Composite Infinite Element for Modeling Unbounded Saturated-Soil Media

By A. P. S. Selvadurai[1] and Rajagopal Karpurapu[2]

Abstract: This paper discusses the development of a mapped-type composite-infinite element for modeling the response of unbounded saturated-soil media. The coordinate ascent mapping technique that uses conventional shape functions and the Gauss-Legendre integration scheme is used for the formulation. The constructed element preserves the compatibility between the variations in effective stress and pore pressure. This compatibility is a basic requirement for the accurate numerical modeling of saturated media. The accuracy of the element has been evaluated by comparing the numerical solutions of problems involving unbounded media with the corresponding analytical solutions. The effects of both the remoteness of the truncated boundary and the location of the infinite element coupling to the finite elements on the solution accuracy has been investigated. It is observed that accurate numerical responses of unbounded media can be obtained by using infinite elements at a much lesser computational effort than the conventional finite-element methods in which unbounded media are represented by large mesh configurations.

Introduction

In the modeling of many geomechanics problems involving soil-structure interaction, the soil medium is represented as a region of either infinite or semi-infinite extent. When these problems are formulated through analytical schemes, the modeling of the semi-infinite nature of the soil domain becomes a natural outcome of the solution procedure. For example, regularity conditions for the decay of stresses and displacements at infinity, etc., can be conveniently handled through the analytical formulations. When considering the numerical modeling of such problems using finite-element techniques, the traditional approach is to achieve the effect of unboundedness by incorporating a large number of elements extending significant distances beyond the range of the loaded region. However, the use of such large finite-element discretizations may result in an inordinate amount of computational effort. Also, the arbitrary location of a truncated boundary may lead to erroneous results. In practice, a compromise is often made between solution accuracy and computational effort. Nevertheless, the level of discretization and the location of the truncated boundary are often selected on a trial and error basis before an acceptable degree of accuracy is achieved.

Many researchers have, in the past, proposed numerical techniques for modeling the unbounded domains. Various boundary-element, coupled-boundary and finite-element methods can be found in the publications by Zienkiewicz et al. (1977), Brebbia and Walker (1980), and Banerjee and Butterfield (1981). More recently, infinite element methods that represent

[1]Prof. and Chmn., Dept. of Civ. Engrg., Carleton Univ., Ottawa, Ont., Canada K1S 5B6.

[2]Res. Assoc., Dept. of Civ. Engrg., Royal Military College of Canada, Kingston, Ont., Canada K7K 5L0.

Note. Discussion open until April 1, 1990. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on March 28, 1988. This paper is part of the ***Journal of Geotechnical Engineering***, Vol. 115, No. 11, November, 1989. ©ASCE, ISSN 0733-9410/89/0011-1633/$1.00 + $.15 per page. Paper No. 24081.

J. Geotech. Engrg., 1989, 115(11): 1633-1646

the unbounded nature of the domain have been proposed by Bettess (1977), Lynn and Hadid (1981), and Curnier (1983). These methods utilize the reciprocal method or the exponential decay terms to ensure the decay of the variables at large distances. Recently Rajapakse and Karasudhi (1985, 1986) have used similar techniques to examine the dynamic behavior of piles embedded in elastic media of infinite extent. All these elements, expressed in integral forms, are problem dependent and the indefinite integrals require specialized numerical-integration techniques. Consequently, it is tedious to implement these infinite elements in standard finite-element programs.

Recently, some of those problems have been overcome by the introduction of mapped infinite elements. There are basically two methods of formulating these elements. One method is the direct approach, or the displacement-descent method, in which the natural-coordinate domain is extended to infinity in the required direction while keeping the standard mapping functions well defined. The unknown variables are expressed in terms of descent-shape functions that decay asymptotically to zero towards infinity. Accounts of these are given by Bettess (1977) and Lynn and Hadid (1981). The second approach is the inverse method, or the coordinate-ascent method, in which the domain of the natural plane (e.g., $-1 \leq \xi \leq +1$) is maintained as usual. Examples in this category are given by Zienkiewicz et al. (1981, 1983), Marquez and Owen (1984), Kumar (1984), Selvadurai and Gopal (1986), Schrefler and Simoni (1987), and Simoni and Schrefler (1987). Ascent-mapping functions are employed for geometries that are singular at an extreme of the natural plane (e.g., $\xi = +1$), causing the physical coordinate to exhibit singular behavior when the natural coordinate approaches that extreme. The physical coordinate value at the extreme of the natural plane (e.g., $\xi = +1$) approaches infinity and thus the element represents an unbounded medium. Conventional-shape functions are employed for interpolating nodal variables. Of these two approaches, the latter is preferred since it uses the conventional Gauss-Legendre numerical integration for element formulations. This feature facilitates their implementation in general-purpose finite-element programs without major modifications.

By using the composite-type mapped infinite elements to model unbounded soil media, Selvadurai and Gopal (1986) obtained the consolidation response of the screw-plate test under various drainage and soil-disturbance conditions. The present paper examines the development of these composite-type infinite elements for applications in the numerical analyses of unbounded saturated-soil media.

Finite Element Equations of Consolidation

Following the standard variational principles discussed in detail by Sandhu and Wilson (1969)and Ghaboussi and Wilson (1972, 1973), the finite element equations of consolidation derived from the governing consolidation equations (Biot 1941) take the form:

$$\begin{bmatrix} \mathbf{K} & \mathbf{C} \\ \mathbf{C}^T & -(\mathbf{E} + \delta\Delta t\mathbf{H}) \end{bmatrix} \begin{pmatrix} \mathbf{u}_{t+\Delta t} \\ \mathbf{p}_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \mathbf{f}_{t+\Delta t} \\ \mathbf{g}_{t+\Delta t} \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

In Eq. 1, $\mathbf{u}$ and $\mathbf{p}$ = the nodal displacements and pore pressures. The remaining symbols are defined in the notation given at the end of the paper.

The finite-element solution of Eq. 1 is accomplished by employing com-

J. Geotech. Engrg., 1989, 115(11): 1633-1646

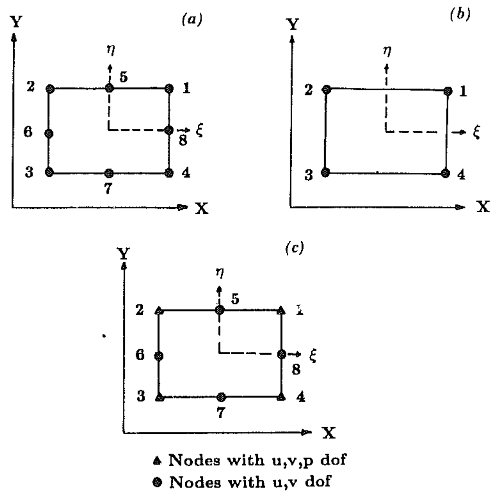

FIG. 1. Composite Quadrilateral Element: (*a*) Quadratic Element for Displacements; (*b*) Linear Element for Pore Pressures; (*c*) Superposed Element

posite-type elements that have both displacements and pore pressures as nodal variables. These elements are obtained by superposition of two elements, one of which has only displacements as nodal variables and the other of which has pore pressures as nodal variables. Fig. 1 shows a typical quadrilateral-composite element that is obtained by superposition of a quadratic element representing the displacements and a linear element representing the pore pressures. The superposed element ensures that both, effective stresses and pore pressures, follow the same degree of variation over the element. It is well known in geomechanics that the variation of these two stress components should be compatible for the element to accurately predict the response of saturated media, (e.g., Sandhu and Wilson 1969; and Christian 1977).

The composite elements are quite effective in providing the instantaneous solutions for incompressible fluids for which the diagonal terms in the lower right quadrant of the coefficient matrix in Eq. 1 are zero. A simple-solution technique is to assign the first few nodes in the mesh to midside nodes that do not have pore pressures as nodal variables. The underlying rationale is that by the time the Gaussian elimination process reaches the zero-diagonal terms, they have changed to nonzero terms during the elimination process.

Composite-Infinite Element

Mapped infinite elements can be constructed to represent various orders of decay rates for the field variables. The nature of the particular decay rate applicable for consolidation-type problems can be obtained from available analytical solutions. Solutions for the instantaneous ($t = 0$) and end of consolidation ($t \rightarrow \infty$) responses indicate that the vertical displacement at the surface of the half-space decays approximately in inverse proportion to the radial distance from load. Selvadurai and Gopal (1986) have reported that

J. Geotech. Engrg., 1989, 115(11): 1633-1646

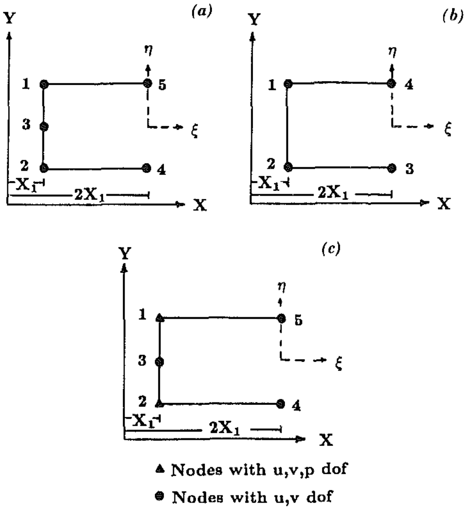

FIG. 2. Composite Infinite Element: (*a*) Five-Noded Serendipity Element for Displacements; (*b*) Two-Noded Superparametric Element for Pore Pressures; (*c*) Superposed Element

elements with a $1/r$-type decay rate yield the most accurate solutions for various consolidation problems. Thus, in the current investigation the same type of decay rate is employed to develop the composite infinite element.

Fig. 2 shows a five-noded composite infinite element that represents an infinite medium in the local-coordinate direction ξ. This element is obtained by superposing a five-noded serendipity infinite element representing quadratic variation for the displacement field and a two-noded superparametric infinite element representing linear variation for the pore-pressure field.

Mapping functions for the element nodes are constructed separately in the infinite direction ξ and the finite direction η, and the global functions are obtained as their product according to:

$$M_i = M_i(\xi)M_i(\eta) \quad \text{(2)}$$

The mapping function in the local coordinate direction ξ is constructed with the singularity at the $\xi = +1$ face that corresponds to an infinite distance. For example, the local-direction mapping functions for node 1 associated with the two-noded superparametric and five-noded serendipity elements can be written as follows:

Two-noded element

$$M_1(\xi) = \frac{-2\xi}{1 - \xi} \quad \text{(3)}$$

$$M_1(\eta) = \frac{(1 + \eta)}{2} \quad \text{(4)}$$

Five-noded element

TABLE 1. Mapping and Shape Functions for Composite-Mapped Infinite Element

Mapping functions (1)	Shape functions (2)
(a) Two-Noded Superparametric Element	
$M_1 = [-2\xi/(1-\xi)][1+\eta)/2]$ $M_2 = [-2\xi/(1-\xi)][(1-\eta)/2]$ $M_3 = [(1+\xi)/(1-\xi)][(1-\eta)/2]$ $M_4 = [(1+\xi)/(1-\xi)][(1+\eta)/2]$	$N_1 = (1/4)(1-\xi)(1+\eta)$ $N_2 = (1/4)(1-\xi)(1-\eta)$
(b) Five-Noded Serendipity Isoparametric Element	
$M_1 = [-2\xi)/(1-\xi)][\eta(1+\eta)/2]$	$N_1 = (1/4)(1-\xi)(1+\eta)$ $-(1/4)(1-\xi^2)(1+\eta)$ $-(1/4)(1-\xi)(1-\eta^2)$
$M_2 = [-2\xi/(1-\xi)][-\eta(1-\eta)/2]$	$N_2 = (1/4)(1-\xi)(1-\eta)$ $-(1/4)(1-\xi)(1-\eta^2)$ $-(1/4)(1-\xi^2)(1-\eta)$
$M_3 = [-2\xi/(1-\xi)](1-\eta^2)$	$N_3 = [(1-\xi)/2](1-\eta^2)$
$M_4 = [(1+\xi)/(1-\xi)][(1-\eta)/2]$	$N_4 = (1-\xi^2)[(1-\eta)/2]$
$M_5 = [(1+\xi)/(1-\xi)][(1+\eta)/2]$	$N_5 = (1-\xi^2)[(1+\eta)/2]$

$$M_1(\xi) = \frac{-2\xi}{1-\xi} \quad \text{(5)}$$

$$M_1(\eta) = \frac{\eta(1+\eta)}{2} \quad \text{(6)}$$

In a similar manner, the mapping functions for other nodes can be obtained as products of two local-direction mapping functions.

The interpolation for the field variables of infinite elements is carried out in terms of nodal variables at the finitely-located nodes, assuming that the variables reduce to zero at an infinite distance. By invoking this assumption, the five- and two-noded infinite elements can be considered as analogs of eight- and four-noded quadrilateral finite elements whose field variables on the nodes at the $\xi = +1$ face are zero. The shape functions of infinite element nodes are then obtained directly from the corresponding nodes of the finite elements.

The pore-pressure shape functions for the two-noded superparametric element are obtained directly from those of the four-noded linear quadrilateral element. Nodes 3 and 4 of this element are used only for mapping and are not used for interpolation. Displacement shape functions for the five-noded serendipity element are obtained from those of the eight-noded quadratic element. Table 1 shows the mapping and shape functions for both the elements.

Both these elements satisfy the requirements for completeness and monotonic convergence and therefore the mapping is independent of the choice of the coordinate system. Selvadurai and Gopal (1986) have shown that a lower order of integration for the infinite elements than that of the finite elements in assemblage yields better results. Thus in the numerical analysis that follows, integration orders of 2×2 and 3×3 have been used respec-

J. Geotech. Engrg., 1989, 115(11): 1633-1646

tively for the infinite and finite elements to formulate various element matrices.

Once the mapping and shape functions are determined for the element, various matrices in Eq. 1 can be formulated using the following equations:

$$\mathbf{K} = \int_V \mathbf{B}_u^T \mathbf{D} \mathbf{B}_u dv \quad (7)$$

$$\mathbf{E} = \int_V \frac{1}{M} \mathbf{N}_p^T \mathbf{N}_p dv \quad (8)$$

$$\mathbf{H} = \int_V \mathbf{B}_p^T \kappa \mathbf{B}_p dv \quad (9)$$

$$\mathbf{C} = \int_V \alpha (\mathbf{B}_u^T \mathbf{1}^T) \mathbf{B}_p dv \quad (10)$$

$$\mathbf{1} = (1\ 1\ 1\ 0) \quad (11)$$

Again, the symbols in these equations are defined in Appendix II. The integrations in the equations are carried out over the finite natural field using the familiar Gauss-Legendre numerical-integration scheme. This procedure is similar to the one followed for the formulation of standard finite-element stiffness matrices. Because of the similarity of formulations, the infinite elements can be implemented in the existing finite-element programs with relatively minor modifications. The following points should be noted in the implementation of the infinite elements in computer programs:

1. The Jacobian is formulated on the basis of the singular mapping functions defined in Table 1.
2. The interpolation is carried out on the basis of the variables at two or five nodes as indicated in Fig. 2.
3. The mapping functions shown in Table 1 refer to the particular geometry of infinite elements for which the nodal coordinates of outer nodes [nodes 5 and 4 in Fig. 2(*c*)] are twice those of the corresponding inner nodes [nodes 1 and 2 in Fig. 2(*c*)].

Numerical Analysis

The purposes of the numerical investigation are twofold. The first is to verify the accuracy of the infinite element developed earlier by comparing the numerical solutions with the well-established analytical solutions. The second aspect of the investigation is to address the often-raised question as to how large a mesh has to be in order to model an unbounded domain. Rather, with the development of infinite element, it becomes necessary to ascertain the location of such elements from the loaded regions to achieve the best solution accuracy. To answer these questions, numerical solutions have been obtained by various conventional and coupled finite-element meshes. While the conventional meshes consist of only regular finite elements, the coupled meshes consist of both the finite and infinite elements. In the con-

J. Geotech. Engrg., 1989, 115(11): 1633-1646

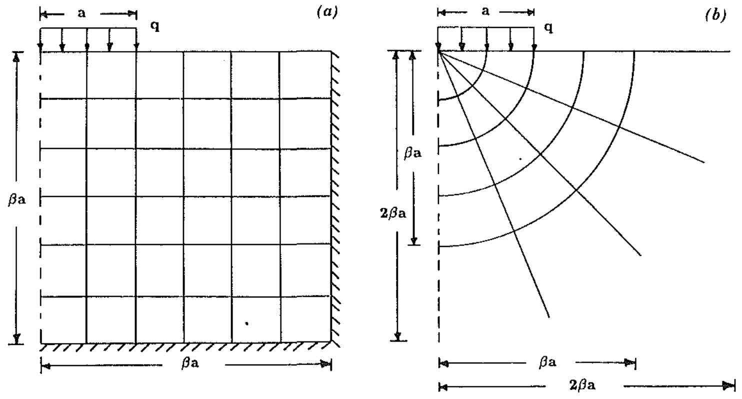

FIG. 3. Typical Meshes for Numerical Analysis: (*a*) Conventional; (*b*) Coupled

ventional meshes, the distance to the truncated boundary is varied, whereas in the coupled meshes, the distance to the coupling location is varied. The problems considered for numerical analysis are the consolidation response of flexible and rigid footings resting on an elastic, semi-infinite saturated-soil media.

In each analysis, a remoteness factor β has been defined in terms of the distance to the boundary (extreme or coupling) and the radius or the half-width of the footing. In both methods of analysis, the number of elements and consequently the number of nodal points have been increased for higher values of β. Aspect ratios of individual elements for meshes in both methods are made approximately equal so that they would have the same errors due to mesh discretization.

Typical finite-element meshes are shown in Fig. 3. For the case of rigid footing, the meshes have been finely discretized at the edge of the footing. All nodes on the truncated boundary of conventional meshes are fixed. In the coupled method, there is no truncated boundary and thus the nodes corresponding to infinite elements are left free. The cases that have been analyzed are summarized in Tables 2 and 3. Here, NUMEL refers to the number

TABLE 2. Analyses Cases for Flexible Footings

Remoteness factor, β (1)	NUMEL/NDOF (2)
(*a*) Conventional Method	
5	49/346
10	64/449
20	100/701
40	144/1,057
(*b*) Coupled Methods	
2	20/135
3	28/197
5	36/259

J. Geotech. Engrg., 1989, 115(11): 1633-1646

TABLE 3. Analyses Cases for Rigid Footings

Remoteness factor, β (1)	NUMEL/NDOF (2)
(*a*) Conventional Method	
5	64/449
10	100/701
20	144/1,057
40	169/1,317
(*b*) Coupled Methods	
2	28/197
3	36/259
5	44/321

of elements in the mesh and NDOF refers to the number of equations.

The accuracy of the conventional and coupled numerical solutions can be easily determined by comparing the ultimate settlement (at the end of consolidation) with the corresponding classical elasticity solutions reported in many standard texts [e.g., Timoshenko and Goodier (1970)]. Figs. 4 and 5 show the variation of percentage error in the surface settlement at the center of a circular footing for the flexible and rigid cases. The Poisson's ratio of soil considered for the two cases is 0.2. It is quite apparent that coupled solutions are far more accurate than those obtained from the conventional methods. Significant errors in the conventional finite-element analyses were observed for remoteness factors less than (say) 10 for the flexible cases and about 20 for the rigid-footing cases. Table 3 indicates that the coupled method with $\beta = 2$ uses 197-NDOF, which is a small fraction 1,057-NDOF used in the conventional analysis with $\beta = 20$. Since the computational effort is approximately proportional to NDOF, the coupled method is about five times more efficient than the conventional method. This efficiency is accompanied by increased accuracy.

Figs. 6–9 show the comparison between the analytical and corresponding

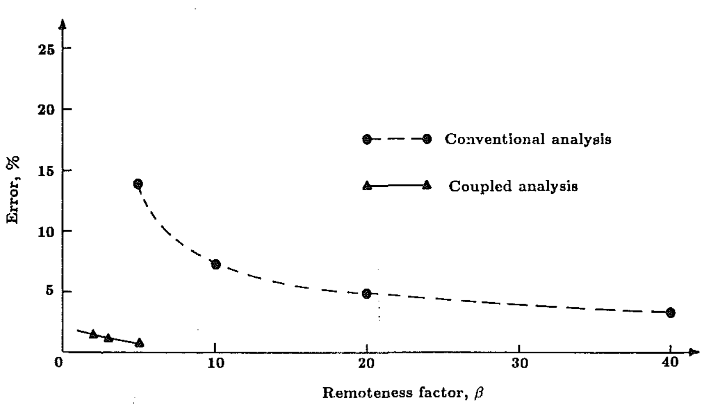

FIG. 4. Variation of Error in Central Settlement of Flexible Circular Footing

J. Geotech. Engrg., 1989, 115(11): 1633-1646

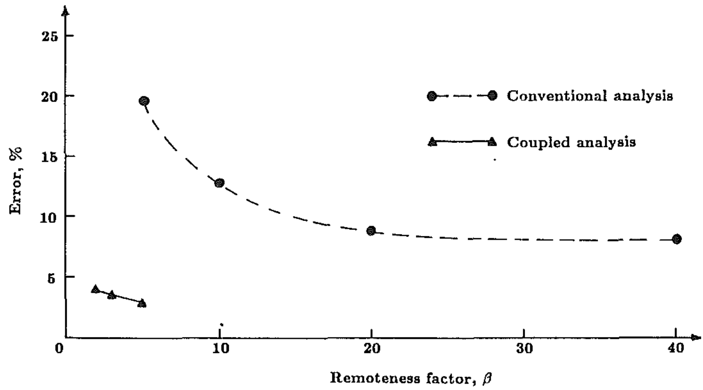

FIG. 5. Variation of Error in Central Settlement of Rigid Circular Footing

numerical prediction of the normalized settlement at the center of a flexible strip and circular footing resting on the surface of a semi-infinite medium. Analytical solutions for these cases have been reported by McNamee and Gibson (1960a, 1960b). The values of β reported are the minimum β values for which good numerical predictions could be obtained. It should be pointed out that although the results of the conventional methods appear to agree well with the analytical results, the predicted settlements still had an error of about 10%. In contrast, the solutions by the coupled methods have an error of approximately 2%. In the case of a drained surface (Figs. 6 and 7), there is a large error at small time factors but this error reduces to less than 5% beyond a time factor of 0.10. This large error at the initial stages could be attributed to the sudden change of drainage conditions at the surface. The instantaneous solution (W_o) was obtained with completely undrained conditions at the surface, which were later changed to drained conditions for the subsequent analysis. The effect of this sudden change of boundary con-

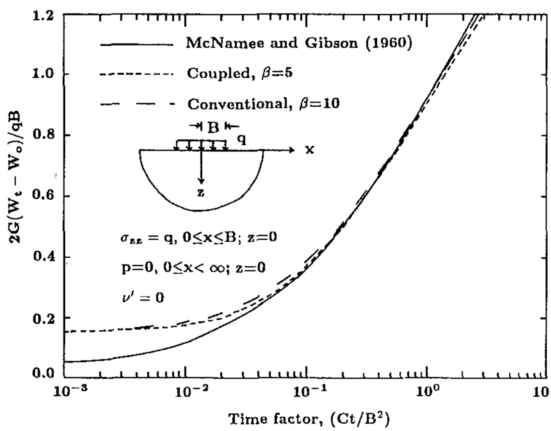

FIG. 6. Consolidation of Center of Flexible Strip Footing (p = 0, z = 0)

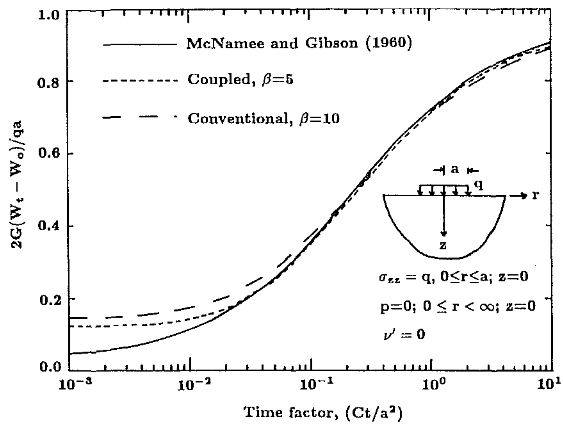

FIG. 7. Consolidation of Center of Flexible Circular Footing (p = 0, z = 0)

ditions is drastic at the beginning and decays as the consolidation progresses. This explanation seems to be justified in view of the relatively small errors at all the time factors for the completely impermeable surface case as shown in Figs. 8 and 9. This case does not have a sudden transformation of boundary conditions that influenced the numerical results in the earlier cases.

Fig. 10 shows the comparison between the analytical and corresponding numerical predictions of the consolidation behavior of a rigid circular footing resting on the surface of a saturated semi-infinite medium. The analytical solutions for this case have been reported by Chiarella and Booker (1975). The rigid behavior of the footing was simulated by constraining the vertical displacements under the footing to be equal; the algorithm proposed by Abel and Shephard (1979) was used for this purpose. The instantaneous solution was obtained with completely impermeable drainage conditions at the sur-

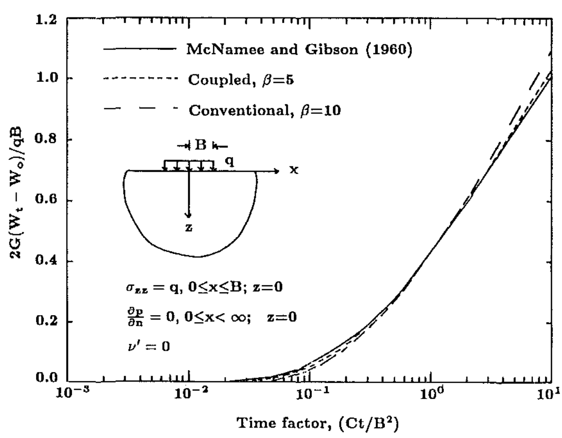

FIG. 8. Consolidation of Center of Flexible Strip Footing ($\partial p/\partial n$ = 0, z = 0)

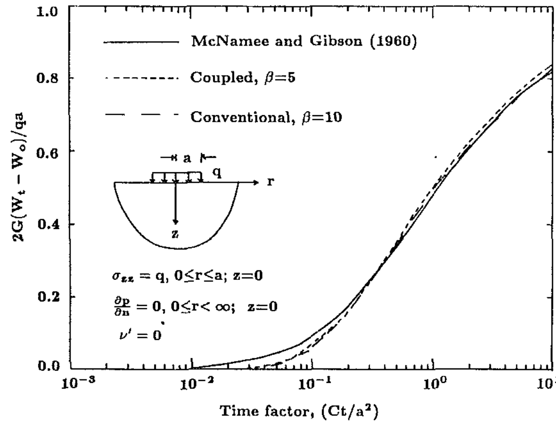

FIG. 9. Consolidation of Center of Flexible Circular Footing ($\partial p/\partial n = 0$, $z = 0$)

face that were later changed to freely draining conditions for subsequent analysis. At small time factors, both numerical methods predicted a faster rate of consolidation when compared with the analytical response. The reason could again be attributed to the sudden change of drainage conditions at the surface. The conventional methods seem to predict a slow rate of consolidation as indicated by the flattening of the response curve. On the other hand, the rate of settlement predicted by the coupled solutions seems to agree well with the analytical values after a time factor of about 0.1.

From the results of Figs. 4–10, it is evident that the coupled methods can predict reasonably good responses for β values of five or more for the cases considered. The conventional methods need β values of more than 10 and 20 for flexible and rigid cases, respectively, to obtain the same degree of accuracy as the coupled methods. Moreover, the coupled solutions have been

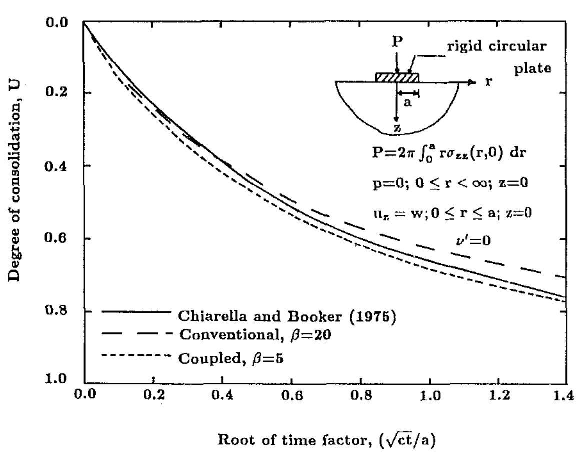

FIG. 10. Consolidation of Center of Rigid Circular Footing

J. Geotech. Engrg., 1989, 115(11): 1633-1646

obtained at much-lesser computational effort than the conventional solutions. From the results of these studies, it is clearly evident that an improper location of the truncated boundary in the conventional methods could lead to significant errors.

Conclusions

The concept of unbounded soil regions is employed quite extensively in the treatment of problems in geotechnical engineering. Although analytical approaches yield the most accurate solutions for problems with such domains, it is difficult to obtain closed-form solutions for problems with variable soil properties and complex boundary conditions. The infinite element developed in the present paper provides a convenient numerical alternative for modeling such problems. The accuracy of the element has been verified by comparing numerical solutions with various bench-mark analytical results. By using mesh configurations of different sizes, it was observed that both computational economy and solution accuracy can be improved by the incorporation of infinite elements.

A reduced integration scheme is recommended for the infinite elements, compared to that used for the finite elements in the mesh. The conventional Gauss-Legendre integration scheme can be used for the formulation of these elements. The location of the infinite element coupling to finite elements should be chosen carefully based upon the required accuracy. The results of the problems treated in this paper can be used as guidelines for such selections. The present paper has considered only linear-elastic behavior for the soil media. For the nonlinear problems, the location of infinite elements should take into consideration the extent of anticipated zones of nonlinearity. Because of the iterative nature of nonlinear problems, infinite elements could be expected to play an important role in improving the computational efficiency of finite element schemes for unbounded domains. Although the paper discusses only the problems in which the nodal unknowns include pore pressures, the technique is equally applicable for the pure displacement formulations also.

Acknowledgment

The work described in this paper was supported by a Natural Sciences and Engineering Council of Canada Grant A3866 awarded to A. P. S. Selvadurai. The writers are also grateful to Professor B. A. Schrefler of University of Padua, Italy, for drawing attention to recent research contributions in this area. The writers would also like to thank the referees for various suggestions that led to considerable improvement in the presentation of the material.

Appendix I. References

Abel, J. F., and Shephard, M. S. (1979). "An algorithm for multipoint constraints in finite element analyses." *Int. J. Num. Methods Eng.*, 11, 464–467.

Banerjee, P. K., and Butterfield, R. (1981). *Boundary element methods in engineering science*. McGraw-Hill Book Co., New York, N.Y.

Bettess, P. (1977). "Infinite elements." *Int. J. Numer. Methods. Eng.*, 11, 53–64.

Biot, M. A. (1941). "General theory of three-dimensional consolidation." *J. Appl. Phys.*, 12, 155–164.
Brebbia, C., and Walker, S. (1980). *Boundary element techniques in engineering*, Newnes Butterworths, London, England.
Chiarella, C., and Booker, J. R. (1975). "The time settlement behaviour of a rigid die resting on a deep clay layer." *Q. J. Mech. Appl. Math.*, 28, 317–328.
Christian, J. T. (1977). "Two and three dimensional consolidation." *Numerical Methods in Geotechnical Engineering*, C. S. Desai and J. T. Christian, eds., McGraw-Hill Book Company, New York, N.Y., 399–426.
Curnier, A. (1983). "A static infinite element." *Int. J. Numer. Methods. Eng.*, 19, 1479–1488.
Ghaboussi, J., and Wilson, E. L. (1972). "Variational formulation of dynamics of fluid-saturated porous elastic solids." *J. Engrg. Mech. Div.*, ASCE, 98, 947–963.
Ghaboussi, J., and Wilson, E. L. (1973). "Flow of compressible fluids in porous elastic media." *Int. J. Numer. Methods. Eng.*, 5, 419–442.
Kumar, P. (1984). "Static infinite element formulation." *J. Struct. Engrg.*, ASCE, 111, 2355–2372.
Lynn, P. P., and Hadid, H. A. (1981). "Infinite elements with $1/r^n$ type decay." *Int. J. Numer. Methods. Eng.*, 17, 347–355.
Marquez, J. M. C., and Owen, D. R. J. (1984). "Infinite elements in quasi-static materially non-linear problems." *Comput. Struct.*, 18, 739–751.
McNamee, J., and Gibson, R. E. (1960a). "Displacement functions and linear transforms applied to diffusion through porous elastic media." *Q. J. Mech. Appl. Math.*, 13, 98–111.
McNamee, J., and Gibson, R. E. (1960b). "Plane strain and axisymmetric problems of the consolidation of a semi-infinite clay stratum." *Q. J. Mech. Appl. Math.*, 13, 210–227.
Rajapakse, R. K. N. D., and Karasudhi, P. (1985). "Elastostatic infinite elements for layered halfspaces." *J. Engrg. Mech.*, ASCE, 111, 1144–1158.
Rajapakse, R. K. N. D., and Karasudhi, P. (1986). "An efficient elastodynamic infinite element." *Int. J. Solids Struct.*, 22, 643–657.
Sandhu, R. S., and Wilson, E. L. (1969). "Finite element analysis of seepage in elastic media." *J. Engrg. Mech. Div.*, ASCE, 95(3), 641–652.
Schrefler, B. A., and Simoni, L. (1987). "Non-isothermal consolidation of unbounded porous media using mapped infinite elements." *Comm. Appl. Numer. Meth.*, 3, 445–452.
Selvadurai, A. P. S., and Gopal, K. R. (1986). "Consolidation analysis of the screw plate test." *Proc. 39th Canadian Geotech. Conference*, Ottawa, Canada, 167–178.
Simoni, L., and Schreffler, B. A. (1987). "Mapped infinite elements in soil consolidation." *Int. J. Numer. Methods Eng.*, 24, 513–527.
Timoshenko, S., and Goodier, J. N. (1970). *Theory of elasticity*. McGraw-Hill Book Co., New York, N.Y.
Zienkiewicz, O. C., Kelly, D. W., and Bettess, P. (1977). "The coupling of the finite element method and boundary solution procedures." *Int. J. Numer. Methods Eng.*, 11, 355–375.
Zienkiewicz, O. C., et al. (1981). "Numerical methods for unbounded field problems and a new infinite element formulation." *ASME Winter Annual Meeting, The Appl. Mech. Div.*, Washington, D.C., 115–148.
Zienkiewicz, O. C., Emson, C., and Bettess, P. (1983). "A novel boundary infinite element." *Int. J. Numer. Methods. Eng.*, 19, 393–404.

Appendix II. Notation

The following symbols are used in this paper:

a = radius of footing
B = half-width of footing;
$\mathbf{B}_p$ = matrix relating pore fluid gradients and pore pressures;

J. Geotech. Engrg., 1989, 115(11): 1633-1646

$\mathbf{B}_u$	=	matrix relating strains and displacements;
C	=	coefficient of consolidation = $2G_k/\gamma_w$;
$\mathbf{C}$	=	interaction stiffness between soil and pore fluid;
$\mathbf{D}$	=	constitutive matrix of solid;
$\mathbf{E}$	=	compressibility matrix of pore fluid;
$\mathbf{F}_{t+\Delta t}$	=	external traction and body force vector;
G	=	shear modulus of soil;
$\mathbf{g}_{t+\Delta t}$	=	dissipation force vector;
$\mathbf{H}$	=	permeability matrix governing dissipation of pore fluid;
$\mathbf{K}$	=	stiffness matrix of soil;
M	=	bulk modulus of pore fluid;
N_u, N_p	=	displacement and pore pressure shape functions;
$\mathbf{p}_{t+\Delta t}$	=	nodal pore pressure vector;
q	=	intensity of footing pressure;
U	=	degree of consolidation = $W_t - W_o/W_\infty - W_o$;
$\mathbf{u}_{t+\Delta t}$	=	nodal displacement vector;
W_o, W_t, W_∞	=	settlements at center of footing at instant of loading;
	=	at time t and at end of consolidation;
α	=	porosity of soil;
γ_w	=	unit weight of pore fluid;
Δt	=	time step increment;
δ	=	convolution integration factor; and
κ	=	permeability coefficient.

J. Geotech. Engrg., 1989, 115(11): 1633-1646