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The Reissner-Sagoci problem for a half-space with a surface constraint

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1. Introduction

The statical Reissner-Sagoci problem [1, 2, 3, 4, 5] is that of determining the components of stress and displacement in the interior of a semi-infinite homogenous isotropic solid $z \geqq 0$ when a circular region $(0 \leqq r < a, z = 0)$ of the boundary surface is forced to rotate through an angle α about an axis normal to the underformed plane surface of the medium. The problem of the torsion of an elastic cylinder embedded in an elastic half-space with a different modulus of rigidity was considered by Dhaliwal, Singh and Sneddon [6]. The statical problem of the torsion of an annular disc attached to a semi-infinite cylinder embedded in an elastic half-space was considered by Singh et al. [7]. Dhaliwal and Singh [8] considered the torsion by an annular die of an elastic layer bonded to a semi-infinite elastic medium. References to static and dynamic torsion problems are found in the recent book by Gladwell [9].

In this paper, we consider the torsion of an elastic half-space $z \geqq 0$ by a rigid disc $(0 < r < a, z = 0)$ bonded to an elastic half-space. The surface $a < r < b, z = 0$ of the half-space is stress free and the remainder of the surface $b < r, z = 0$ is rigidly constrained. The geometry of the problem is shown in Figure 1. To the best of the authors' knowledge, this problem is new to the literature.

The solution of this three-part boundary value problem is reduced to solving triple integral equations. By modifying the method discussed by Cooke [10], the solution of the triple integral equations is obtained so as to obtain the expression for the resultant torque needed to produce the prescribed displacements.

2. Boundary conditions and derivation of triple integral equations

Expressed in cylindrical co-ordinates (r, ϕ, z), the displacement vector has only one non-vanishing component $u_\phi(r, z)$ and the stress tensor has only two non-vanishing components, $\sigma_{r\phi}(r, z)$ and $\sigma_{\phi z}(r, z)$. The stress-strain relations reduce to the two equations

$$\sigma_{r\phi} = Gr\frac{\partial}{\partial r}\left(\frac{u_\phi}{r}\right), \quad \sigma_{\phi z} = G\frac{\partial u_\phi}{\partial z}, \tag{1}$$

where G is the shear modulus of the material. From these equations, it follows that the

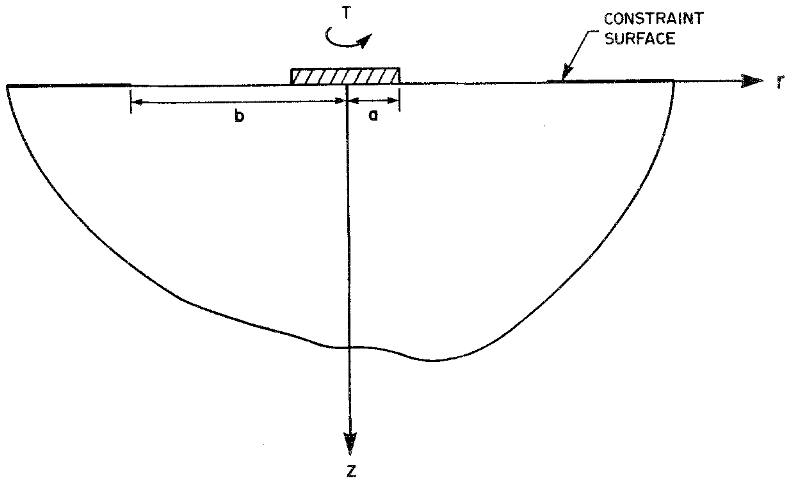

Figure 1

equilibrium equation is satisfied, provided that the function $u_\phi(r, z)$ is a solution of the partial differential equation

$$\frac{\partial^2 u_\phi}{\partial r^2} + \frac{1}{r}\frac{\partial u_\phi}{\partial r} - \frac{u_\phi}{r^2} + \frac{\partial^2 u_\phi}{\partial z^2} = 0. \tag{2}$$

The boundary conditions can be written in the form

$$u_\phi(r, 0) = \alpha r, \quad 0 \leqq r < a, \tag{3}$$

$$\sigma_{\phi z}(r, 0) = 0, \quad a < r < b, \tag{4}$$

$$u_\phi(r, 0) = 0, \quad b < r, \tag{5}$$

where α is the angle of rotation of the disc. We assume that, as $r \to \infty$, u_ϕ, $\sigma_{r\phi}$ and $\sigma_{\phi z}$, all tend to zero. Solving equation (2), we find that

$$u_\phi(r, 0) = \int_0^\infty A(\xi) J_1(\xi r)\, d\xi, \tag{6}$$

$$\sigma_{\phi z}(r, 0) = -G \int_0^\infty \xi A(\xi) J_1(\xi r)\, d\xi, \tag{7}$$

where $J_\nu(\xi r)$ are Bessel functions of first kind with $\nu \geqq 0$. $A(\xi)$ is an unknown function which is to be determined. The boundary conditions (3), (4) and (5) reduced to the triple integral equations

$$\int_0^\infty A(\xi) J_1(\xi r)\, d\xi = \alpha r, \quad 0 \leqq r < a, \tag{8}$$

$$\int_0^\infty \xi A(\xi) J_1(\xi r)\, d\xi = 0, \quad a < r < b, \tag{9}$$

$$\int_0^\infty A(\xi) J_1(\xi r)\, d\xi = 0, \quad b < r < \infty. \tag{10}$$

3. Solution of the triple integral equations

Let us assume that

$$\int_0^\infty \xi A(\xi) J_1(\xi r)\, d\xi = \begin{cases} f_1(r), & 0 < r < a, \\ f_2(r), & b < r < \infty. \end{cases} \tag{11}$$

Making use of the Hankel inversion theorem, we obtain from equations (9) and (11) that

$$A(\xi) = \int_0^a t f_1(t) J_1(t\xi)\, dt + \int_b^\infty t f_2(t) J_1(t\xi)\, dt. \tag{12}$$

Substituting the value of $A(\xi)$ from equation (12) into equations (8) and (10) yields

$$\int_0^a t f_1(t) L(r, t)\, dt + \int_b^\infty t f_2(t) L(r, t)\, dt = \alpha r, \quad 0 < r < a, \tag{13}$$

$$\int_0^a t f_1(t) L(r, t)\, dt + \int_b^\infty t f_2(t) L(r, t)\, dt = 0, \quad b < r < \infty, \tag{14}$$

where

$$L(r, t) = \int_0^\infty J_1(\xi r) J_1(\xi t)\, d\xi. \tag{15}$$

Making use of the paper of Cooke [10], we find that

$$\begin{aligned} L(r, t) &= \frac{2}{\pi r t} \int_0^{\min(r,t)} \frac{w^2\, dw}{[(r^2 - w^2)(t^2 - w^2)]^{1/2}}, \\ &= \frac{2rt}{\pi} \int_{\max(r,t)}^\infty \frac{w^{-2}\, dw}{[(w^2 - r^2)(w^2 - t^2)]^{1/2}} \end{aligned} \tag{16}$$

$$\int_a^b dt \int_0^{\min(r,t)} dw = \int_0^r dw \int_w^b dt + \int_0^a dw \int_a^b dt, \tag{17}$$

and

$$\int_a^b dt \int_{\max(r,t)}^\infty dw = \int_r^b dw \int_a^w dt + \int_b^\infty dw \int_a^b dt. \tag{18}$$

With the help of the equation (16), we write equation (13) in the following form.

$$\begin{aligned} &\int_0^a f_1(t)\, dt \int_0^{\min(r,t)} \frac{w^2\, dw}{[(r^2 - w^2)(t^2 - w^2)]^{1/2}} + r^2 \int_b^\infty t^2 f_2(t)\, dt \\ &\times \int_t^\infty \frac{w^{-2}\, dw}{[(w^2 - r^2)(w^2 - t^2)]^{1/2}} = \frac{\pi \alpha r^2}{2}, \quad 0 < r < a. \end{aligned} \tag{19}$$

Using equation (17) and changing the order of integration in the second integral, we get

$$\int_0^r \frac{w^2 F_1(w)\, dw}{(r^2 - w^2)^{1/2}} = \frac{\pi \alpha r^2}{2} - r^2 \int_b^\infty \frac{w^{-2} F_2(w)\, dw}{(w^2 - r^2)^{1/2}}, \quad 0 < r < a, \tag{20}$$

where

$$F_1(w) = \int_w^a \frac{f_1(t)\,dt}{(t^2 - w^2)^{1/2}}, \quad 0 < w < a, \tag{21}$$

$$F_2(w) = \int_b^w \frac{t^2 f_2(t)\,dt}{(w^2 - t^2)^{1/2}}, \quad b < w < \infty. \tag{22}$$

Regarding the right hand side of this equation as a known function of r, equation (20) is of an Abel type form. Hence, its solution can be written as

$$wF_1(w) = 2w\alpha - \frac{1}{\pi}\int_b^\infty \left[\frac{2vw}{v^2 - w^2} - \log\left|\frac{v - w}{v + w}\right|\right] v^{-2} F_2(v)\,dv, \quad 0 < w < a, \tag{23}$$

In obtaining equation (23), we have made use of the following integrals:

$$\frac{d}{dw}\int_0^w \frac{r^3\,dr}{(w^2 - r^2)^{1/2}} = 2w^2, \tag{24}$$

$$\frac{d}{dw}\int_0^w \frac{r^3\,dr}{[(w^2 - r^2)(v^2 - r^2)]^{1/2}} = \frac{w}{2}\left[\frac{2vw}{v^2 - w^2} - \log\left|\frac{v - w}{v + w}\right|\right]. \tag{25}$$

Using equation (16), we write equation (14) in the form

$$\int_b^\infty t^2 f_2(t)\,dt \int_{\max(r,t)}^\infty \frac{w^{-2}\,dw}{[(w^2 - r^2)(w^2 - t^2)]^{1/2}} + \frac{1}{r^2}\int_0^a f_1(t)\,dt$$

$$\times \int_0^t \frac{w^2\,dw}{[(r^2 - w^2)(t^2 - w^2)]^{1/2}} = 0, \quad b < r < \infty. \tag{26}$$

Making use of the result (18), changing the order of integration in second integral and using the equations (21) and (22), we get

$$\int_r^\infty \frac{w^{-2} F_2(w)\,dw}{(w^2 - r^2)^{1/2}} = -\frac{1}{r^2}\int_0^a \frac{w^2 F_1(w)\,dw}{(r^2 - w^2)^{1/2}}, \quad b < r < \infty. \tag{27}$$

Solving the above Abel type integral equation gives

$$w^{-2} F_2(w) = \frac{2}{\pi}\frac{d}{dw}\int_w^\infty \frac{dr}{r(r^2 - w^2)^{1/2}} \int_0^a \frac{v^2 F_1(v)\,dv}{(r^2 - v^2)^{1/2}}, \quad b < w < \infty, \tag{28}$$

Making use of the integral

$$\frac{d}{dw}\int_w^\infty \frac{dr}{r[(r^2 - w^2)(r^2 - v^2)]^{1/2}} = \frac{1}{2vw^2}\log\left|\frac{w - v}{w + v}\right| - \frac{1}{w(w^2 - v^2)}, \tag{29}$$

this in turn gives

$$w^{-1} F_2(w) = \frac{1}{\pi}\int_0^a vF_1(v)\left[\frac{1}{w}\log\left|\frac{w - v}{w + v}\right| - \frac{2v}{(w^2 - v^2)}\right] dv, \quad b < w < \infty. \tag{30}$$

Assuming

$$vF_1(v) = 2\alpha a X_1(v), \quad v^{-1} F_2(v) = 2\alpha a X_2(v), \quad \varepsilon = \frac{a}{b}, \tag{31}$$

and changing the variable

$$v = bv_1, \quad w = aw_1,$$

the integral equation (23) can be written as

$$X_1(aw_1) = w_1 - \frac{1}{\pi}\int_1^\infty \left[\frac{2w_1 c}{v_1^2 - c^2 w_1^2} - \frac{1}{v_1}\log\left| \frac{1 - \dfrac{cw_1}{v_1}}{1 + \dfrac{cw_1}{v_1}} \right| \right] X_2(bv_1)\,dv_1, \quad 0 < w_1 < 1. \tag{32}$$

In a similar way, equation (30) can be written as

$$X_2(bw_1) = \frac{1}{\pi}\int_0^1 \left[\frac{c}{w_1}\log\left| \frac{1 - \dfrac{cv_1}{w_1}}{1 + \dfrac{cv_1}{w_1}} \right| - \frac{2v_1 c^2}{w_1^2 - c^2 v_1^2} \right] X_1(av_1)\,dv_1, \quad 1 < w_1 < \infty. \tag{33}$$

For small values of $c \ll 1$, we find that

$$\frac{2w_1 c}{v_1^2 - w_1^2 c^2} - \frac{1}{v_1}\log\left| \frac{1 - \dfrac{cw_1}{v_1}}{1 + \dfrac{cw_1}{v_1}} \right| = \frac{4cw_1}{v_1^2} + \frac{8w_1^3 c^3}{3v_1^4} + \frac{15c^5 w_1^5}{5v_1^6}$$
$$+ \frac{16c^7 w_1^7}{7v_1^8} + \frac{20c^9 w_1^9}{9v_1^{10}} + O(c^{11}), \tag{34}$$

and

$$\frac{c}{w_1}\log\left| \frac{1 - \dfrac{cv_1}{w_1}}{1 + \dfrac{cv_1}{w_1}} \right| - \frac{2v_1 c^2}{w_1^2 - c^2 v_1^2} = -\frac{4c^2 v_1}{w_1^2} - \frac{8c^4 v_1^3}{3w_1^4} - \frac{12c^6 v_1^5}{5w_1^6} - \frac{16c^8 v_1^7}{7w_1^8} + O(c^{10}), \tag{35}$$

Let us assume that

$$X_1(aw_1) = \sum_{i=0}^{p} c^i m^i(w_1), \tag{36}$$

$$X_2(bw_1) = \sum_{i=0}^{p} c^i n^i(w_1). \tag{37}$$

Substituting the value of $X_1(aw_1)$ and $X_2(bw_1)$ from equations (36) and (37) into equations (32) and (33) respectively, making use of the equations (34) and (35) and on comparing the like terms of c, we find that $X_1(aw_1)$ and $X_2(bw_1)$ can be expressed as

$$X_1(aw_1) = w_1 + \frac{16w_1}{9\pi^2}c^3 + \frac{32}{\pi^2}\left(\frac{w_1}{75} + \frac{w_1^3}{45}\right)c^5 + \frac{256}{81\pi^4}w_1 c^6 + O(c^7), \tag{38}$$

$$X_2(bw_1) = -\frac{4c^2}{3\pi w_1^2} - \frac{8c^4}{15\pi w_1^4} - \frac{64c^5}{27\pi^3 w_1^2} - \frac{12c^6}{35\pi w_1^6} + O(c^8). \tag{39}$$

Knowing $X_1(aw_1)$ and $X_2(bw_1)$, we can find $F_1(w)$ and $F_2(w)$ from equation (31). Equation (21) and (22) are of Abel type. Hence the solutions can be written as

$$f_1(t) = -\frac{2}{\pi}\frac{d}{dt}\int_t^a \frac{wF_1(w)\,dw}{(w^2 - t^2)^{1/2}}, \quad 0 < t < a, \tag{40}$$

$$f_2(t) = \frac{2}{\pi}t^{-2}\frac{d}{dt}\int_b^t \frac{wF_2(w)\,dw}{(t^2 - w^2)^{1/2}}, \quad b < t < \infty. \tag{41}$$

Therefore, knowing $F_1(w)$ and $F_2(w)$, we can find $f_1(t)$ and $f_2(t)$ from equations (40) and (41) and finally $A(\xi)$ from equation (12).

4. Expression for torque

We find from equation (7) that

$$\sigma_{\phi z}(r, 0) = G \frac{\partial}{\partial r} \int_0^\infty A(\xi) J_0(\xi r)\, d\xi, \quad 0 < r < a, \tag{42}$$

Substituting the value of $A(\xi)$ from equation (12) into equation (42) and using the relationship

$$\int_0^\infty J_0(\xi r) J_1(t\xi)\, d\xi = \begin{cases} \frac{1}{t}, & r < t, \\ 0, & t < r. \end{cases} \tag{43}$$

we get

$$\sigma_{\phi z}(r, 0) = -G f_1(r). \tag{44}$$

Also on using equation (40), we find that

$$\sigma_{\phi z}(r, 0) = \frac{2}{\pi} G \frac{d}{dr} \int_r^a \frac{w F_1(w)\, dw}{(w^2 - r^2)^{1/2}}, \quad 0 < r < a. \tag{45}$$

The torque-twist relationship is

$$T = -2\pi \int_0^a r^2 \sigma_{\phi z}(r, 0)\, dr. \tag{46}$$

Substituting the value of $\sigma_{\phi z}(r, 0)$ from equation (45) into equation (46), integrating by parts and changing the order of integrations, we get

$$T = 8G \int_0^a r^2 F_1(r)\, dr. \tag{47}$$

With the help of relationship (31), it follows that

$$T = 16\alpha G a^3 \int_0^1 w_1 X_1(a w_1)\, dw_1, \tag{48}$$

Substituting the value $X_1(aw_1)$ from equation (38) into (48) gives

$$T = \frac{16\alpha G a^3}{3} \left[1 + \frac{16c^3}{9\pi^2} + \frac{64c^5}{75\pi^2} + \frac{256}{81\pi^4} c^6 + O(c^7) \right]. \tag{49}$$

As $b \to \infty$, we recover the expression for the torque in the absence of the constrained surface; viz,

$$T_\infty = \frac{16\alpha G a^3}{3}. \tag{50}$$

This expression is well known in the literature and its derivation is found in references [1–5].

Finally, with the use of the relationship (50), it follows that

$$\frac{T}{T_\infty} = 1 + \frac{16c^3}{9\pi^2} + \frac{64c^5}{75\pi^2} + \frac{256}{81\pi^4} c^6 + O(c^7), \tag{51}$$

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Abstract

In this paper, the Reissner-Sagoci problem for a half-space with a surface constraint is considered. The problem is reduced to the triple integral equation by use of integral transforms. The triple integral equations are solved for small values of parameters characterizing the geometry of the problem. An expression for the torque required to rotate the disc through a fixed angle is obtained.

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