) + \frac{1}{v_S^3} G'''(t) + \frac{3}{av_S^2} G''(t) + \frac{6}{a^2 v_S} G'(t) + \frac{6}{a^3} G(t) = 0 \quad (12)$$

In view of the steady rectilinear oscillations of the smoothly embedded inclusion, we can write

$$\zeta(t) = \delta a e^{i\omega t}, \qquad \text{for all } t \quad (13)$$

and assume that $F(t)$ and $G(t)$ can be represented in the forms

$$F(t) = \delta A a^4 e^{i\omega t} \quad (14a)$$

$$G(t) = \delta B a^4 e^{i\omega t} \quad (14b)$$

where A and B = constants. Making use of Eqs. 13 and 14, the reduced boundary conditions can be expressed as follows:

$$[-q^2\chi^2 + 2 + 2q\chi i]A + [2 + 2\chi i]B = 1 \quad (15)$$

$$[-2q^2\chi^2 + 6 + 6q\chi i]A + [-3\chi^2 + 6 + (6\chi - \chi^3)i]B = 0 \quad (16)$$

where

$$q = \frac{v_S}{v_P} = \left(\frac{\mu}{\lambda + 2\mu}\right)^{1/2} = \left\{\frac{1 - 2\nu}{2 - 2\nu}\right\}^{1/2} \quad (17)$$

$$\chi = \frac{aw}{v_S} \quad (18)$$

and ν = Poisson's ratio for the elastic material. Solution of Eqs. 15 and 16 gives the following expressions for the constants A and B:

$$A = \frac{1}{D}[(-3\chi^2 + 6) + (6\chi - \chi^3)i] \quad (19)$$

$$B = \frac{1}{D}[(-2q^2\chi^2 + 6) + 6q\chi i] \quad (20)$$

and

$$D = \chi^2\{[(3q^2 + 2q)\chi^2 - (2q^2 + 6)] + \chi[q^2\chi^2 - (2q^2 + 6q + 2)]i\} \quad (21)$$

For future reference, we note that A and B can also be represented in the forms

$$A = \frac{(A_1 + iA_2)}{\chi^2 D^*} \quad \text{(22a)}$$

$$B = \frac{(B_1 + iB_2)}{\chi^2 D^*} \quad \text{(22b)}$$

where

$$A_1 = (-3\chi^2 + 6)[(3q^2 + 2q)\chi^2 - (2q^2 + 6)]$$
$$+ (-\chi^3 + 6\chi)\chi[q^2\chi^2 - (2q^2 + 6q + 2)] \quad \text{(23a)}$$

$$A_2 = -(-3\chi^2 + 6)\chi[q^2\chi^2 - (2q^2 + 6q + 2)]$$
$$+ (-\chi^3 + 6\chi)[(3q^2 + 2q)\chi^2 - (2q^2 + 6)] \quad \text{(23b)}$$

$$B_1 = -(-2q^2\chi^2 + 6)[(3q^2 + 2q)\chi^2 - (2q^2 + 6)]$$
$$- (6q\chi)\chi[q^2\chi^2 - (2q^2 + 6q + 2)] \quad \text{(23c)}$$

$$B_2 = (-2q^2\chi^2 + 6)\chi[q^2\chi^2 - (2q^2 + 6q + 2)]$$
$$- (6q\chi)[(3q^2 + 2q)\chi^2 - (2q^2 + 6)] \quad \text{(23d)}$$

and

$$D^* = [(3q^2 + 2q)\chi^2 - (2q^2 + 6)]^2 + \chi^2[q^2\chi^2 - (2q^2 + 6q + 2)]^2 \quad \text{(23e)}$$

The real part of the functions $F(t)$ and $G(t)$ can be expressed in the form

$$F(t) = \frac{\delta a^4}{\chi^2 D^*}\{A_1 \cos \omega t - A_2 \sin \omega t\} \quad \text{(24)}$$

$$G(t) = \frac{\delta a^4}{\chi^2 D^*}\{B_1 \cos \omega t - B_2 \sin \omega t\} \quad \text{(25)}$$

The algebraic computations presented in the preceding sections have been systematized with speed and accuracy by using computer-based symbolic manipulation programs such as SMP (Kudera 1987) and MATHEMATICA (Wolfram 1988). Similar techniques have been used with considerable success in the analytical formulation and solution of problems in dynamics, fluid mechanics, heat and mass transfer, nonlinear elasticity, and structural mechanics [see, e.g., Griffiths and Morrison (1978), Rand (1984), Selvadurai (1987), Lee and Dasgupta (1988), and Selvadurai et al. (1988)].

Dynamic Compliance

The non-zero traction at the rigid sphere-elastic medium interface due to the dynamic loading is the radial stress σ_{RR}. By making use of Eq. 3, the result (Eq. 7) can be expressed in the form

$$\sigma_{RR} = -(\lambda + 2\mu)\left[\frac{1}{Rv_P^3}F'''(t_P) + \frac{1}{R^2 v_P}F''(t_P)\right]\cos\Theta - 4\mu\left[\frac{1}{R^2 v_P^2}F''(t_P)\right.$$
$$\left. + \frac{3}{R^3 v_P}F'(t_P) + \frac{3}{R^4}F(t_P) + \frac{1}{R^2 v_S^2}G''(t_S) + \frac{3}{R^3 v_S}G'(t_S) + \frac{3}{R^4}G(t_S)\right]\cos\Theta \quad (26)$$

On the boundary of the rigid inclusion, Eq. 26 reduces to the result

$$\frac{\sigma_{RR}(a,\Theta,t)}{\mu\delta} = \frac{\cos\Theta\cos\omega t}{\chi^2 D^*}\{\chi^3 q A_2 - \chi^2 A_1 - 4\chi^2 q^2 A_1 - 12\chi q A_2 + 12A_1$$

$$- 4\chi^2 B_1 - 12\chi B_2 + 12B_1\} - \frac{\cos\Theta\sin\omega t}{\chi^2 D^*}\{\chi^3 q A_1 + \chi^2 A_2 + 4\chi^2 q^2 A_2$$

$$- 12\chi q A_1 - 12A_2 + 4\chi^2 B_2 - 12\chi B_1 - 12B_2\} \qquad (27)$$

Considering the equilibrium of the rigid spherical inclusion of mass M, which is subjected to the rectilinear oscillations by the periodic force $Q(t)$, we have

$$M\frac{d^2\zeta(t)}{dt^2} = Q(t) + 2\pi a^2 \int_0^{\pi} \sigma_{RR}(a,\Theta,t)\sin\Theta\cos\Theta d\Theta \qquad (28)$$

Considering the real part of $\zeta(t)$, we can reduce Eq. 28 to the form

$$\frac{Q(t)}{\frac{4}{3}\pi\delta a^2\mu} = \hat{Q}\cos(\omega t + \pi + \alpha) \qquad (29)$$

where

$$\hat{Q} = \chi^2[(l_1 + l)^2 + m_1^2]^{1/2} \qquad (30)$$

where

$$l_1 = -\frac{1}{\chi^4 D^*}[(-\chi^2 - 4\chi^2 q^2 + 12)A_1 + (\chi^3 q - 12\chi q)A_2$$

$$+ (-4\chi^2 + 12)B_1 - 12\chi B_2] \qquad (31a)$$

$$m_1 = \frac{1}{\chi^4 D^*}[(\chi^3 q - 12\chi q)A_1 + (\chi^2 + 4\chi^2 q^2 - 12)A_2 - 12\chi B_1$$

$$+ (4\chi^2 - 12)B_2] \qquad (31b)$$

$$\tan\alpha = \frac{m_1}{l_1 + l} \qquad (31c)$$

$$l = \frac{M}{\frac{4}{3}\pi a^3 \rho_0} = \frac{\bar{\rho}}{\rho_0} \qquad (31d)$$

and $\bar{\rho}/\rho_0$ = the mass ratio or density mismatch.

In the developments presented here, it is explicitly assumed that the rigid spherical inclusion maintains smooth bilateral contact at the interface during the application of the steady dynamic oscillation $\delta a e^{i\omega t}$. It is likely that for certain combinations of the initial stress σ_0 and the dynamic stresses induced by the steady oscillation, contact may be lost at the smooth interface. It is possible to establish a criterion that will assign the limits of applicability of the developed solution. Here again, computer algebra can be used to advantage to perform the computations.

Consider the problem in which the elastic solid containing the smoothly embedded rigid inclusion is subjected to a radial compressive stress of magnitude σ_0 at infinity. It can be shown that the stress σ^S_{RR} developed at the smooth interface due to σ_0 is given by

$$\sigma^S_{RR}(a,\Theta) = -\sigma_0\left(\frac{3 + 5\nu}{1 - \nu}\right) \quad (32)$$

Here, compressive stresses are considered to be negative. The dynamic radial stress σ^D_{RR} induced at the bilateral interface due to the steady oscillations is given by Eq. 27. This, in turn, can be rewritten in the form

$$\sigma^D_{RR}(a,\Theta) = \delta\mu\chi^2(l_1^2 + m_1^2)^{1/2} \cos(\omega t + \alpha^*) \cos\Theta \quad (33)$$

where l_1 and m_1 are defined by Eq. 31, and $\tan\alpha^* = (m_1/l_1)$.

Considering the maximum value of Eq. 33, we can obtain a consistency condition that will establish the occurrence of tensile stresses or the initiation of separation at the bilateral interface, i.e.

$$-\sigma_0\left(\frac{3 + 5\nu}{1 - \nu}\right) + \mu\delta\chi^2(l_1^2 + m_1^2)^{1/2} = 0 \quad (34)$$

Alternatively, it can be concluded that separation will not occur provided the static initial stress field σ_0 satisfies the condition

$$\frac{\sigma_0}{\mu\delta} \geq \chi^2(l_1^2 + m_1^2)^{1/2}\left\{\frac{3}{11 - 4q^2}\right\} \quad (35)$$

The right-hand side of the inequality Eq. 35 is purely a function of the non-dimensional frequency $k = [\rho_0\omega^2a^2/\mu]^{1/2}$ and Poisson's ratio ν.

Numerical Results

Specific numerical results pertaining to the dynamic compliance of the smoothly embedded rigid spherical inclusion can be obtained by simplifying the result (Eq. 29) in the form

$$\frac{Q(t)}{\frac{4}{3}\pi\delta a^2\mu} = \chi^2[(l_1 + l)^2 + m_1^2]^{1/2} \cos(\omega t + \pi + \alpha) \quad (36)$$

where

$$l_1 = -\frac{1}{\chi^2 D^*}[q^2(4q^2 - 1)\chi^6 + (4q^4 + 16q^3 - q^2 - 2)\chi^4 - (8q^4 - 60q^2 - 18)\chi^2 + 12(3q^2 + q)] \quad (37)$$

and

$$m_1 = -\frac{1}{\chi D^*}[q^3\chi^6 + q^3(8q - 3)\chi^4 + 24q^2\chi^2 + 12(3q^3 + 6)] \quad (38)$$

In the limit as the frequency ω reduces to zero, the result (Eq. 36) yields

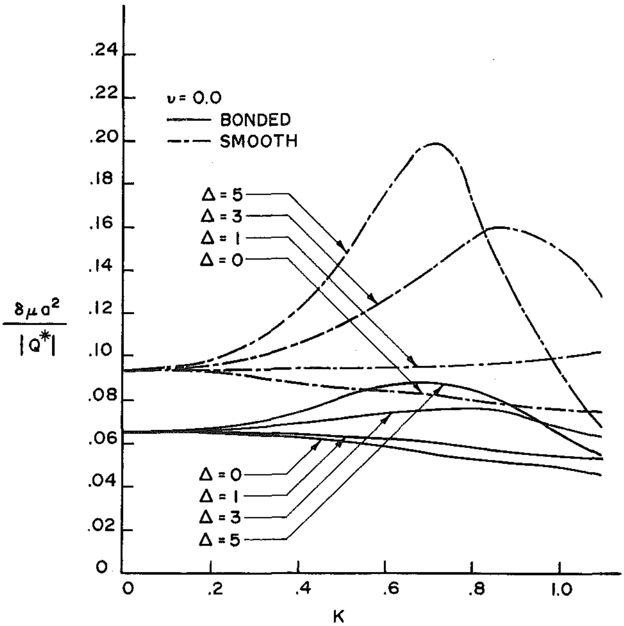

FIG. 3. Dynamic Compliance of Embedded Rigid Spherical Inclusion

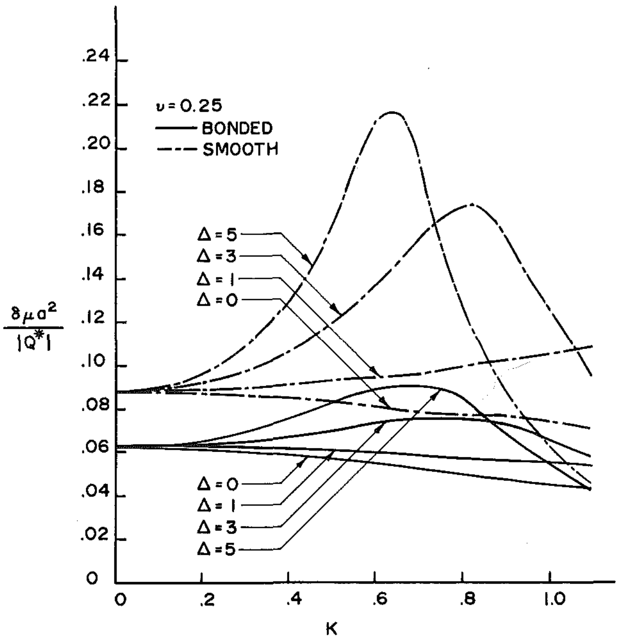

FIG. 4. Dynamic Compliance of Embedded Rigid Spherical Inclusion

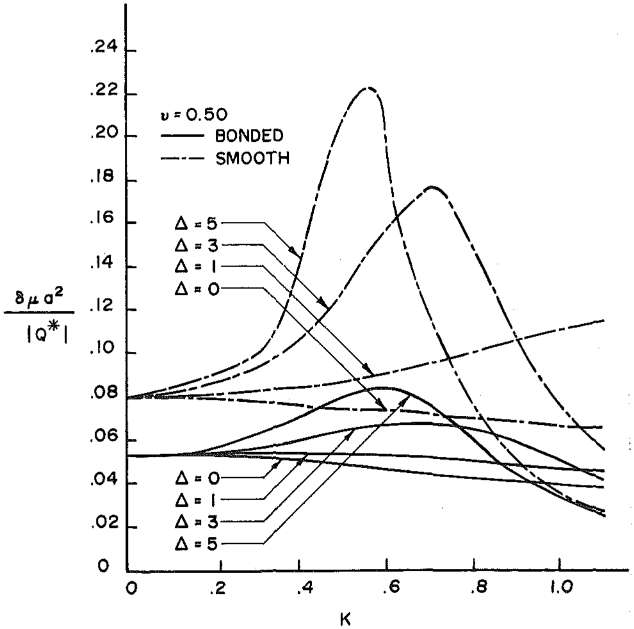

FIG. 5. Dynamic Compliance of Embedded Rigid Spherical Inclusion

the solution to the problem of the static loading (force Q_0) of a rigid spherical inclusion that is smoothly embedded in an isotropic elastic solid of infinite extent, i.e.

$$\frac{Q_0}{\frac{4}{3}\pi\delta a^2\mu} \doteq \frac{18(1-\nu)}{(7-8\mu)} \quad \text{(39)}$$

The presented result is in agreement with the exact closed-form result derived by Selvadurai and Au (1985) for the smoothly embedded rigid spherical inclusion problem.

The exact closed-form solution (Eq. 36) can be evaluated to obtain specified results for the nondimensional relationship between the dynamic displacement δa and maximum value Q^* of the periodic force $Q(t)$. Figs. 3–5 show the manner in which the dynamic compliance of the smoothly embedded rigid spherical inclusion is influenced by the modified mass ratio Δ [$= M(1-\nu)/4\rho_0 a^3$] and the nondimensional frequency k [$= (\rho_0\omega^2 a^2/\mu)^{1/2}$]. Also presented in these figures are the analogous results for the case in which the rigid spherical inclusion is embedded in full bonded contact with the elastic medium of infinite extent.

The result (Eq. 34) can be evaluated to establish the combinations of $\sigma_0/\mu\delta$, ν, and k that will either initiate or inhibit the occurrence of separation at the bilateral contact region. Fig. 6 shows the permissible combinations of these parameters that will satisfy the interface boundary conditions implicit in the theoretical developments. The aforementioned analytical expressions

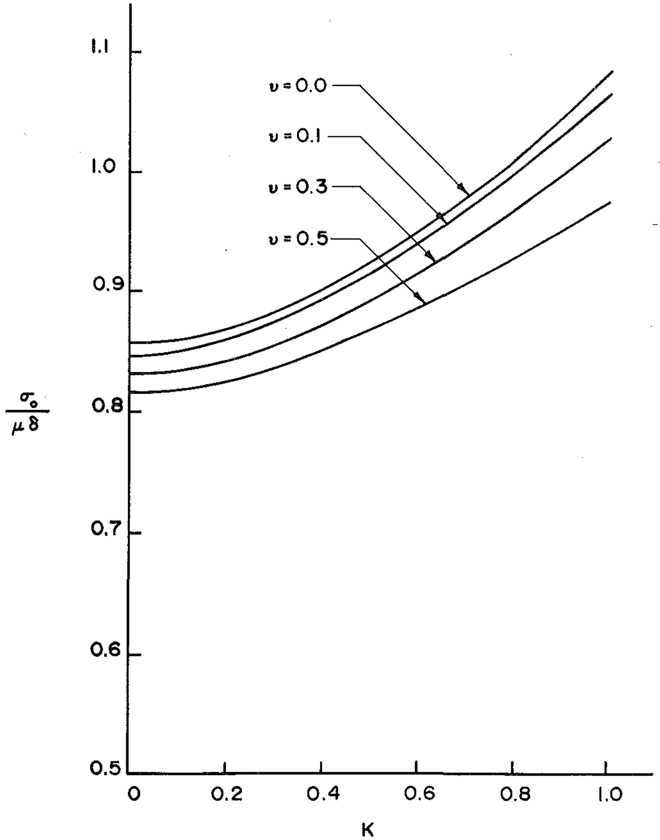

FIG. 6. Combinations of Initial Stress, Amplitude, and Frequency that Will Induce Separation at Smooth Bilateral Interface between Elastic Medium and Inclusion

have been derived for idealized elastic soil media. In the case of material damping losses in the soil, a constant hysteretic or a Voigt model is very often considered. Numerical forms of the elastic-viscoelastic correspondence principle, such as the one proposed by Dasgupta and Sackman (1977), can be employed to convert the numerical compliance coefficients presented herein. It should be mentioned that by using computer algebra packages such as SMP or MATHEMATICA, it is possible to isolate the real and imaginary parts of responses from the closed-form viscoelastic responses. The latter can be obtained by using the equivalent complex frequency (for viscoelasticity) in the algebraic expressions for responses in the place of ω.

Conclusions

The classical elastodynamic problem pertaining to the rigid spherical inclusion that is embedded in smooth bilateral contact with an isotropic elastic medium provides a useful bound for the investigation of the dynamic behavior of certain in situ testing techniques and embedded foundations. In the case in which the rigid spherical inclusion is subjected to steady oscillations, the solution to the elasto-dynamic problem can be evaluated in exact closed form. The numerical results presented in the paper examine the manner in which the boundary conditions at the inclusion-elastic medium interface (either fully bonded or smooth) influence the dynamic compliance of the embedded rigid spherical inclusion. As is evident, the relaxation of the bond conditions leads to a substantial increase in the dynamic compliance, and such changes are at a maximum when the Poisson's ratio of the elastic medium $\nu = 1/2$. The analytical results presented in the paper can also be used to establish the combinations of initial stress, amplitude of oscillations, frequency, and elastic material parameters that will initiate or suppress the development of separation zones.

In the development of the expressions for stresses and displacements in terms of potentials, computer algebra packages have been used extensively. All the intermediate results presented herein are verified by symbolic manipulation programs. The numerical results are finally obtained as simulations with particular input data sets. Such procedures are convenient, practical applications of the theoretical developments presented herein.

Acknowledgments

The writers are grateful for the comments of the referees. The work described in this paper was supported in part by a Natural Sciences and Engineering Research Council of Canada Grant A-3866. The support of NCEER at SUNY Buffalo, New York through grants 87-1010 and 87-1313 is appreciated.

Appendix I. References

Chadwick, P. and Trowbridge, E. A. (1967). "Oscillations of a rigid sphere embedded in an infinite elastic solid, II. Rectilinear oscillations," *Proc. Cambridge Philosoph. Soc.*, 63, 1207–1227.

Dasgupta, G., and Chopra, A. K. (1979). "Dynamic stiffness matrices for viscoelastic half-planes." *J. Engrg. Mech. Div.*, ASCE, 105(5), 729–745.

Dasgupta, G., and Sackman, J. L. (1977). "An alternative representation of the elastic-viscoelastic correspondence principle for harmonic oscillations." *J. Appl. Mech.*, 99, 57–60.

Datta, S. K. (1970). "The diffraction of a plane compressional wave by a rigid circular disc." *Q. Appl. Math.*, 37, 86–91.

Eringen, A. C., and Suhubi, E. (1975). *Elastodynamics. Vol. II; Linear theory.* Academic Press, New York, N.Y.

Gazetas, G. (1983). "Analysis of machine foundation vibration: State of the art," *Soil Dyn. Earthquake Engrg.*, 2, 2–42.

Griffiths, S. K., and Morrison, F. A. (1970). "Low Peclet number heat and mass transfer from a drop in an electric field." *J. Heat Transfer,* ASME, 101, 484–488.

Gurtin, M. E. (1972). "The linear theory of elasticity." *Handbuch der Physik,* C. Truesdell, ed., Band VIa/2, Springer-Verlag, Berlin, 1–295.

Kanwal, R. P. (1965). "Dynamical displacements of an infinite elastic space and matched asymptotic expansions." *J. Math. Phys.*, 44, 275–283.

Kudera, J. R. (1987). *SMP users reference guide with an introduction to computer mathematics.* Computer Mathematics Associates, Marina del Rey, Calif.

Lee, X., and Dasgupta, G. (1988). "Analysis of structural variability with computer

algebra." *J. Engrg. Mech.*, ASCE, 114(1), 161–171.
Mal, A. K. (1971). "Motion of a rigid disc in an elastic solid." *Bull. Seismol. Soc. Am.*, 61, 1717–1729.
Mal, A. K., Ang, D. D., and Knopff, L. (1968). "Diffraction of elastic waves by a rigid circular disc." *Proc. Cambridge Philosoph. Soc.*, 64, 237–247.
Rand, R. H. (1984). "Computer algebra in applied mathematics: An introduction to MAC-SYMA." *Research notes in mathematics*, Pitman Advanced Publishing Program, Boston, Mass.
Selvadurai, A. P. S. (1980). "The dynamic response of a rigid circular foundation embedded in an isotropic medium of infinite extent." *Proc. Int. Symp. on Soils under Cyclic and Transient Loading*, G. N. Pande and O. C. Zienkiewicz, eds., A. A. Balkema, Rotterdam, the Netherlands, 597–608.
Selvadurai, A. P. S. (1981a). "Rotary oscillations of a rigid disc inclusion embedded in an isotropic elastic infinite space." *Int. J. Solids Struct.*, 17, 493–498.
Selvadurai, A. P. S. (1981b). "The influence of viscoelastic damping on the dynamic behaviour of an embedded foundation." *Proc., Int. Conf. on Recent Advances in Geotechnical Earthquake Engineering and Soil Dynamics*, S. Prakash, ed., University of Missouri, Rolla, Mo., 345–350.
Selvadurai, A. P. S. (1987). "Computer algebra in applied mathematics; applications to problems in non-linear elasticity." *Studies in applied mechanics*, A. P. S. Selvadurai, ed., 16, Elsevier Scientific Publishing Co., Amsterdam, the Netherlands, 69–88.
Selvadurai, A. P. S., and Au, M. C. (1985). "Response of inclusions with interface separation, friction and slip." *Proc., 7th Int. Conf. on Boundary Elements*, C. A. Brebbia and G. Maier, eds., Springer-Verlag, Berlin, 14.109–14.127.
Selvadurai, A. P. S., and Rahman, M. (1981). "Engineering approximations for the dynamic behaviour of foundations embedded in elastic media." *Proc., ASCE/EMD Specialty Conf. on Dynamic Response of Structures*, Atlanta, Ga., G. C. Hart, ed., 361–375.
Selvadurai, A. P. S., Spencer, A. J. M., and Rudgyard, M. A. (1988). "Second-order elasticity with axial symmetry, II. Spherical cavity and spherical rigid inclusion problems." *Int. J. Engrg. Sci.*, 25, 1477–1490.
Williams, W. E. (1966). "A note on the slow vibrations in an elastic medium." *Q. J. Mech. Appl. Math.*, 19, 413–416.
Wolfram, S. (1988). *Mathematica: A system for doing mathematics by computer.* Addison-Wesley Publishing Company, Mass.

APPENDIX II. NOTATION

The following symbols are used in this paper:

A, B = complex constants;
a = radius of sphere;
$F(t_P), G(t_S)$ = substitution functions;
$k = [\rho_0 \omega^2 a^2/\mu]^{1/2}$ = nondimensional frequency;
$l = 3M/4\pi a^3 \rho_0 = \bar{\rho}/\rho_0$ = density mismatch;
M = mass of rigid spherical inclusion;
Q^* = maximum value of $Q(t)$;
Q_0 = static force on rigid spherical inclusion;
$Q(t)$ = periodic force on rigid spherical inclusion;
$q = v_S/v_P$ = $[(1 - 2\nu)/(2 - 2\nu)]^{1/2}$;
R, Θ, Φ = spherical polar coordinates;
t = time variable;
u_R, u_Θ = displacement components;
$v_P = [(\lambda + 2\mu)/\rho_0]^{1/2}$ = compressional wave velocity;
$v_S = [\mu/\rho_0]^{1/2}$ = shear wave velocity;

$\mathbf{x}$ = coordinate vector;

$\Delta = M(1 - \nu)/4\rho_0 a^3$ = nondimensional modified mass ratio;

δa = amplitude of harmonic motion;

$\zeta(t)$ = generalized time-dependent axial displacement of sphere;

λ, μ = Lame's constants;

ν = Poisson's ratio;

ρ_0 = mass density;

$\sigma_{RR}, \sigma_{R\Theta}$ = stress components;

σ_{RR}^S = static radial stress at inclusion-elastic medium interface;

σ_{RR}^D = dynamic radial stress at inclusion-elastic medium interface;

σ_0 = radial compressive stress at infinity;

ϕ, ψ = scalar wave functions;

χ = $a\omega/v_S$; and

ω = frequency.