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THE INFLUENCE OF A BOUNDARY FRACTURE ON THE ELASTIC STIFFNESS OF A DEEPLY EMBEDDED ANCHOR PLATE

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SUMMARY

The problem of the axial loading of a rigid disk-shaped anchor plate embedded in an isotropic elastic medium of infinite extent is examined. At the boundary of the disk anchor plate the elastic medium contains a cracked region of finite extent. The presence of the cracked region decreases the elastic stiffness of the anchor plate. The mathematical formulation of the problem is developed, and a numerical scheme is presented which can be used to solve the resulting coupled integral equations. The numerical technique is used to evaluate the results, which illustrate the manner in which the elastic stiffness of the anchor plate is influenced by the extent of cracking. Similar results are developed for the flaw shearing mode stress intensity factor at the external boundary of the cracked region.

INTRODUCTION

The class of problems which deal with the loading of plate shaped objects embedded in elastic media provides a useful basis for the modelling of the short-term or working load range stiffnesses of anchor plates. A number of researchers have investigated a variety of elastostatic problems involving anchor plates in which account is taken of complete bonding or partial debonding at the anchor plate/elastic medium interface, flexibility of the anchor plate, transverse isotropy of the soil medium, geometrical features of the anchor plate and the influence of the boundaries (see e.g. References 1–10). Accounts of the elastostatic analysis of disk-shaped anchor problems are given in References 11 and 12.

In the present paper we focus on the problem of a rigid circular anchor plate embedded in bonded contact with an isotropic elastic soil mass of infinite extent. The elastic medium beyond the boundary of the rigid disk anchor contains an in-plane cracked region of finite extent (Figure 1). The development of such cracks can be attributed to the use of grout materials which are injected at high pressure. Similar cracking features can occur during the penetration of single-helix ground anchors into stiff soil masses such as overconsolidated clays. This study is concerned specifically with the evaluation of the elastostatic stiffness of the embedded anchor in the presence of a cracked region emanating from the boundary of the anchor region. In this problem the inner boundary of the cracked region terminates at the boundary of the rigid circular anchor plate. It is well known (see e.g. References 13–15) that in such situations involving linearly elastic media, the stress singularity at the discontinuity can exhibit oscillatory phenomena. Consequently, in situations where the exact stress distributions at the inner boundary of the cracked region are of interest, it is necessary to perform the analysis by appeal to a formulation based on the Hilbert problem. On the other hand, if the primary interest is the evaluation of global results pertaining to

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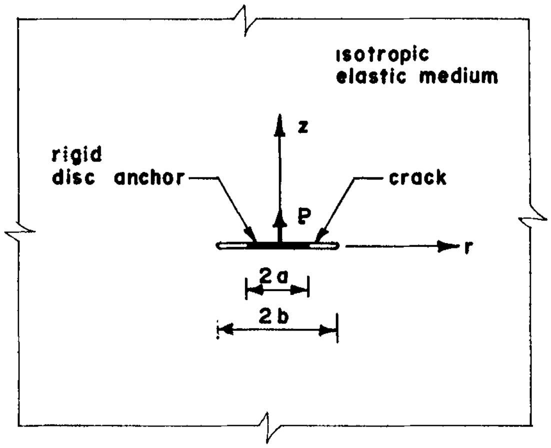

Figure 1. Geometry of the disk inclusion and the extent of cracking

the stiffness of the embedded anchor, then the problem can be formulated by using an integral transform approach (References 16 and 17). The basis for the adoption of the latter approach in the solution of the embedded disk anchor problem can be illustrated by considering a rigid circular punch in adhesive contact with a half-space region. If the bonded circular punch, of radius a, is subjected to a central force P which induces a displacement Δ in the axial direction, the mixed boundary conditions are

$$u_z(r, 0) = \Delta, \quad 0 \leq r \leq a \tag{1}$$

$$u_r(r, 0) = 0, \quad 0 \leq r \leq a \tag{2}$$

$$\sigma_{zz}(r, 0) = 0, \quad a < r < \infty \tag{3}$$

$$\sigma_{rz}(r, 0) = 0, \quad a < r < \infty \tag{4}$$

The solution of the mixed boundary value problem, defined by equations (1)–(4), which incorporates the oscillatory form of the stress singularity at the boundary of the rigid punch was developed by Ufliand,[18] and further expositions of the method of solution were given by Mossakovskii[19] and Gladwell.[14] The result of primary interest to the present discussion concerns the load–displacement behaviour of the rigid circular punch. Using the formulation based on the Hilbert problem, this result can be evaluated in the exact closed form

$$\frac{P}{4G\Delta a} = \frac{\ln(3-4\nu)}{1-2\nu} \tag{5}$$

where G is the linear elastic shear modulus and ν is Poisson's ratio. In the Hankel integral transform approach, the mixed boundary value problem associated with the bonded circular punch is effectively reduced to a single Fredholm integral equation of the second kind. This integral equation can be solved numerically to generate the load–displacement relationship for the bonded punch. The comparison between the results obtained by the two schemes is shown in Table I. It is evident that the two approaches yield virtually the same result for the axial stiffness of the bonded punch. The maximum difference between the two sets of results does not exceed half a per cent. Guided by this observation, we adopt a Hankel integral transform based on an integral

Table 1. Axial stiffness $P/8G\Delta a$ of a rigid circular punch bonded to an isotropic elastic half-space

ν	Hilbert problem approach	Hankel transform approach
0	0·549	0·546
0·1	0·597	0·596
0·2	0·657	0·658
0·3	0·736	0·737
0·4	0·841	0·842
0·5	1·000	1·000

equation formulation of the axially loaded disk anchor problem in the presence of boundary cracking. The mathematical analysis of the mixed boundary value problem yields a system of three coupled integral equations. These in turn are reduced to a pair of coupled Fredholm integral equations of the second kind. These are solved in a numerical fashion to generate the load–displacement relationship for the disk anchor. The numerical results illustrate the manner in which the extent of cracking influences the stiffness of the disk anchor. The numerical procedure can also be used to evaluate the flaw shearing mode (or mode II) stress intensity factor at the outer boundary of the cracked region. Again, the numerical results illustrate the influence of the extent of cracking on the stress intensity factor at the crack boundary.

FUNDAMENTAL EQUATIONS

Since the anchor problem exhibits axial symmetry, it is convenient to use the method of solution which is based on the strain potential approach taken by Love.[20] It can be shown that the solution to the displacement equations of equilibrium can be expressed in terms of a single function $\Phi(r, z)$ which satisfies the equation

$$\nabla^2\nabla^2\Phi(r, z) = 0 \tag{6}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{7}$$

is the axisymmetric form of Laplace's operator in cylindrical polar co-ordinates. The displacement and stress components in the elastic medium may be evaluated in terms of the strain potential Φ; we have

$$2Gu_r = -\frac{\partial^2\Phi}{\partial r\partial z} \tag{8}$$

$$2Gu_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \tag{9}$$

and

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2\Phi - \frac{\partial^2\Phi}{\partial r^2}\right) \tag{10}$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2\Phi - \frac{1}{r}\frac{\partial\Phi}{\partial r}\right) \tag{11}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2 - \frac{\partial^2\Phi}{\partial z^2}\right\} \tag{12}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right\} \tag{13}$$

Owing to the asymmetry of the state of deformation about the plane $z = 0$, the disk anchor problem can be formulated as a mixed boundary value problem associated with a half-space region. For this purpose, and for convenience, we restrict our attention to the half-space region defined by $z \geq 0$. The solution of equation (6) applicable to the formulation of the disk anchor problem can be obtained by a Hankel transform development. The relevant result can be expressed in the integral form[21]

$$\Phi(r, z) = \int_0^\infty \xi[A(\xi) + B(\xi)z]\, e^{-\xi z} J_0(\xi r)\, d\xi \tag{14}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions, and $J_0(\xi r)$ is the zeroth-order Bessel function of the first kind. It may be noted that the displacement and stress fields derived from equation (14) reduce to zero as $(r^2 + z^2)^{1/2} \to \infty$.

THE DISK ANCHOR PROBLEM

We consider a disk anchor of radius a which is embedded in bonded contact with the surrounding elastic medium. The cracked region beyond the outer boundary of the disk anchor has radius b. The surfaces of the cracked region are traction-free. The disk anchor is subjected to a central force P which induces a rigid-body displacement Δ in the z direction. Since the displacement of the disk anchor induces an asymmetry, the mixed boundary conditions become

$$u_z(r, 0) = \Delta, \quad 0 \leq r \leq a \tag{15}$$

$$u_r(r, 0) = 0, \quad 0 \leq r \leq a \tag{16}$$

$$u_r(r, 0) = 0, \quad b \leq r \leq \infty \tag{17}$$

$$\sigma_{zz}(r, 0) = 0. \quad a < r < \infty \tag{18}$$

$$\sigma_{rz}(r, 0) = 0, \quad a < r < b \tag{19}$$

Using the integral form of the solution for $\Phi(r, z)$ given by equation (14), the mixed boundary conditions (15)–(19) can be expressed in the forms

$$H_0[\xi\{\xi A(\xi) + 2(1-2\nu)B(\xi)\}; r] = -2G\Delta, \quad 0 \leq r \leq a \tag{20}$$

$$H_1[\xi\{-\xi A(\xi) + B(\xi)\}; r] = 0, \quad 0 \leq r \leq a \tag{21}$$

$$H_1[\xi\{-\xi A(\xi) + B(\xi)\}; r] = 0, \quad b \leq r < \infty \tag{22}$$

$$H_0[\xi^2\{\xi A(\xi) + (1-2\nu)B(\xi)\}; r] = 0, \quad a < r < \infty \tag{23}$$

$$H_1[\xi^2\{\xi A(\xi) - 2\nu B(\xi)\}; r] = 0, \quad a < r < b \tag{24}$$

where $H_n[\Omega(\xi); r]$ is the Hankel transform of order n defined by

$$H_n[\Omega(\xi); r] = \int_0^\infty \xi\Omega(\xi) J_n(\xi r)\, d\xi \tag{25}$$

By introducing the substitutions

$$2(1-\nu)\xi^3 A(\xi) = -(1-2\nu)M(\xi) + N(\xi) \tag{26}$$

$$2(1-\nu)\xi^2 B(\xi) = M(\xi) + N(\xi) \tag{27}$$

we can express the integral equations (20)–(24) as follows:

$$H_0\left[\xi^{-1}\left\{N(\xi) + \frac{(1-2\nu)}{(3-4\nu)} M(\xi)\right\}; r\right] = -\frac{4G\Delta(1-\nu)}{3-4\nu}, \quad 0 \le r \le a \tag{28}$$

$$H_1[\xi^{-1}M(\xi); r] = 0, \quad 0 \le r \le a \tag{29}$$

$$H_1[\xi^{-1}M(\xi); r] = 0, \quad b \le r < \infty \tag{30}$$

$$H_0[N(\xi); r] = 0, \quad a < r < \infty \tag{31}$$

$$H_1[\{(1-2\nu)N(\xi) - M(\xi)\}; r] = 0, \quad a < r < b \tag{32}$$

From equation (28) we note that

$$H_0[\xi^{-1}N(\xi); r] = -\frac{4G\Delta(1-\nu)}{3-4\nu} - \frac{(1-2\nu)}{3-4\nu} H_0[\xi^{-1}M(\xi); r], \quad 0 \le r \le a \tag{33}$$

With reference to equation (31), we assume that $N(\xi)$ admits a representation of the form

$$N(\xi) = \int_0^a \phi(t)\cos(\xi t)\, dt = \frac{\phi(a)\sin(\xi a)}{\xi} - \frac{1}{\xi}\int_0^a \phi'(t)\sin \xi t\, dt \tag{34}$$

where the prime denotes the derivative with respect to t. By substituting equation (34) into equation (33) we obtain

$$\int_0^r \frac{\phi(t)\, dt}{(r^2-t^2)^{1/2}} = -\frac{4G(1-\nu)}{3-4\nu} - \frac{(1-2\nu)}{3-4\nu} H_0[\xi^{-1}M(\xi); r], \quad 0 \le r \le a \tag{35}$$

which is an integral equation of the Abel type; its solution may be written as

$$\phi(t) = -\frac{8G\Delta(1-\nu)}{\pi(3-4\nu)} - \frac{2(1-2\nu)}{\pi(3-4\nu)} \int_0^\infty M(\xi)\cos(\xi t)\, d\xi, \quad 0 \le t \le a \tag{36}$$

Equations (29), (30) and (32) may be written in the following forms:

$$H_1[\xi^{-1}M(\xi); r] = 0, \quad 0 \le r \le a, \quad b \le r < \infty \tag{37}$$

$$H_1[M(\xi); r] = (1-2\nu)H_1[N(\xi); r] = G_1(r), \quad a < r < b \tag{38}$$

Substituting for $N(\xi)$ defined by equation (34), equation (38) may be expressed in the form

$$G_1(r) = r(1-2\nu)\int_0^a \frac{\phi(t)\, dt}{(r^2-t^2)^{3/2}}, \quad a < r < b \tag{39}$$

To obtain the solution of the system of triple integral equations defined by equations (37) and (38), we assume that

$$H_1[M(\xi); r] = f_1(r), \quad a < r < b \tag{40}$$

$$H_1[M(\xi); r] = f_3(r), \quad b < r < \infty \tag{41}$$

Making use of the Hankel inversion theorem, we obtain from equations (38), (40) and (41)

$$M(\xi) = \int_0^a u f_1(u) J_1(\xi u)\, du + \int_a^b u G_1(u) J_1(\xi u)\, du + \int_b^\infty u f_3(u) J_1(\xi u)\, du \tag{42}$$

By substituting equation (42) into equation (37) and making use of the techniques outlined by Cooke,[16] we obtain the following set of integral equations:

$$\tfrac{1}{2}\pi r^{-1}(a^2-r^2)^{1/2} f_1(r) = -\int_a^b \frac{(t^2-a^2)^{1/2}}{t^2-r^2} G_1(t)\, dt - \int_b^\infty \frac{(t^2-a^2)^{1/2}}{t^2-r^2} f_3(t)\, dt, \quad 0 < r < a \tag{43}$$

$$\tfrac{1}{2}\pi r(r^2-b^2)^{1/2} f_3(r) = -\int_0^a \frac{t^2(b^2-t^2)^{1/2}}{(r^2-t^2)} f_1(t)\, dt - \int_a^b \frac{t^2(b^2-t^2)^{1/2}}{r^2-t^2} G_1(t)\, dt, \quad b < r < \infty \tag{44}$$

With the aid of the result (42), we obtain

$$\int_0^\infty M(\xi)\cos(\xi t)\, d\xi = \int_0^a f_1(u)\, du - t\int_0^t \frac{f_1(u)\, du}{(t^2-u^2)^{1/2}} + \int_a^b G_1(u)\, du + \int_b^\infty f_3(u)\, du, \quad 0 < t < a \tag{45}$$

By using equations (36) and (45), the result (39) may be expressed as

$$G_1(r) = -\frac{2}{\pi}\frac{(1-2\nu)^2 a}{(3-4\nu) r (r^2-a^2)^{1/2}}\left[\int_0^a f_1(u)\, du + \int_a^b G_1(u)\, du + \int_b^\infty f_3(u)\, du\right] - \frac{8G\Delta(1-2\nu)(1-\nu)a}{\pi(3-4\nu)r(r^2-a^2)^{1/2}} + \frac{2(1-2\nu)^2 r}{\pi(3-4\nu)}\int_0^a \frac{f_1(u)}{(r^2-u^2)}\left(\frac{a^2-u^2}{r^2-a^2}\right)^{1/2} du, \quad a < r < b \tag{46}$$

Introducing the substitutions

$$\frac{\pi(a^2-r^2)^{1/2} f_1(r)}{2r} = \frac{8G\Delta(1-2\nu)(1-\nu)a}{\pi(3-4\nu)} F_1(r) \tag{47}$$

$$\frac{\pi r(r^2-b^2)^{1/2} f_3(r)}{2} = \frac{8G\Delta(1-2\nu)(1-\nu)a}{\pi(3-4\nu)} F_3(r) \tag{48}$$

$$r(r^2-a^2)^{1/2} G_1(r) = \frac{8G\Delta(1-2\nu)(1-\nu)a}{\pi(3-4\nu)} G_1^*(r) \tag{49}$$

$$\phi(r) = \frac{8G\Delta(1-2\nu)(1-\nu)a}{\pi(3-4\nu)} \phi^*(r) \tag{50}$$

Using the above, the integral equations (43), (44) and (46) can be expressed as

$$F_1(r) = -\int_a^b \frac{G_1^*(t)\, dt}{t(t^2-r^2)} - \frac{2}{\pi}\int_b^\infty \left(\frac{t^2-a^2}{t^2-b^2}\right)^{1/2} \frac{F_3(t)\, dt}{t(t^2-r^2)}, \quad 0 < r < a \tag{51}$$

$$F_3(r) = -\frac{2}{\pi}\int_0^a \left(\frac{b^2-t^2}{a^2-t^2}\right)^{1/2} \frac{t^3 F_1(t)\, dt}{r^2-t^2} - \int_a^b \left(\frac{b^2-t^2}{t^2-a^2}\right)^{1/2} \frac{t G_1^*(t)\, dt}{r^2-t^2}, \quad b < r < \infty \tag{52}$$

$$G_1^*(r) = -1 - \frac{2a(1-2\nu)^2}{\pi\ 3-4\nu}\left[\frac{2}{\pi}\int_0^a \frac{u F_1(u)\, du}{(a^2-u^2)^{1/2}} + \int_a^b \frac{G_1^*(u)\, du}{u(u^2-a^2)^{1/2}}\right.$$

$$+\frac{2}{\pi}\int_b^\infty \frac{F_3(u)\,\mathrm{d}u}{u(u^2-b^2)^{1/2}}\Bigg]+\frac{4(1-2\nu)^2r^2}{(3-4\nu)\pi^2}\int_0^a \frac{uF_1(u)\,\mathrm{d}u}{r^2-u^2},\quad a<r<b \tag{53}$$

Also, note that $\phi^*(t)$ may be expressed in terms of F_1, F_3 and G_1^* in the form

$$\phi^*(t)=-\frac{1}{(1-2\nu)a}-\frac{2}{\pi}\frac{1-2\nu}{3-4\nu}\Bigg[\frac{2}{\pi}\int_0^a \frac{uF_1(u)\,\mathrm{d}u}{(a^2-u^2)^{1/2}}-\frac{2t}{\pi}\int_0^t \frac{uF_1(u)\,\mathrm{d}u}{(a^2-u^2)^{1/2}(t^2-u^2)^{1/2}}$$
$$+\int_a^b \frac{G_1^*(u)\,\mathrm{d}u}{u(u^2-a^2)^{1/2}}+\frac{2}{\pi}\int_b^\infty \frac{F_3(u)\,\mathrm{d}u}{u(u^2-b^2)^{1/2}}\Bigg],\quad 0<r<\infty \tag{54}$$

This completes the formal reduction of the mixed boundary value problem defined by equations (15)–(19) effectively to the solution of the coupled integral equations (51)–(53). The numerical solution of these equations is discussed below.

LOAD–DISPLACEMENT RELATIONSHIP FOR THE ANCHOR

The axial stress distribution at the disk anchor/elastic medium interface can be used to evaluate the load–displacement relationship for the rigid disk anchor.

From equation (31) we note that

$$\sigma_{zz}(r,0)=-\int_0^\infty \xi N(\xi)J_0(\xi r)\,\mathrm{d}\xi,\quad 0<r<a \tag{55}$$

Substituting the value of $N(\xi)$ defined by equation (34) into equation (55), we have

$$\sigma_{zz}(r,0)=-\frac{\phi(a)}{(a^2-r^2)^{1/2}}+\int_r^a \frac{\phi'(t)\,\mathrm{d}t}{(t^2-r^2)^{1/2}},\quad 0<r<a \tag{56}$$

where the prime denotes the derivative with respect to the argument. The load–displacement relationship for the rigid disk anchor may be evaluated by considering the equilibrium equation for inclusion. Taking account of tractions which act on both faces of the disk anchor, we obtain the total force P on the inclusion as

$$P=4\pi\int_0^a r\sigma_{zz}(r,0)\,\mathrm{d}r \tag{57}$$

By substituting (56) into (57) we obtain

$$P=-4\pi\int_0^a \phi(t)\,\mathrm{d}t \tag{58}$$

and with (50) substituted in (58),

$$\frac{P}{16G\Delta a}=-\frac{2(1-\nu)(1-2\nu)}{3-4\nu}\int_0^a \phi^*(t)\,\mathrm{d}t \tag{59}$$

The expression for the load–displacement relationship may be evaluated by making use of the numerical solutions developed for F_1, F_3 and G_1^*.

THE STRESS INTENSITY FACTOR AT THE BOUNDARY OF THE CRACK

The flaw shearing mode (or mode II) stress intensity factor at the boundary of the crack may be evaluated by considering the stress distribution σ_{rz} in the region $r\in(b,\infty)$. The integral expression for the shear stresses in the elastic medium is

$$\sigma_{r,z}(r,0)=\frac{f_3(r)}{2(1-\nu)}+\frac{r(1-2\nu)}{2(1-\nu)}\int_r^a \frac{\phi(t)\,dt}{(t^2-r^2)^{3/2}} \tag{60}$$

By making use of (48) and (50), the result (60) may be expressed in the form

$$\sigma_{r,z}(r,0)=-\frac{8G\Delta a(1-2\nu)F_3(r)}{\pi^2 r(3-4\nu)(r^2-b^2)^{1/2}}+\frac{r(1-2\nu)}{2(1-\nu)}\int_r^a \frac{\phi(t)\,dt}{(t^2-r^2)^{3/2}},\quad b<r<\infty \tag{61}$$

The flaw shearing mode stress intensify factor K_{II} is defined by

$$K_{\mathrm{II}}=\lim_{r\to b^+}\{2(r-b)\}^{1/2}\sigma_{rz}(r,0) \tag{62}$$

Using (61) in (62) and taking the appropriate limits, we obtain

$$\frac{K_{\mathrm{II}}}{G\Delta}=-\frac{8a(1-2\nu)F_3(b)}{\pi^2(3-4\nu)b^{3/2}} \tag{63}$$

Again, the stress intensity factor may be evaluated upon numerical solution of the coupled system of integral equations (51)–(53).

NUMERICAL SOLUTION OF THE INTEGRAL EQUATIONS

The system of triple integral equations (51)–(53) may be reduced to a system of coupled integral equations by substituting the result for $F_3(r)$ given by (52) into (51) and (53). Performing this operation, and introducing the substitutions

$$\xi=\frac{t}{b},\qquad \eta=\frac{r}{b},\qquad c=\frac{a}{b},\qquad f(\eta)=b^2F_1(r),\qquad g(\eta)=G_1^*(r) \tag{64}$$

we obtain the following system of coupled integral equations;

$$\begin{aligned} f(\eta)=&\frac{4}{\pi^2}\int_0^c\left(\frac{1-\xi^2}{c^2-\xi^2}\right)^{1/2}\xi^3\chi(\xi,\eta)f(\xi)\,d\xi\\ &+\int_c^1\left[\frac{2}{\pi}\left(\frac{1-\xi^2}{\xi^2-c^2}\right)^{1/2}\xi\chi(\xi,\eta)\right.\\ &\left.+\frac{1}{\xi(\eta^2-\xi^2)}\right]g(\xi)\,d\xi,\quad 0<\eta<c \end{aligned} \tag{65}$$

and

$$\begin{aligned} g(\eta)=&-1-\frac{4}{\pi^2}\left(\frac{1-2\nu}{3-4\nu}\right)^2\\ &\times\left\{\int_0^c\left[\frac{\eta^2}{(\eta^2-\xi^2)}+c\left(\frac{1-\xi^2}{c^2-\xi^2}\right)\right]\xi f(\xi)\,d\xi\right.\\ &\left.+\frac{\pi}{2}c\int_c^1\left(\frac{1-\xi^2}{\xi^2-c^2}\right)^{1/2}\frac{g(\xi)}{\xi}\,d\xi\right\},\quad c<\eta<1 \end{aligned} \tag{66}$$

where the function $\chi(\xi, \eta)$ is defined as

$$\chi(\xi, \eta) = \int_0^1 \left(\frac{1-c^2S^2}{1-S^2}\right)^{1/2} \frac{S^3\,dS}{(1-\eta^2S^2)(1-\xi^2S^2)} \tag{67}$$

The expression for the axial stiffness of the anchor corresponding to equation (59) may now be written as

$$\begin{aligned}\frac{P}{16G\Delta} &= -\frac{2(1-\nu)}{3-4\nu} \\ &\times\left\{1+\frac{4}{\pi^2}\frac{(1-2\nu)^2}{3-4\nu}\left[\int_0^c\left\{c\left(\frac{1-\xi^2}{c^2-\xi^2}\right)^{1/2}-1\right\}\xi f(\xi)\,d\xi\right.\right. \\ &\left.\left.+\frac{\pi}{2}c\int_c^1\left(\frac{1-\xi^2}{\xi^2-c^2}\right)^{1/2}\frac{g(\xi)}{\xi}\,d\xi\right]\right\}\end{aligned} \tag{68}$$

Similarly, the flaw shearing mode stress intensity factor at the boundary of the cracked region, equation (63), may be written as

$$\frac{K_{\mathrm{II}}}{G\Delta} = \frac{8(1-2\nu)}{\pi^2(3-4\nu)}\frac{c}{\sqrt{b}}\left[\frac{2}{\pi}\int_0^c \frac{\xi^3 f(\xi)\,d\xi}{[(c^2-\xi^2)(1-\xi^2)]^{1/2}} + \int_c^1 \frac{\xi g(\xi)\,d\xi}{[(\xi^2-c^2)(1-\xi^2)]^{1/2}}\right] \tag{69}$$

A quadrature scheme of order N is used to solve the coupled system (65)–(66). The scheme is applied to the intervals $[0, c]$ and $[c, 1]$ separately. Therefore a matrix equation of order $2N$ is obtained, in the form

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}\begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \tag{70}$$

where the coefficient submatrices are obtained from equations (65) and (66). Upon solution, the results for the axial stiffness of the anchor and the stress intensity factor at the crack tip can be obtained via equations (68) and (69), respectively. The order of the quadrature scheme can be varied to ensure convergence of the results to limiting values. In the numerical treatments presented here, we employ $N = 16$ and $\nu \in (0, 0{\cdot}5)$. The techniques adopted for the numerical solution of the coupled integral equations are sufficiently accurate for the purposes of evaluation of the load–displacement relationship and the stress intensity factor defined by equations (68) and (69). Other numerical procedures that can be adopted for the solution of integral equations are documented by Baker.[22]

NUMERICAL RESULTS AND CONCLUSIONS

Before examining the results derived from the numerical analysis of the coupled integral equations governing the disk anchor problem, it is instructive to establish certain limiting cases associated with the extent of boundary cracking. In the limiting case when $b \to a$, the cracking is absent and the problem reduces to that of the axial loading of a disk anchor which is embedded in bonded contact with an elastic medium of infinite extent. The exact closed-form results for the axial stiffness of the embedded rigid disk anchor were derived by Collins,[23] Selvadurai,[2] and Kanwal and Sharma[24] by using, respectively, integral equation techniques, spheroidal harmonic function techniques and singularity methods. As $a/b \to 1$,

$$\frac{P}{16G\Delta a} = \frac{2(1-\nu)}{3-4\nu} \tag{71}$$

As the cracking extends to infinity, the problem reduces to the case where the disk anchor is embedded in bonded contact with two identical half-space regions. The exact analytical solution for the axial stiffness of the disk anchor may then be obtained by simply considering the result, given in the introduction, for the axial stiffness of a rigid punch bonded to an isotropic elastic half-space. Consequently, as $a/b \to 0$,

$$\frac{P}{16G\Delta a} = \frac{\ln(3-4\nu)}{2(1-2\nu)} \tag{72}$$

The results for the axial stiffness of the disk anchor derived by using the numerical analysis scheme are shown in Figure 2. The exact closed-form results obtained from equations (71) and (72) are also shown for comparison. It is evident that the numerical procedures yield results which compare

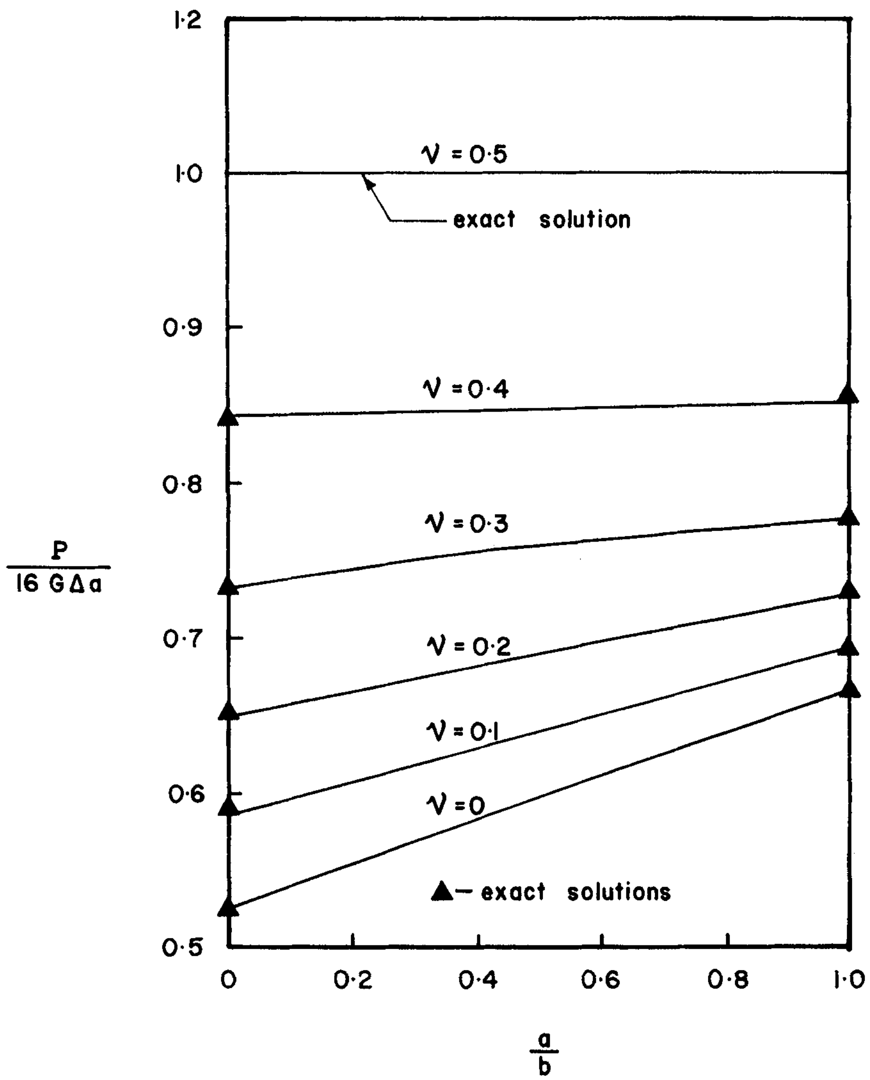

Figure 2. Variation of the axial stiffness of the disk anchor with the extent of cracking

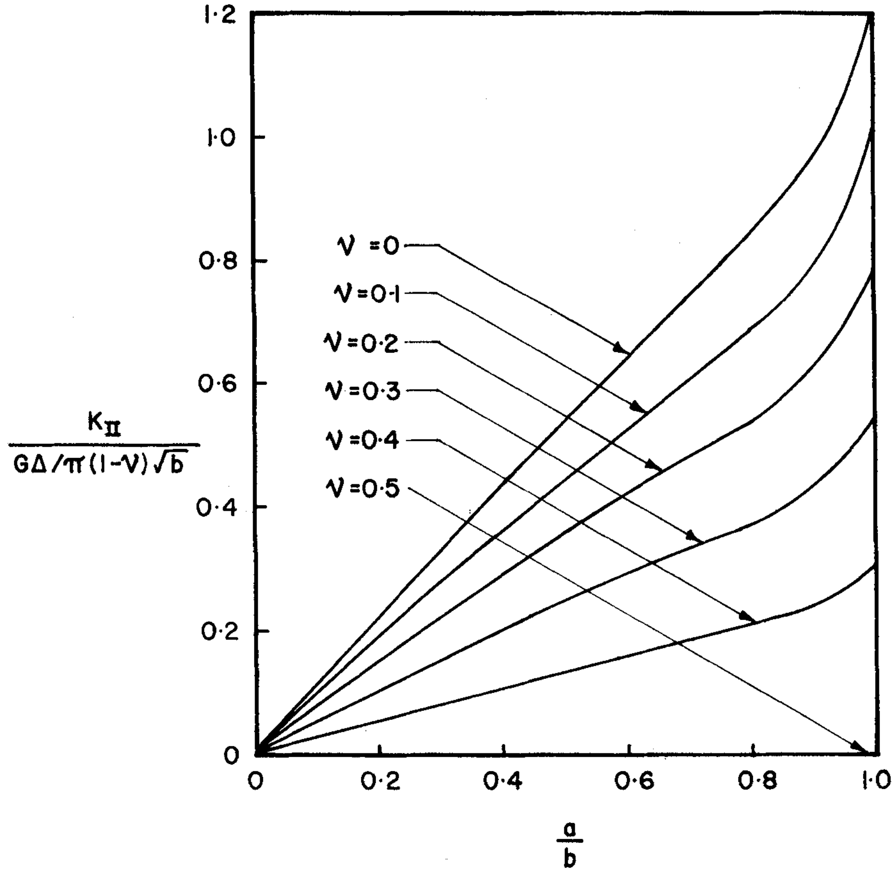

Figure 3. Variation of the flaw shearing mode stress intensity factor at the boundary of the cracked region

very accurately with the exact closed-form solutions cited in equations (71) and (72). As is evident, in the limit of material incompressibility the extent of cracking has no influence on the axial stiffness of the disk anchor. The maximum influence of the cracking on the axial stiffness of the anchor occurs when $\nu = 0$. In this case the elastostatic stiffness can be reduced by as much as approximately 25 per cent of the stiffness for the uncracked case. However, for most naturally occurring soils and rocks $\nu \in (0{\cdot}2, 0{\cdot}5)$. In this case the reduction in stiffness due to the boundary cracking is much smaller and may be considered to be of little or no practical significance. Figure 3 illustrates the manner in which the flaw shearing mode stress intensity factor at the boundary of the crack is influenced by the extent of cracking. Again, in the limit of material incompressibility, the flaw shearing mode stress intensity factor at the boundary of the crack reduces to zero for all choices of $a/b \in (0, 1)$.

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