Int. J. mech. Sci., Vol. 18, pp. 463–468. Pergamon Press 1976. Printed in Great Britain

THE RESPONSE OF A RIGID CIRCULAR PLATE RESTING ON AN IDEALIZED ELASTIC–PLASTIC FOUNDATION

A. P. S. SELVADURAI
Department of Civil Engineering, Carleton University, Ottawa, Canada

(*Received* 16 *March* 1976, *and in revised form* 27 *July* 1976)

Summary—This paper investigates the load–deflection and contact stress distribution beneath an axisymmetrically loaded rigid circular plate resting on an idealized elastic–plastic soil medium. The plate is subjected to a monotonically increasing load. The theoretical contact stress distributions exhibit trends consistent with measurements encountered beneath actual structural foundations.

NOTATION

r, θ, z	cylindrical polar co-ordinates
w	surface displacement of modified foundation
w_0	displacement of rigid plate
w_1, w_2	displacements in Winkler and Pasternak foundations
c, k	spring constants for the Winkler media
G	rigidity constant for the shear layer
∇^2	Laplace's operator
$q, q_0, \bar{q}$	contact stressses
N	shear force in the shear layer
N_y	yield value of shear force
n	ratio of spring constants
a	radius of circular plate
Φ, λ, μ	substitution parameters
I_n, K_n	nth order modified Bessel functions of the first and second kind, respectively
P	external load on the rigid circular plate
$\bar{\gamma}$	coefficient with dimensions of length^{-1}
γ, ξ	non-dimensional parameters
u_r, u_z	displacement components
H	thickness of Vlazov layer
E_0, ν_0	modified elastic constants
E_s, ν_s	elastic constants
Ψ_k, Ψ_G	material constant of the Vlazov layer
τ_y	yield stress in simple shear

Subscripts

i	foundation region occupied by plate
e	foundation region exterior to plate
y	value at yield
el.	elastic
el. pl.	elastic–plastic

INTRODUCTION

THE ANALYSIS of structural foundations resting on soil media constitutes an important branch of geotechnical engineering. Since most naturally occurring soils exhibit complex stress–strain characteristics, it becomes both necessary and expedient to idealize their response to external loads. A natural first approximation would be to represent the soil medium as a linearly deformable elastic medium. The two extreme types of linear elastic response commonly used in soil–foundation interaction analyses range from the perfectly continuous elastic solid to the Winkler medium, which exhibits completely discontinuous behaviour in the surface deflection pattern. Other intermediate models of elastic behaviour include those of Reissner[1] and Vlazov and Leontiev[2] which impose deformational constraints on the elastic continuum model or the Pasternak[3] and Kerr[4] models which achieve a certain degree of continuity between the individual spring elements of the Winkler medium. For example, the Pasternak foundation is composed of a Winkler medium in which interaction between the individual spring elements is achieved by incorporating an elastic layer which deforms in shear only. As such, mathematical equivalence can be established between the constants characterizing the Winkler model, its generalized versions and the elastic constants of the continuum. Recently Gibson[5] has shown that the surface deformational response of an incompressible isotropic elastic halfspace whose shear modulus varies linearly with depth from zero at the surface, is identical to that of the Winkler medium. A comprehensive study of the application of the linear elastic soil models to the analysis of soil–foundation interaction is presented by Selvadurai[6].

In relation to the behaviour of highly compressible soils such as silts, soft clays and compressible loose sands, the representation of the soil as an idealized linear elastic medium has several limitations. By far the most prominent deficiency lies in the physical non-linear behaviour observed in the load–deflection characteristics of mainly rigid foundations resting on compressible media. Further, investigations relating to the measurement of contact stresses beneath model and prototype structural foundations resting on such compressible soils indicate that the observed contact stress distributions are different in character to those predicted by the linear elastic models.

For example, the contact stresses beneath the edges of a rigid foundation are considerably lower than those predicted by the elastic continuum model and unlike the response predicted by the Winkler model the contact stress distribution is generally non-uniform. A summary of these experimental investigations suggests that any mathematical representation of the mechanical behaviour of such highly compressible soil media should take into account the finite strength characteristics of the soil.

An extension to the purely elastic behaviour can thus be achieved by incorporating the effects of ideal plastic behaviour. Again, the two basic types of elastic–plastic soil response can be developed either by using an elastic–plastic continuum idealization or by incorporating plasticity effects into the mechanical models such as the Winkler or Pasternak types. The analytical treatment of the interaction problem which employs the elastic–plastic continuum idealization is generally quite complex. The more important problem relating to the flexural response of structural elements resting on elastic–plastic continua has received little attention. An alternative to this continuum approach is achieved by incorporating elastic–plastic effects in the purely mechanical models of soil behaviour. In this connection it should be mentioned that the classical one-dimensional models of elastic–plastic behaviour (see e.g. Prager,[7] Whiteman,[8] Wells and Paslay[9]) correspond to a generalization of the Winkler model to take into account elastic–plastic effects. Of particular interest here, is the model proposed by Rhines[10] which extends the Pasternak model to include elastic–plastic effects.

In this paper we consider the interaction problem related to a rigid circular plate resting on a modified Pasternak foundation. The shear interaction layer in this modified foundation is assumed to exhibit elastic–perfectly plastic effects. The axisymmetric external load is increased up to and beyond the point at which failure occurs in the shear interaction layer. The results for the monotonic loading case are presented in exact closed form and numerical results are given to illustrate the load–deflection pattern and the distribution of contact stresses at the interface.

The basic problem is of interest in connection with foundations of grain silos resting on soft marine clay deposits. It can also be used to examine the plate loading test commonly used to evaluate the *in situ* properties of the soil medium.

FORMULATION OF THE PROBLEM

The modified Pasternak type of idealized soil behaviour is phenomenologically represented by two layers of independent linearly deformable spring elements interconnected by an elastic layer which deforms in shear only. (Fig. 1.). The surface deflection $w(r)$ experienced by this model due to the action of a continuously distributed load of stress intensity $q(r)$ is given by

$$w(r) = w_1(r) + w_2(r) \tag{1}$$

where $w_1(r)$ is the deflection of the upper Winkler medium and $w_2(r)$ is the deflection in the Pasternak foundation. By considering the constitutive relations of the top Winkler layer and the equilibrium of the Pasternak layer it can be shown that

$$\begin{aligned} q(r) &= cw_1(r) \\ q(r) &= kw_2(r) - G\nabla^2 w_2(r) \end{aligned} \tag{2}$$

where c is the spring constant of the upper layer, k is the spring constant for the lower layer of springs, G is the constant for the shear layer and

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} \tag{3}$$

is Laplace's operator referred to the axisymmetric cylindrical polar coordinate system. Using the above equations it can be shown that the response of the complete foundation due to a continuously distributed load $q(r)$ is given by

$$\left(1 + \frac{k}{c}\right) q(r) - \frac{G}{c}\nabla^2 q(r) = kw(r) - G\nabla^2 w(r). \tag{4}$$

The above equation describes the deformational characteristics of the modified Pasternak foundation up to the point at which the plastic limit load is reached in the shear layer. In the elastic domain the constitutive relation for

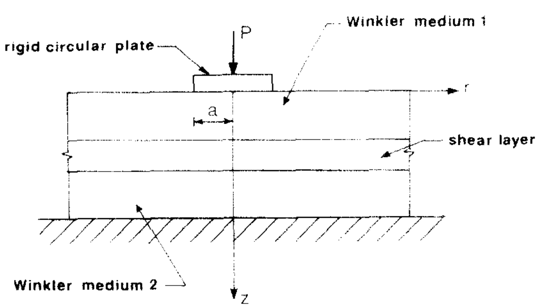

FIG. 1. The modified Pasternak foundation (spring constant for Winkler medium 1 = c; spring constant for Winkler medium 2 = k).

the shear layer is given by

$$N = G\frac{\mathrm{d}w_2}{\mathrm{d}r}. \tag{5}$$

It is postulated that when the shearing force at any point in the shear layer reaches its yield value N_y, slip may occur between the vertical elements of the shear layer thus producing a discontinuity in the shear layer deflection w_2. It is further noted that owing to the assumed elastic–perfectly plastic behaviour, the critical shear force N_y is maintained irrespective of the magnitude of the slip discontinuity. The critical shear strain at which yielding commences in the shear layer is given by

$$\left[\frac{\mathrm{d}w_2}{\mathrm{d}r}\right]_y = \frac{N_y}{G}. \tag{6}$$

From the point of view of stress analysis which incorporates the idealized elastic–plastic soil model thus described, the following points should be noted: (i) when the response of the idealized medium is elastic, the displacements of the interconnecting shear layer, $w_2(r)$, and its derivative are both continuous functions which satisfy the elastic constitutive equation given by the second equation of (2). (ii) When yield occurs at a point in the elastic–perfectly plastic shear layer, the displacement field $w_2(r)$ becomes discontinuous at that point whereas its derivative remains continuous to provide compatibility of shearing force in the layer at that location. Thus the displacements w_2 may change in the rest of the elastic regions but to satisfy the constancy of the shearing force implied by the yield condition (6) (i.e. N_y/G is a material constant) the derivative of the displacement w_2 at a yielded point should remain constant. For all intensive purposes the stress analysis of the yielded foundation can be performed by separating the foundation into separate regions about the location at which yield occurs; each of these regions would satisfy the elastic stress–strain relations together with the shear boundary conditions at the yield location. If no further external loads $q(r)$ are applied to the surface of the region separated by a yield location, then no *further* yield will occur in that zone. A similar situation occurs in the analysis of the title problem. Here, yield first occurs at the edge of the rigid circular plate and once yield takes place at this location the supporting medium beyond the rigid plate remains unaffected for further deformations of the plate.

AXISYMMETRIC LOADING OF THE RIGID CIRCULAR PLATE

Elastic behaviour

We first consider the problem of a rigid circular plate resting on a modified Pasternak foundation and subjected to a central force P. The equations governing the settlement of the plate in the elastic range are given by (1) and (4). Since the loading is axisymmetric the surface settlement under the rigid foundation $w(r) = w_0 =$ constant. For sufficiently small values of P the shear layer exhibits an elastic response and the relationship between the surface deflection w_0 and the contact stress $q(r)$ is given by (4), i.e.

$$\nabla^2 q - (1+n)\frac{k}{G}q = -\frac{nk^2}{G}w_0 \tag{7}$$

where $n = c/k$.

Since no boundary conditions can be prescribed at the plate boundary $r = a$, (7) cannot be integrated to generate $q(r)$ within the contact region. However, the solution of the problem can be achieved by considering the deformations of the shear layer inside (w_{2i}) and outside (w_{2e}), respectively. Since $w(r) = w_0$ and $w_{1i}(r) = q(r)/c$ for $r < a$, we can, by using (1) and (7) show that the differential equation governing w_{2i} is given by

$$\nabla^2 w_{2i} - \frac{k}{G}(1+n)w_{2i} = -n\frac{k}{G}w_0. \tag{8a}$$

Also, since $q(r) = 0$ for $r > a$, the deflections outside the contact region are governed by

$$\nabla^2 w_{2e} - \frac{k}{G}w_{2e} = 0. \tag{8b}$$

The four boundary conditions governing (8a) and (8b), in relation to the circular plate problem are

$$\text{(i)}\ w_{2i}(a) = w_{2e}(a) \qquad \text{(ii)}\ \left[\frac{\mathrm{d}w_{2i}}{\mathrm{d}r}\right]_{r=a} = \left[\frac{\mathrm{d}w_{2e}}{\mathrm{d}r}\right]_{r=a}$$
$$\text{(iii)}\ \left[\frac{\mathrm{d}w_{2i}}{\mathrm{d}r}\right]_{r=0} = 0 \qquad \text{(iv)}\ [w_{2e}]_{r\to\infty} = 0. \tag{9}$$

The complete solutions of (8a) and (8b) which satisfy these boundary conditions are given by

$$\begin{aligned} w_{2i} &= \frac{nw_0}{(1+n)}\{1 - \Phi I_0(\lambda r/a)\} \\ w_{2e} &= \frac{nw_0}{(1+n)}\left[\frac{\lambda I_1(\lambda)}{\mu K_1(\mu)}K_0(\mu r/a)\right] \end{aligned} \tag{10}$$

where I_n and K_n ($n = 0, 1$) are the nth order modified Bessel functions of the first and second kind, respectively, and

$$\lambda = \left[\frac{(1+n)ka^2}{G}\right]^{1/2}; \qquad \mu = \left[\frac{ka^2}{G}\right]^{1/2} \tag{11}$$
$$\Phi = \mu K_1(\mu)[\mu K_1(\mu)I_0(\lambda) + \lambda I_1(\lambda)K_0(\mu)]^{-1}.$$

The contact stresses beneath the rigid foundation are given by the second equation of (2); we have

$$q(r) = \frac{knw_0}{(1+n)}[1 + n\Phi I_0(\lambda r/a)]. \tag{12}$$

The load–deflection relationship for the rigid plate, valid for the elastic range, can now be obtained by considering the vertical equilibrium of the plate: i.e.

$$P = \int_0^a \int_0^{2\pi} q(r) r \,\mathrm{d}r\,\mathrm{d}\theta = \frac{\pi a^2 knw_0}{(1+n)\lambda}[\lambda + 2n\Phi I_1(\lambda)]. \tag{13}$$

Using (13) we can write (12) as

$$\frac{q(r)}{q_0} = \lambda\left[\frac{1 + n\Phi I_0(\lambda r/a)}{\lambda + 2n\Phi I_1(\lambda)}\right] \tag{14}$$

where q_0 $(= P/\pi a^2)$ is the average contact stress beneath the rigid plate. The preceding analysis completely describes the elastic response of the rigid circular plate resting on a modified Pasternak medium.

Elastic–plastic behaviour

In order to investigate the elastic–plastic response of the modified foundation, it is necessary to establish the location at which the strain in the shear layer is a maximum. Using (10) we note that for $0 < r < a$

$$w_{2i} = \frac{P\lambda}{k\pi a^2}\left[\frac{1 - \Phi I_0(\lambda r/a)}{\lambda + 2n\Phi I_1(\lambda)}\right]. \tag{15}$$

From (14) we note that the maximum value of the derivative $(\mathrm{d}w_{2i}/\mathrm{d}r)$ occurs at $r = a$; and from (5) we obtain the maximum shear force as

$$\max|N| = \frac{PG\lambda^2\Phi I_1(\lambda)}{k\pi a^3[\lambda + 2n\Phi I_1(\lambda)]}. \tag{16}$$

At yield conditions $\max|N| = N_y$ and the corresponding external load P_y is given by

$$P_y = \frac{N_y\pi a}{(1+n)}\left[2n + \frac{\lambda}{\Phi I_1(\lambda)}\right]. \tag{17a}$$

Using (13) and (17a), the deflection of the rigid plate at initiation of yield in the shear layer is given by

$$[w_0]_y = \frac{N_y\lambda}{nka\Phi I_1(\lambda)}. \tag{17b}$$

In post-yield behaviour, the deformations of the foundation are governed by the differential equations (8a) and (8b); however, the appropriate boundary conditions are

$$\text{(i)}\ \left[\frac{\mathrm{d}w_{2i}}{\mathrm{d}r}\right]_{r=a} = -\frac{N_y}{G}; \qquad \text{(ii)}\ \left[\frac{\mathrm{d}w_{2e}}{\mathrm{d}r}\right]_{r=a} = -\frac{N_y}{G}$$

$$\text{(iii)}\ \left[\frac{\mathrm{d}w_{2i}}{\mathrm{d}r}\right]_{r=0} = 0; \qquad \text{(iv)}\ [w_{2e}]_{r\to\infty} = 0. \tag{18}$$

The complete solutions of (8a) and (8b) corresponding to boundary conditions (18) can be shown to be

$$w_{2i} = \frac{nw_0}{(1+n)} - \frac{N_y a I_0(\lambda r/a)}{\lambda G I_1(\lambda)}$$

$$w_{2e} = \frac{N_y a K_0(\mu r/a)}{\mu G K_1(\mu)}. \tag{19}$$

Again, using the first of these equations in the second equation of (2) it can be shown that the contact stress distribution beneath the rigid foundation subsequent to yield in the shear layer at $r = a$ corresponds to

$$[q(r)]_y = \frac{n}{(1+n)}\left[kw_0 + \frac{\lambda N_y I_0(\lambda r/a)}{aI_1(\lambda)}\right]. \tag{20}$$

The force–deformation relationship for the rigid foundation at post-yield conditions can be obtained by considering its overall vertical equilibrium condition, i.e.,

$$P = \frac{\pi n a^2}{(1+n)}\left[kw_0 + \frac{2N_y}{a}\right]. \tag{21}$$

NUMERICAL RESULTS

The load–deflection relationship for the rigid foundation subjected to a monotonically increasing load is represented by equation (13) for the elastic range and by equation (21) for the elastic–plastic range. The non-dimensional form of the load–deflection curve of $P/k\pi a^3$ vs w_0/a is shown in Fig. 2. A distinct change in the slope of the load–deflection curve occurs as yield commences in the interconnecting shear layer. In order to illustrate the effects of yielding on the contact stress distribution we consider the following representation. The average contact stress beneath the circular plate at the initiation of yield in the shear layer q_y is given by

$$q_y = \frac{P_y}{\pi a^2} = \frac{N_y}{(1+n)a}\left[\frac{\lambda + 2n\Phi I_1(\lambda)}{\Phi I_1(\lambda)}\right]. \tag{22}$$

We can now represent the contact stress beneath the rigid plate at any load P as a multiple of q_y; i.e. $(P/\pi a^2) = \xi(P_y/\pi a^2)$. The contact stress distribution obtained for $\xi < 1$ corresponds to purely elastic behaviour of the foundation. Similarly the contact stresses obtained for $\xi > 1$ corresponds to the elastic–plastic behaviour of the foundation. Using (22), the elastic contact stress distribution (14) can be written as

$$\frac{q(r)_{el}}{\{N_y\lambda/a(1+n)\Phi I_1(\lambda)\}} = \xi\{1 + n\Phi I_0(\lambda r/a)\}. \tag{23}$$

Similarly the contact stresses beneath the rigid foundation, obtained for elastic–plastic behaviour (20) can be written as

$$\frac{q(r)_{el\,pl}}{\{N_y\lambda/a(1+n)\Phi I_1(\lambda)\}} = \xi\{1 + n\Phi I_0(\lambda r/a)\} - (\xi - 1)\Phi\frac{n}{\lambda}\{\lambda I_0(\lambda r/a) - 2I_1(\lambda)\}. \tag{24}$$

We note that when $\xi = 1$, (24) becomes identical to (23). The form of the equation for the contact stress distribution (24), suggests that the magnitude of the deviation of $q(r)_{el\,pl}$ from the purely elastic distribution (23) depends upon the sign of $\{\lambda I_0(\lambda r/a) - 2I_1(\lambda)\}$. If we consider the series representations for the modified

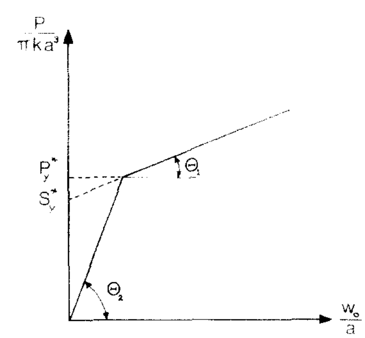

FIG. 2. Load–deflection relationship for the rigid circular plate.

$$\left[P_y^* = \frac{N_y}{ka^2(1+n)}\left\{2n + \frac{\lambda}{\Phi I_1(\lambda)}\right\};\right.$$

$$S_y^* = \frac{2N_y n}{ka^2(1+n)}; \qquad \theta_1 = \tan^{-1}\left(\frac{n}{1+n}\right);$$

$$\left.\theta_2 = \tan^{-1}\left[\frac{n}{(1+n)}\left\{1 + \frac{2n}{\lambda}\Phi I_1(\lambda)\right\}\right]\right].$$

Bessel functions

$$I_0(\lambda) = 1 + \frac{\lambda^2}{2^2} + \frac{\lambda^4}{2^2 \cdot 4^2} + \frac{\lambda^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

$$I_1(\lambda) = \frac{\lambda}{2} + \frac{\lambda^3}{2^2 \cdot 4} + \frac{\lambda^5}{2^2 \cdot 4^2 \cdot 6} + \frac{\lambda^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} + \cdots \tag{25}$$

it can be shown that

$$\{\lambda I_0(\lambda) - 2I_1(\lambda)\} > 0 \quad \text{for all } \lambda > 0.$$

The above consistency condition implies that when elastic–ideal plastic effects of the foundation are taken into consideration, the contact stress at the boundary of the rigid plate is lower than the corresponding elastic values. Accordingly the contact stress at the centre of the plate is increased. In order to obtain specific numerical results for the contact stress distributions (23) and (24) it becomes necessary to assign values to n, μ and λ. This may be achieved by making use of Vlazov and Leontiev's[2] interpretation of the Pasternak model in terms of the surface deflection of an elastic layer with the constrained displacement field

$$u_r = 0; \quad u_z = w_2(r)\frac{\sinh \gamma(H-z)}{\sinh \gamma H}. \tag{26}$$

Where u_r and u_z are the radial and axial components of the displacement vector, H is the thickness of the layer and γ is a coefficient with dimensions of $(\text{length})^{-1}$ determining the variation of u_z with depth. Using such an approach we can express k and G as

$$k = \frac{E_0}{H(1-\nu_0^2)}\Psi_k; \qquad G = \frac{E_0 H}{6(1+\nu_0)}\Psi_G. \tag{27}$$

The modified constants E_0 and ν_0 are related to the true elastic constants of the elastic layer E_s, ν_s by

$$E_0 = \frac{E_s}{(1-\nu_s^2)}; \quad \nu_0 = \frac{\nu_s}{(1-\nu_s)} \tag{28a}$$

and also,

$$\left.\begin{matrix}\Psi_k \\ \Psi_G\end{matrix}\right\} = \frac{\sinh \gamma H \cosh \gamma H \pm \gamma H}{\sinh^2 \gamma H}\left\{\begin{matrix}\dfrac{\gamma H}{2} \\[2mm] \dfrac{3}{2\gamma H}.\end{matrix}\right. \tag{28b}$$

By substituting the expressions for G and k given by (27) into the second equation of (11) we obtain

$$\mu = \frac{a}{H}\bar{\gamma}\left[\frac{2(1-\nu_s)}{(1-2\nu_s)}\left\{\frac{\sinh \bar{\gamma} \cosh \bar{\gamma} + \bar{\gamma}}{\sinh \bar{\gamma} \cosh \bar{\gamma} - \bar{\gamma}}\right\}\right]^{1/2}. \tag{29}$$

Where $\bar{\gamma} = \gamma H$ is a non-dimensional parameter. For the purpose of calculating the contact stress distributions beneath the rigid plate we adopt the following values for the variables encountered: $(a/H) = 1/5$; $\bar{\gamma} = 2$; $\nu_s = 0.15$; $n = 1$. The corresponding values for μ, λ and Φ are obtained from (11) and (29); we have $\mu \approx 0.71$, $\lambda = 1.0$, $\Phi = 0.615$. The variation of contact stresses beneath rigid plates resting on both elastic and elastic–plastic foundations are shown in Fig. 3. These contact stresses are presented in a non-dimensional form which makes use of

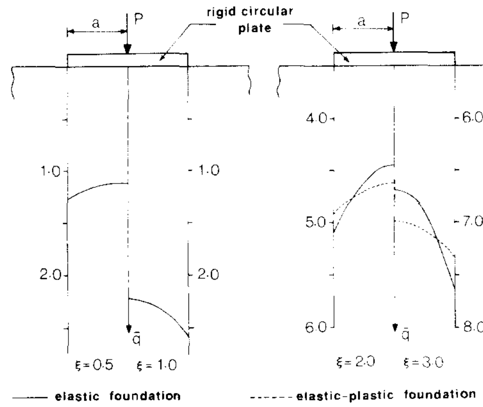

FIG. 3. The contact stress distributions $\bar{q}(r)$ beneath the rigid circular plate.

$$\left[\bar{q}(r) = \frac{q(r)}{\{N_y\lambda/a(1+n)\Phi I_1(\lambda)\}}; \quad \xi = \frac{P}{P_y}; \quad P_y = \frac{\pi a N_y}{(1+n)}\left\{2n + \frac{\lambda}{\Phi I_1(\lambda)}\right\}\right].$$

the yield value (N_y) of the interconnecting shear layer. This value of N_y may be identified to a first approximation, as the total shear force in the constrained elastic layer in which the limiting strength of the material in simple shear (τ_y) is reached at the surface. Considering the above criterion we obtain

$$N_y = \frac{H\tau_y}{\bar{\gamma}\sinh \bar{\gamma}}[\cosh \bar{\gamma} - 1]. \tag{30}$$

Results similar to (30) may be obtained by applying other limiting strength criteria to the behaviour of the elastic layer with constrained deformations.

CONCLUSIONS

This paper presents an analysis of a rigid circular plate resting on an idealized elastic–plastic Pasternak foundation. This foundation model is intended to describe the mechanical behaviour of highly compressible soil media such as soft clays, silts and loose granular media in which failure occurs as a result of punching shear rather than the general type of bearing capacity failure associated with the Prandtl–Hill[11] type of classical plasticity solution. The elastic–plastic behaviour of the idealized foundation is restricted to the interconnecting shear layer. It is found that yielding of the foundation initiates at the edges of the rigid circular plate. The load–deflection characteristics and contact stresses beneath the circular rigid plate are considerably altered by the inclusion of the elastic–plastic effects. These computed distributions represent characteristic trends observed in model and prototype foundation behaviour. A direct comparison with the experimental data, however, cannot be made owing to the lack of detailed experimental information.

Acknowledgements—The work described in this paper was supported in part by a Research Grant (No. A3866) awarded by the National Research Council of Canada.

REFERENCES

1. E. REISSNER, *J. appl. Mech.* **25**, 144 (1958).
2. V. Z. VLAZOV and N. N. LEONTIEV, *Beams, Plates and Shells on Elastic Foundations.* Israel Program for Scientific Translations, Jerusalem (1966).
3. P. L. PASTERNAK, *On a New Method of Analysis of an Elastic Foundation by Means of Two Foundation Constants* (in Russian) Gost. Izd. Lit. Stroit. Arkhi., Moscow (1954).
4. A. D. KERR, *Acta Mech.* **1**, 135 (1965).
5. R. E. GIBSON, *Geotechnique* **17**, 58 (1967).
6. A. P. S. SELVADURAI, *Elastic Analysis of Soil-Foundation Interaction.* Elsevier, Amsterdam (in press).
7. W. PRAGER, *Introduction to Plasticity.* Addison-Wesley, Reading, Mass. (1959).
8. I. R. WHITEMAN, *J. appl. Mech.* **26**, 95 (1959).
9. C. H. WELLS and P. R. PASLAY, *J. appl. Mech.* **36**, 15 (1969).
10. W. J. Rhines, *J. Soil Mech. Fdn. Eng. Div. Proc. A.S.C.E.* **SM3**, 819 (1969).
11. R. HILL, *The Mathematical Theory of Plasticity.* Oxford University Press, London (1950).