Elastic Stiffness of Flat Anchor Region at Cracked Geological Interface

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Abstract: The present paper examines the problem of a flat circular anchoring region that is embedded at an elastic medium-rigid surface interface. The compliance of the anchor is enhanced by the presence of a debonded region. The paper develops a mathematical analysis of the ensuing elastostatic contact problem. A numerical scheme coupled with a bounding technique is used to develop a set of bounds for the elastic stiffness of the anchor region. The numerical results presented in the paper illustrate the manner in which the elastic stiffness of the anchor is influenced by the extent of the cracked region and Poisson's ratio of the elastic medium.

Introduction

Flat anchoring devices are used quite extensively in structural and geotechnical engineering to provide either temporary or permanent reactive support against uplift loads, prestressing forces, and gravity loads. In structural engineering, such anchoring devices embedded in concrete foundations provide anchorages for structural columns. In geotechnical engineering, earth and rock anchors with flat shapes can be created by hydraulic fracturing of the geological medium. Alternatively, flat plate anchors can be used at interfaces between grouted linings and the intact rock (Fig. 1).

This paper focuses on the analytical estimation of the elastic stiffness of a flat anchoring region embedded either at a geological interface or at the interface between grouted concrete and a rock mass. The anchor is modeled as a rigid disc that is embedded in partially bonded contact with the deformable medium (Fig. 2). This partial bonding can occur either during the installation of the anchor or during the repeated application of anchor loads. The partial bonding and the detachment of the anchor region affects the load-displacement response of the anchor.

In the ensuing analysis, we examine the category of problem in which the anchor is located at the boundary between two geological media with contrasting deformability characteristics, especially in the elastic range. In this sense, the anchor is assumed to be located at a deformable geological medium-rigid boundary interface. The anchor exhibits contact with the deformable geological medium, and the latter exhibits contact with the rigid boundary beyond the cracked region (Fig. 3). The anchor is subjected to an axial force P, which induces a rigid body displacement Δ in the axial direction. The elasticity formulation of the anchor-geological medium interaction considered in this paper is a simplified idealization of an otherwise complex problem. Features such as nonlinear and time-dependent responses of the geological medium and the interfaces, partial contact, cracking and sepa-

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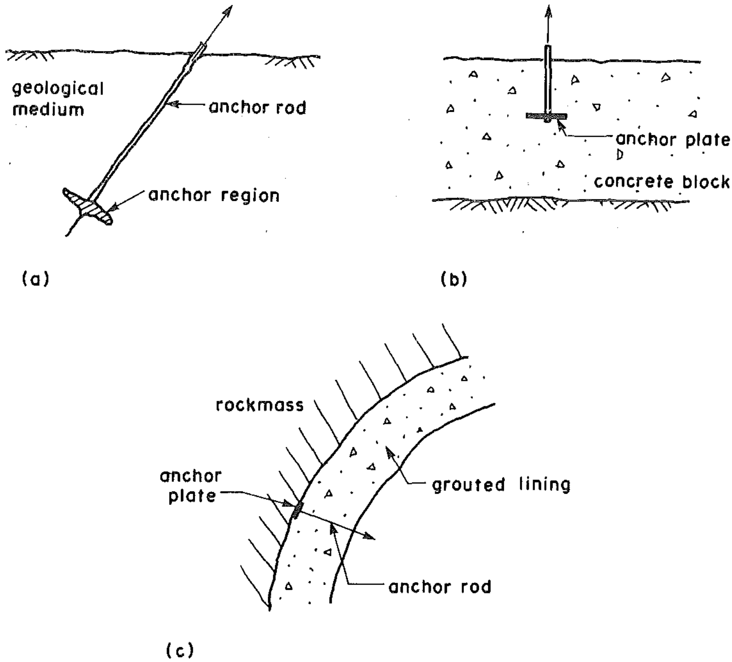

FIG. 1. Anchoring Systems: (*a*) Ground Anchor; (*b*) Plate Anchor Used for Column Connection; (*c*) Plate Anchor at Interface

ration at the anchor-geological medium interface, material nonhomogeneities, etc., can have a significant influence on the anchor response [see, e.g., Jaeger (1972), Littlejohn (1970), Hobst and Zajic (1983), Selvadurai (1978), and Stephanson (1984)]. The elastic model, however, provides a useful first approximation for the analysis of anchor problems and results derived from such analysis indicate the typical anchor behavior at working load levels. Examples of such applications are given by Coates and Yu (1970), Selvadurai (1976, 1978, 1980), Luk and Keer (1980), Rowe and Davis (1982), and Selvadurai and Rajapakse (1985). Recent research by Ballarini et al. (1986) investigate the development of failure at the boundary of anchors embedded in brittle media.

The mathematical formulation of the anchor problems yields a mixed boundary value problem, the solution of which is complicated and non-routine. In this paper, we employ a method of analysis first proposed by Selvadurai (1984), wherein a set of bounds are developed for the elastic stiffness of the embedded anchor region. These bounds are developed by imposing displacement and/or traction constraints at the interface containing the anchor. The introduction of these constraints reduces the mixed boundary value problem to a system of triple integral equations that are solved in an approximate series fashion. The numerical results presented in the paper illustrate the manner in which the elastic stiffness of the embedded anchor is influenced by Poisson's ratio of the deformable geological medium and the extent of the cracked region.

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Fundamental Equations

Considering the axisymmetric state of the deformation induced by the indenting anchor (Fig. 3), it is convenient to employ the formulation based on the strain potential approach of Love (1927). In the absence of body forces, the solution of the displacement equations of equilibrium can be represented in terms of a biharmonic function $\phi(r,z)$, i.e.

$$\nabla^2\nabla^2\phi(r,z) = 0 \quad (1)$$

where

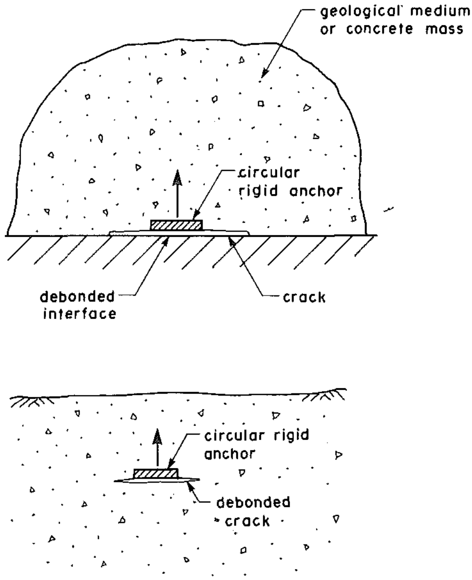

FIG. 2. Debonding and Cracking at Disc-Shaped Anchor Region

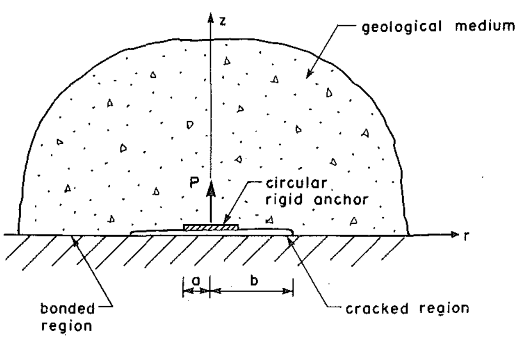

FIG. 3. Geometry of Anchor Located at Debonded Interface

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \quad (2)$$

The components of the displacement vector **u** and the Cauchy stress tensor $\boldsymbol{\sigma}$ referred to the cylindrical polar coordinate system and can be expressed in terms of the derivatives of $\phi(r,z)$. We have

$$2Gu_r = -\frac{\partial^2\phi}{\partial r \partial z} \quad (3)$$

$$2Gu_z = 2(1 - \nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2} \quad (4)$$

where G and ν = the linear elastic shear modulus and Poisson's ratio, respectively. Similarly, the components of the stress tensor are given by

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right) \quad (5)$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}\right) \quad (6)$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left[(2 - \nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \quad (7)$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left[(1 - \nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right] \quad (8)$$

Anchor Problem

We examine the problem of a rigid circular plate anchor of radius a which is embedded at a geological interface. It is assumed that the elastic properties of the regions are such that one region behaves effectively, as a rigid boundary. The interface region surrounding the anchor region experiences debonding over a circular region of radius b (Fig. 3). The rigid plate anchor is subjected to a central axial force P which induces a rigid body displacement Δ in the axial direction. During the application of the load P, the anchor remains bonded to the deformable elastic solid. The relevant mixed boundary conditions associated with the indentation problem are as follows:

$$u_z(r,0) = \Delta, \qquad 0 \le r \le a \quad (9)$$

$$u_r(r,0) = 0, \qquad 0 \le r \le a \quad (10)$$

$$\sigma_{zz}(r,0) = 0, \qquad a < r < b \quad (11)$$

$$\sigma_{rz}(r,0) = 0, \qquad a < r < b \quad (12)$$

$$u_r(r,0) = 0, \qquad b \le r < \infty \quad (13)$$

$$u_z(r,0) = 0, \qquad b \le r < \infty \quad (14)$$

The exact analytical solution of the category of mixed boundary value problems defined by Eqs. 9–14 entails an inordinate amount of computational effort that may not be altogether justified in view of the approximate nature of the basic model. Also, since the analysis concentrates primarily on the estimation of results of a global nature (namely, the load-displacement response of the anchor), it is desirable to explore alternative techniques that can be used to solve the basic problem. It is of course possible to employ numerical methods of stress analysis, such as finite element, boundary element, and boundary integral equation techniques to solve the problem posed. Even with these numerical schemes, it is necessary to take into account variable mesh refinement as the ratio a/b changes and to incorporate singular behavior of the stress and displacement fields at the boundaries $r = a$ and $r = b$.

In the following, we adopt the technique proposed by Selvadurai (1984) to develop a set of bounds for the elastic stiffness of the anchor embedded at the cracked interface. These bounds are developed by imposing displacement and/or traction constraints at the interface. The upper bound imposes an inextensibility constraint at the interface, and the lower bound is obtained by imposing a frictionless bilateral contact at the interface.

Upper Bound Analysis

In the development of the upper bound, we assume that the plane containing the disc anchor exhibits inextensibility conditions in the radial direction, in the region $a \le r \le b$. Consequently, the mixed boundary conditions Eqs. 9–14 reduce to the following:

$$u_z(r,0) = \Delta, \qquad 0 \le r \le a \qquad (15)$$

$$u_r(r,0) = 0, \qquad r \ge 0 \qquad (16)$$

$$\sigma_{zz}(r,0) = 0, \qquad a < r < b \qquad (17)$$

$$u_z(r,0) = 0, \qquad b \le r < \infty \qquad (18)$$

For the solution of the mixed boundary conditions defined by Eqs. 15–18, we seek solutions of Eq. 1 that can be obtained by a Hankel transform development of the governing differential equation. Furthermore, the displacement and stress fields derived from $\phi(r,z)$ should reduce to zero as $(r^2 + z^2)^{1/2} \to \infty$. Following Sneddon (1975), the relevant solution to Eq. 1 is given by

$$\phi(r,z) = \int_0^\infty \xi[A(\xi) + zB(\xi)]e^{-\xi z}J_0(\xi r)d\xi \qquad (19)$$

where $A(\xi)$ and $B(\xi)$ = arbitrary functions which are to be determined by satisfying the mixed boundary conditions Eqs. 15–18. These mixed boundary conditions yield the following system of triple integral equations for a single unknown function $R(\xi)$, i.e.

$$H_0[\xi^{-1}R(\xi);r] = -\frac{2G\Delta}{(3-4\nu)}, \qquad 0 \le r \le a \qquad (20)$$

$$H_0[R(\xi);r] = 0, \qquad a < r < b \qquad (21)$$

$$H_0[\xi^{-1}R(\xi);r] = 0, \qquad b \le r < \infty \qquad (22)$$

where $H_n[\psi(\xi);r]$ = the Hankel transform of order n, which is defined by

$$H_n[\psi(\xi);r] = \int_0^\infty \xi\psi(\xi)J_n(\xi r)d\xi \quad (23)$$

and

$$\xi^3 A(\xi) = \xi^2 B(\xi) = R(\xi) \quad (24)$$

For the solution of the triple system Eqs. 20–22, we assume that Eq. 21 admits a representation of the form

$$H_0[R(\xi);r] = f_1(r), \qquad 0 < r < a \quad (25a)$$

$$H_0[R(\xi);r] = f_2(r), \qquad b < r < \infty \quad (25b)$$

By using the Hankel inversion theorem, we observe that

$$R(\xi) = \int_0^a \lambda f_1(\lambda)J_0(\xi\lambda)d\lambda + \int_b^\infty \lambda f_2(\lambda)J_0(\xi\lambda)d\lambda \quad (26)$$

Using Eq. 26, the integral Eqs. 20 and 22 can be written in the forms

$$\int_0^a \lambda f_1(\lambda)L(\lambda,r)d\lambda + \int_b^\infty \lambda f_2(\lambda)L(\lambda,r)d\lambda = -\frac{2G\Delta}{(3-4\nu)}, \qquad 0 \le r \le a \quad (27)$$

and

$$\int_0^a \lambda f_1(\lambda)L(\lambda,r)d\lambda + \int_b^\infty \lambda f_2(\lambda)L(\lambda,r)d\lambda = 0, \qquad b \le r < \infty \quad (28)$$

respectively, where

$$L(\lambda,r) = \int_0^\infty J_0(\xi\lambda)J_0(\xi r)d\xi = \frac{2}{\pi}\int_0^{\min(\lambda,r)} \frac{ds}{[(\lambda^2 - s^2)(r^2 - s^2)]^{1/2}} \quad (29a)$$

$$L(\lambda,r) = \int_0^\infty J_0(\xi\lambda)J_0(\xi r)d\xi = \frac{2}{\pi}\int_{\max(\lambda,r)}^\infty \frac{ds}{[(s^2 - \lambda^2)(s^2 - r^2)]^{1/2}} \quad (29b)$$

where $\min(\lambda,r)$ and $\max(\lambda,r)$ = the minimum and the maximum values of λ and r, respectively.

Using the following results:

$$\int_a^b d\lambda \int_0^{\min(\lambda,r)} ds = \int_a^r ds \int_s^\lambda d\lambda + \int_0^a ds \int_a^b d\lambda \quad (30)$$

$$\int_a^b d\lambda \int_{\max(\lambda,r)}^\infty ds = \int_r^b ds \int_a^s d\lambda + \int_b^\infty ds \int_a^b d\lambda \quad (31)$$

Eq. 27 can be expressed in the following form:

$$\int_0^r \frac{ds}{(r^2 - s^2)^{1/2}} \int_s^a \frac{\lambda f_1(\lambda)d\lambda}{(\lambda^2 - s^2)^{1/2}} + \int_b^\infty \frac{ds}{(s^2 - r^2)^{1/2}} \int_b^s \frac{\lambda f_2(\lambda)d\lambda}{(s^2 - \lambda^2)^{1/2}}$$

[*Continued*]

$$= -\frac{G\Delta\pi}{(3 - 4\nu)}, \qquad 0 \le r \le a \qquad (32)$$

We now introduce the substitutions

$$\int_s^a \frac{\lambda f_1(\lambda)d\lambda}{(\lambda^2 - s^2)^{1/2}} = F_1(s) \qquad (33)$$

$$\int_b^s \frac{\lambda f_2(\lambda)d\lambda}{(s^2 - \lambda^2)^{1/2}} = F_2(s) \qquad (34)$$

Using the given and the properties of integral equations of the Abel-type [see, e.g., Sneddon (1960)], Eqs. 32 and 28 can be reduced to the form

$$F_1(r) = -\frac{2}{\pi}\int_b^\infty \frac{sF_2(s)ds}{(s^2 - r^2)} - \frac{2G\Delta}{(3 - 4\nu)}, \qquad 0 \le r \le a \qquad (35)$$

$$\int_r^\infty \frac{F_2(s)ds}{(s^2 - r^2)^{1/2}} + \int_0^a \frac{F_1(s)ds}{(r^2 - s^2)^{1/2}} = 0, \qquad b \le r < \infty \qquad (36)$$

Observing that Eq. 36 is an integral equation of the Abel-type, it can be shown that

$$F_2(s) = -\frac{2}{\pi}\frac{d}{ds}\int_s^\infty \frac{rdr}{(r^2 - s^2)^{1/2}}\int_0^a \frac{F_1(u)du}{(r^2 - u^2)^{1/2}}; \qquad b \le s \qquad (37)$$

If we make a change in the order of integration in Eq. 37, we find that the inner integral is divergent. To render the integral convergent, we make use of the following result:

$$\frac{2}{\pi}\int_r^\infty \frac{sds}{(s^2 - a^2)(s^2 - r^2)^{1/2}}\int_0^a F_1(t)dt = \frac{1}{(r^2 - a^2)^{1/2}}\int_0^a F_1(t)dt \qquad (38)$$

Using Eq. 38, the integral Eq. 36 can be reduced to the form

$$F_2(s) = -\frac{2}{\pi}s\int_0^a \frac{F_1(u)du}{(s^2 - u^2)}; \qquad b \le s \qquad (39)$$

Introduce the substitutions

$$s = as_1 \qquad (40a)$$

$$u = bu_1 \qquad (40b)$$

and

$$[F_1(s); F_2(s)] = \frac{2G\Delta}{(3 - 4\nu)}[M(s_1); N(s_1)] \qquad (41)$$

The integral Eqs. 35 and 39 can be written in the forms

$$M(s_1) = -\frac{2}{\pi}\int_1^\infty \frac{u_1 N(u_1)du_1}{(u_1^2 - c^2 s_1^2)} - 1, \qquad 0 \le s_1 \le 1 \qquad (42)$$

$$N(s_1) = -\frac{2}{\pi}s_1 c\int_0^1 \frac{M(u_1)du_1}{(s_1^2 - c^2 u_1^2)}; \qquad s_1 \ge 1 \qquad (43)$$

where $c = a/b$.

For the solution of the integral Eq. 42 and 43, we assume that $M(s_1)$ and $N(s_1)$ admit power series expansion of the form

$$M(s_1) = \sum_{i=1}^{t^*} c^i m_i(s_1) \quad \text{(44a)}$$

$$N(s_1) = \sum_{i=1}^{t^*} c^i n_i(s_1) \quad \text{(44b)}$$

where $c < 1$. Using Eq. 44 in Eqs. 42 and 43 and expanding certain terms in the integrands in power series form, we obtain the following:

$$\sum_{i=1}^{t^*} c^i m_i(s_1) = -\frac{2}{\pi}\int_1^{\infty}\left(\frac{1}{u_1}+\frac{c^2 s_1^2}{u_1^3}+\frac{c^4 s_1^4}{u_1^5}+\frac{c^6 s_1^6}{u_1^7}+\dots\right)\sum_{i=1}^{t^*} c^i n_i(u_1)du_1 - 1,$$
$$0 \le s_1 < 1 \quad \text{(45)}$$

$$\sum_{i=1}^{t^*} c^i n_i(s_1) = -\frac{2}{\pi} s_1 \int_0^1\left(\frac{1}{s_1^2}+\frac{c^2 u_1^2}{s_1^4}+\frac{c^4 u_1^4}{s_1^6}+\frac{c^6 u_1^6}{s_1^8}+\dots\right)\sum_{i=1}^{t^*} c^i m_i(u_1)du_1,$$
$$s \ge 1 \quad \text{(46)}$$

Comparing like terms in Eqs. 45 and 46, it is possible to determine the components of $M(s_1)$ and $N(s_1)$ to any required order. The associated integral expressions for $m_i(s_1)$ and$n_i(s_1)$ are given in Appendix I. It is sufficient to record here the series expansions for $M(s_1)$ and $N(s_1)$. We have

$$M(s_1) = -1 - \frac{4}{\pi^2}c - \frac{16}{\pi^4}c^2 - \left[\frac{2}{\pi^2}\left(\frac{32}{\pi^4}+\frac{2}{9}\right)+\frac{4s_1^2}{3\pi^2}\right]c^3 - \left[\frac{16s_1^2}{3\pi^4}+\frac{48}{9\pi^4}+\frac{256}{\pi^8}\right]c^4$$
$$-\left[\frac{2}{\pi}\left(\frac{160}{9\pi^5}+\frac{2}{25\pi}+\frac{512}{\pi^9}\right)+\frac{2}{\pi}s_1^2\left(\frac{2}{15\pi}+\frac{32}{3\pi^5}\right)+\frac{4s_1^4}{5\pi^2}\right]c^5 + 0(c^6) \quad \text{(47)}$$

$$N(s_1) = \frac{2c}{\pi s_1} + \frac{8c^2}{\pi^3 s_1} + \left(\frac{2}{3\pi s_1^3}+\frac{32}{\pi^5 s_1}\right)c^3 + \left[\frac{2}{\pi s_1}\left(\frac{64}{\pi^6}+\frac{8}{9\pi^2}\right)+\frac{8}{3\pi^3 s_1^3}\right]c^4$$
$$+\left(\frac{32}{3\pi^5 s_1^3}+\frac{2}{5\pi s_1^5}+\frac{512}{\pi^9 s_1}+\frac{128}{9\pi^5 s_1}\right)c^5 + 0(c^6) \quad \text{(48)}$$

This formally completes the analysis of the upper bound problem. Results for the displacement and stress fields in the medium can be evaluated by making use of Eqs. 47, 48, and the appropriate expressions.

The result of primary interest to geomechanics concerns the load-displacement response of the disc anchor that is embedded at the interface. By considering the equilibrium of the disc, we have

$$P = -\int_0^{2\pi}\int_0^a r\sigma_{zz}(r,0)drd\theta \quad \text{(49)}$$

Considering the Abel integral Eq. 33, it can be shown that

$$\lambda f_1(\lambda) = -\frac{2}{\pi}\frac{d}{d\lambda}\int_\lambda^a \frac{sF_1(s)ds}{(s^2-\lambda^2)^{1/2}}, \qquad 0 < \lambda < a \quad \text{(50)}$$

Using Eqs. 25, 26, 41, and 50, Eq. 49 can be reduced to the form

$$P = -\frac{16G\Delta a(1 - \nu)}{(3 - 4\nu)} \int_0^1 M(s_1)ds_1 \qquad (51)$$

Using Eq. 47 in Eq. 51, we obtain the upper bound for the load-displacement relationship as follows:

$$P = \frac{16G\Delta a(1 - \nu)}{(3 - 4\nu)} \Omega(c) \qquad (52)$$

where

$$\Omega(c) = 1 + \frac{4}{\pi^2} c + \frac{16}{\pi^4} c^2 + \frac{1}{\pi^2}\left(\frac{64}{\pi^4} + \frac{8}{9}\right)c^3 + \left(\frac{64}{9\pi^4} + \frac{256}{\pi^8}\right)c^4$$

$$+ \frac{1}{\pi^2}\left(\frac{1024}{\pi^8} + \frac{384}{9\pi^4} + \frac{92}{225}\right)c^5 + 0(c^6) \qquad (53)$$

Lower Bound Analysis

In the development of the lower bound, we assume that the plane containing the disc anchor exhibits a smooth bilateral contact in the regions $0 \leq r \leq a$ and $b \leq r < \infty$. The bilateral contact can be maintained by subjecting the deformable geological medium to a suitable precompression and maintaining traction-free boundary conditions in the region $a < r < b$. The mixed boundary conditions Eqs. 9–14 now reduce to the following:

$$u_z(r,0) = \Delta, \qquad 0 \leq r \leq a \qquad (54)$$

$$\sigma_{rz}(r,0) = 0, \qquad r \geq 0 \qquad (55)$$

$$\sigma_{zz}(r,0) = 0, \qquad a < r < b \qquad (56)$$

$$u_z(r,0) = 0, \qquad b \leq r < \infty \qquad (57)$$

Considering the stress function Eq. 19 and the expressions Eqs. 3–8, the mixed boundary conditions Eqs. 54–57 can be reduced to the following system of triple integral equations for a single function $S(\xi)$:

$$H_0[\xi^{-1}S(\xi); r] = -\frac{2G\nu\Delta}{(1 - \nu)}, \qquad 0 \leq r \leq a \qquad (58)$$

$$H_0[S(\xi); r] = 0, \qquad a < r < b \qquad (59)$$

$$H_0[\xi^{-1}S(\xi); r] = 0, \qquad b \leq r < \infty \qquad (60)$$

It is evident that the mixed boundary conditions associated with the lower bound analysis yield a three-part boundary value problem that is identical to that which describes the upper bound analysis. The analysis follows the procedures outlined in the preceding section. Avoiding details of analysis, it can be shown that the lower bound result for the load-displacement relationship takes the form

$$P = \frac{4G\Delta a}{(1 - \nu)} \Omega(c) \qquad (61)$$

where $\Omega(c)$ is defined by Eq. 53.

Limiting Cases and Numerical Results

The upper and lower bound estimates for the load-displacement relationship for the disc anchor embedded interface can be combined to form the following inequality:

$$\frac{2\Omega(c)}{(1-\nu^2)} \leq \frac{P}{E\Delta a} \leq \frac{8(1-\nu)\Omega(c)}{(1+\nu)(3-4\nu)} \quad (62)$$

1. In the limiting case of material incompressibility (i.e., $\nu = 1/2$), the bounds Eq. 62 reduce to the single result

$$\frac{P}{E\Delta a} = \frac{8}{3}\,\Omega(c) \quad (63)$$

From this result, it is evident that when the deformable geological medium exhibits incompressible behavior, the interface satisfies the inextensibility and shear stress boundary conditions simultaneously.

2. As $c \rightarrow 0$, the boundary value problem reduces to that of a disc anchor that is embedded in bonded contact at an interface where extensive cracking or separation is present. In this case, the bounds yield the following results for $\nu = 0$:

$$2\Omega(0) \leq \frac{P}{E\Delta a} \leq \frac{8}{3}\,\Omega(0) \quad (64)$$

where $\Omega(0) = 1$.

The exact result for this case can be obtained from the solution to the problem of the axial loading of a rigid circular punch that is bonded to a halfspace [see, e.g., Gladwell (1980)]. We have

$$\frac{P}{E\Delta a} = \frac{2ln(3-4\nu)}{(1+\nu)(1-2\nu)} \quad (65)$$

As $\nu \rightarrow 1/2$, Eq. 65 reduces to $(P/E\Delta a) = 8/3$, and when $\nu = 0$, Eq. 65 gives $(P/E\Delta a) \simeq 2.2$.

3. In the special case in which the disc anchor exhibits a smooth contact at the anchor-geological medium interface, the associated mixed boundary value problem for the debonded interface yield the following boundary conditions:

$$u_z(r,0) = \Delta, \qquad 0 \leq r \leq a \quad (66)$$

$$\sigma_{zz}(r,0) = 0, \qquad a < r < b \quad (67)$$

$$\sigma_{rz}(r,0) = 0, \qquad 0 < r < b \quad (68)$$

$$u_r(r,0) = 0, \qquad b \leq r < \infty \quad (69)$$

$$u_z(r,0) = 0, \qquad b \leq r < \infty \quad (70)$$

The mixed boundary value problem defined by Eqs. 66–70 can be formulated as a set of integral equations by employing the Hankel transform approach. These, in turn, can be reduced to two coupled Fredholm integral equations of the second

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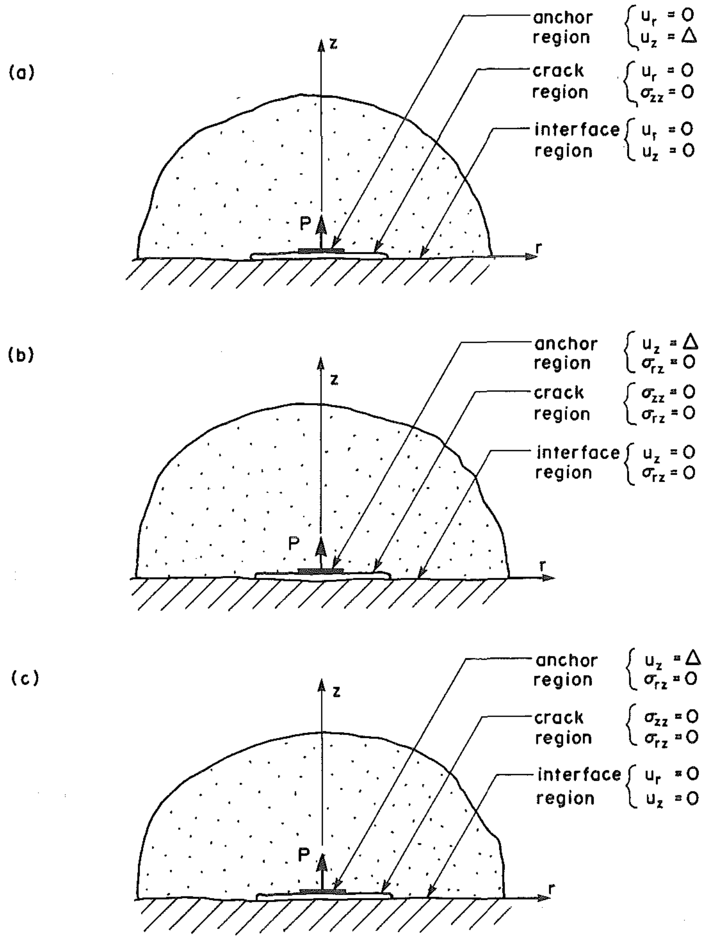

FIG. 4. Summary of Mixed Boundary Conditions Associated with Models Used in Estimation of Elastic Stiffness of Disc Anchor

kind that can be solved in a numerical fashion. Details of the method are given by Selvadurai et al. (1989). The load-displacement relationship for the smoothly indenting anchor can be evaluated from the preceding numerical scheme. The relevant numerical results will be shown in Figs. 5 and 6.

4. An alternative model for the dislocation-type discontinuity can be obtained by making use of the results for the internal indentation of a penny-shaped crack developed by Selvadurai and Singh (1984) and Selvadurai (1986). In this model, a penny-shaped crack of radius b is internally indented by a smooth disc inclusion of radius a and thickness 2Δ. The problem can be easily generalized to include an inextensible boundary on the plane $z = 0$. The mixed boundary conditions associated with the reduced halfspace problem are as follows:

$$u_z(r,0) = \Delta, \qquad 0 \leq r \leq a \quad \text{(71)}$$

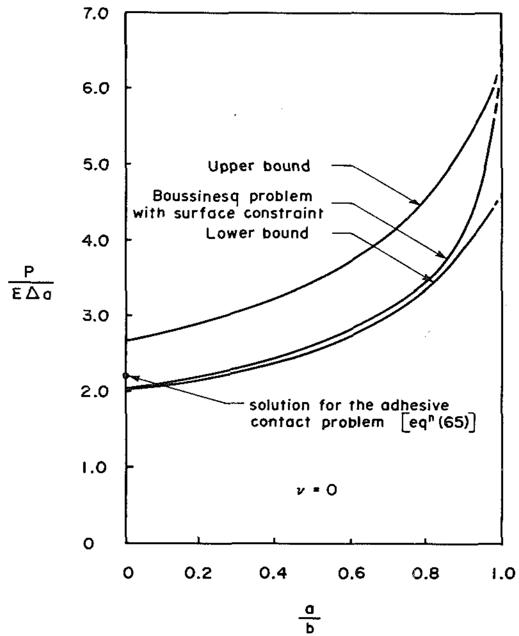

FIG. 5. Elastic Stiffness of Disc Anchor Embedded at Geological Interface

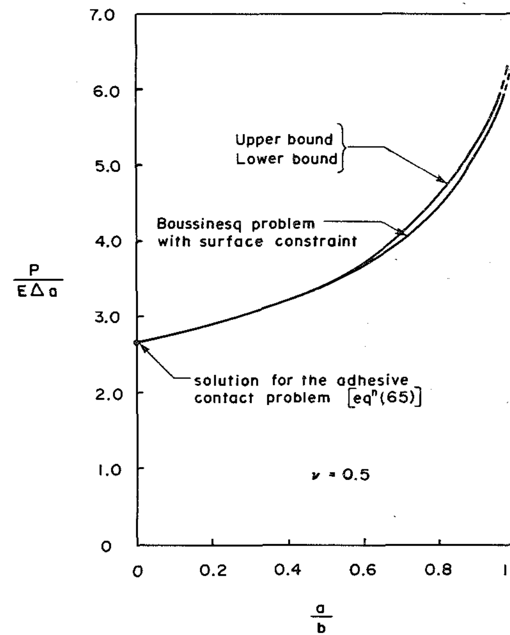

FIG. 6. Elastic Stiffness of Disc Anchor Embedded at Geological Interface

$$\sigma_{zz}(r,0) = 0, \qquad a < r < b \quad (72)$$

$$u_z(r,0) = 0, \qquad b \leq r < \infty \quad (73)$$

$$u_r(r,0) = 0, \qquad 0 \leq r < \infty \quad (74)$$

The details of the solution of this mixed boundary value problem are given in the references cited previously. The relationship between the displacement Δ and the total force exerted on the indenting inclusion can be expressed as a power series in terms of the radii ratio $a/b(= c) << 1$:

$$\frac{P}{E\Delta a} = \frac{8\,(1-\nu)}{(1+\nu)(3-4\nu)}\left[1 + \frac{4}{\pi^2}c + \frac{16}{\pi^4}c^2 + c^3\left(\frac{8}{9\pi^2} + \frac{64}{\pi^6}\right) + c^4\left(\frac{64}{9\pi^4} + \frac{256}{\pi^8}\right) + c^5\left(\frac{92}{225\pi^2} + \frac{128}{3\pi^6} + \frac{1{,}024}{\pi^{10}}\right) + c^6\left(\frac{7{,}104}{2{,}025\pi^4} + \frac{2{,}048}{9\pi^8} + \frac{4{,}096}{\pi^{12}}\right) + 0(c^7)\right] \quad (75)$$

Fig. 4 shows the mixed boundary conditions associated with the models that are used to estimate the elastic stiffness of the rigid disc anchor that is embedded at the boundary of the geological interface. The boundary conditions Eqs. 66–70 essentially deal with a Boussinesq-type punch problem in which displacement constraints are imposed at the bonded interface region $b \leq r \leq \infty$. As $b \rightarrow \infty$ (or $c \rightarrow 0$), the results derived from the mixed boundary value problem Eqs. 66–70 yield the exact solution for the indentation of a halfspace region by a rigid punch with a smooth interface. Fig. 5 and 6 show the manner in which the elastic stiffness of the anchor is influenced by the extent of the cracked region, the constraints imposed by the bounding scheme, and Poisson's ratio of the deformable geological medium. The maximum deviation between the two bounds occurs when $\nu = 0$. The results derived for the Boussinesq problem with the surface constraint indicates that as $(a/b) \rightarrow 1$, the smooth indentation effectively coincides with the upper bound estimate where inextensibility is imposed at the entire interface.

Conclusions

In summary, the paper examines a plausible model for the elastostatic interaction between a rigid circular anchoring region and a deformable geological medium. The mathematical analysis focuses on the evaluation of the elastic stiffness of the anchor when partial debonding occurs over a circular region. Perfect adhesive contact is maintained at the deformable geological medium-rigid boundary interface beyond the debonded region. The numerical results derived from the mathematical analysis take into account a variety of interface conditions at the contact between the anchor and the geological medium. The results of the elastostatic analysis can be considered as a necessary prelude to more advanced treatments of the anchor problem which can involve fracture and yielding of the geological medium and flexibility characteristics of the disc anchor.

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Appendix I. Associated Integral Expressions for $m_i(s_1)$ and $n_i(s)$

The functions $m_i(s_1)$ and $n_i(s_1)(i = 1, 2, \ldots, 5)$ take the forms

$$m_0(s_1) = -1 \quad (76)$$

$$m_1(s_1) = -\frac{2}{\pi}\int_1^\infty \frac{n_1(u_1)du_1}{u_1} \quad (77)$$

$$m_2(s_1) = -\frac{2}{\pi}\int_1^\infty \frac{n_2(u_1)du_1}{u_1} - \frac{2}{\pi}s_1^2\int_1^\infty \frac{n_0(u_1)du_1}{u_1^3} \quad (78)$$

$$m_3(s_1) = -\frac{2}{\pi}\int_1^\infty \frac{n_3(u_1)du_1}{u_1} - \frac{2}{\pi}s_1^2\int_1^\infty \frac{n_1(u_1)du_1}{u_1^3} \quad (79)$$

$$m_4(s_1) = -\frac{2}{\pi}s_1^2\int_1^\infty \frac{n_2(u_1)du_1}{u_1^3} - \frac{2}{\pi}s_1^4\int_1^\infty \frac{n_0(u_1)du_1}{u_1^5} - \frac{2}{\pi}\int_1^\infty \frac{n_4(u_1)du_1}{u_1} \quad (80)$$

$$m_5(s_1) = -\frac{2}{\pi}\int_1^\infty \frac{n_5(u_1)du_1}{u_1} - \frac{2}{\pi}s_1^2\int_1^\infty \frac{n_3(u_1)du_1}{u_1^3} - \frac{2}{\pi}s_1^4\int_1^\infty \frac{n_1(u_1)du_1}{u_1^5} \quad (81)$$

and

$$n_0(s_1) = 0 \quad (82)$$

$$n_1(s_1) = -\frac{2}{\pi s_1}\int_0^1 m_0(u_1)du_1 \quad (83)$$

$$n_2(s_1) = -\frac{2}{\pi s_1}\int_0^1 m_1(u_1)du_1 \quad (84)$$

$$n_3(s_1) = -\frac{2}{\pi s_1}\int_0^1 m_2(u_1)du_1 - \frac{2}{\pi s_1^3}\int_0^1 u_1^2 m_0(u_1)du_1 \quad (85)$$

$$n_4(s_1) = -\frac{2}{\pi s_1}\int_0^1 m_3(u_1)du_1 - \frac{2}{\pi s_1^3}\int_0^1 u_1^2 m_1(u_1)du_1 \quad (86)$$

$$n_5(s_1) = -\frac{2}{\pi s_1^5}\int_0^1 u_1^4 m_0(u_1)du_1 - \frac{2}{\pi s_1^3}\int_0^1 u_1^2 m_2(u_1)du_1 - \frac{2}{\pi s_1}\int_0^1 m_4(u_1)du_1 \quad (87)$$

respectively.

Appendix II. References

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