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Torsion of foundations embedded in a non-homogeneous soil with a weathered crust

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The torsional response of rigid foundations embedded in a non-homogeneous elastic soil with a weathered crust is investigated. The shear modulus of the weathered crust is assumed to decrease linearly with depth and that of the underlying non-homogeneous soil increases linearly with depth. The problems related to both rigid circular and cylindrical foundations are examined by using an integral equation method based on displacement and traction Green's functions of the weathered non-homogeneous medium. Numerical solutions for torsional stiffness of rigid circular foundations are presented for different depths of embedment, gradients of shear modulus profile and depths of weathered crust. A closed form solution based on a simplified stress field in the soil is also developed to evaluate the torsional stiffness of a rigid cylindrical foundation. The torsional stiffness derived by using the approximate solution is compared with the rigorous solution based on exact Green's functions.

KEYWORDS: elasticity; footings/foundations; shear modulus; soil–structure interaction; stiffness; torsion.

L'article examine la résistance à la torsion de fondations rigides encastrées dans un sol élastique non-homogène avec une croûte altérée. On admet que le module de cisaillement de la croûte altérée décroît linéairement avec la profondeur et que celui du sol sous-jacent non-homogène augmente avec la profondeur. On étudie les problèmes associés aux fondations rigides circulaires et cylindriques en employant une méthode d'équations intégrale basée sur des fonctions de Green du déplacement et de la traction de la matière altérée non-homogène. Des solutions numériques pour la rigidité torsionnelle des fondations rigides circulaires sont présentées pour différentes profondeurs d'encastrement, différents gradients de profil de module de cisaillement et différentes épaisseurs de croûte altérée. Une solution basée sur un champ de contrainte simplifié dans le sol est développée afin d'évaleur la rigidité torsionnelle d'une fondation rigide cylindrique. La rigidité torsionnelle estimée par l'emploi de la solution approximative est comparée avec la solution rigoureuse basée sur des fonctions précises de Green.

NOTATION

a	radius of the embedded circular/cylindrical foundation
$G(r, z; s, z')$	displacement Green's function
h	depth of the weathered crust
H	depth of embedment
H_1	height of the foundation
I_n	modified Bessel function of the first kind and order n
J_n	Bessel function of the first kind and order n
K_n	modified Bessel function of the second kind and order
m_1, m_2	gradients of shear modulus profile
$\bar{N}, N_1, N_2$	number of node points used to discretize $\bar{S}$, S respectively
s	radius of ring load
s_1, s_2	boundary surfaces
$\bar{S}, S_1^*, S_2^*$	fictitious surfaces used in boundary integral equations
$T(r, z)$	traction in θ-direction on the contact surface
T_0	torque acting on the foundation
T_a	solution of torque based on approximate analysis
v	circumferential displacement
μ	shear modulus
μ_0	shear modulus at $z = 0$
$\psi(r, z; s, z')$	traction Green's functions
$\sigma_{\theta z}, \sigma_{\theta r}$	shear stresses
ξ	Hankel transform parameter

INTRODUCTION

Solutions based on the classical theory of elasticity have been used quite extensively in the analysis of a variety of problems related to soil–structure interaction (e.g. Harr, 1966; Poulos & Davis, 1974; Desai & Christian, 1977; Scott, 1978; Selvadurai, 1979). In many situations that involve geotechnical engineering practice,

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however, it becomes necessary to modify the assumptions of isotropy and homogeneity which are inherent in the classical theory of elasticity. Investigations by Bjerrum (1954), Skempton & Henkel (1957), Ward *et al.* (1959, 1965) and others have established the fact that effects of deposition, overburden, desiccation and so on, can lead to geological media which exhibit both anisotropic and non-homogeneous deformability characteristics. These observations have also been supplemented by the experimental investigations of Symons & Murray (1971), Simon, Christian & Ladd (1974), Graham & Houlsby (1983), Wroth, Randolph, Houlsby & Fahey (1984) and others.

Owing to the potential importance of material non-homogeneity to the study of both static and dynamic problems in soil–structure interaction, several researchers have focused attention on the extension of many classical solutions to include effects of elastic non-homogeneities. Investigations by Klein (1956), Korenev (1957), Mossakovskii (1958), Popov (1959), Rakov & Rvachev (1961), Rostovtsev (1964) and others examined various classes of boundary value problems in which half-space regions exhibit exponential and power law variations in the shear modulus and Poisson's ratio is invariably kept constant. Detailed accounts of the early works related to non-homogeneous elastic media are given by Olszak (1959) and Golecki & Knops (1969) and further advances in this area are documented by Gladwell (1980) and Selvadurai, Singh & Vrbik (1986).

In applications related to geoetechnical engineering, a seminal paper on the problem concerning a non-homogeneous elastic medium was presented by Gibson (1967). This study particularly focused on the stress analysis of an isotropic incompressible elastic half-space in which the shear modulus varies with depth in a linear fashion. The characteristic discontinuous displacement profile observed at the surface of a normal loaded half-space region provided a valuable basis for establishing the relevance of simplified models such as the Winkler model to geotechnical engineering. Gibson and co-workers (e.g. Gibson, 1974; Gibson & Sills, 1969, 1975; Gibson, Brown & Andrews, 1971; Gibson & Kalsi, 1974; Awojobi & Gibson, 1973) have examined a variety of problems related to both compressible and incompressible media in which the shear modulus exhibits a linear variation with depth. The linearly varying inhomogeneity has also been re-examined by Calladine & Greenwood (1978).

An examination of the relevant literature in geotechnical engineering indicates that most problems which deal with the non-homogeneity in the elasticity properties focus on situations where the half-space region is subjected to loads which are applied at its surface. The class of problems which deals with situations where the loads are transmitted to the interior of the non-homogeneous region is of particular interest to geotechnical engineering. Deeply embedded foundations, piles, ground anchors, in situ testing devices and so on primarily deal with loads that are imparted at the interior of the soil region.

In practice one often encounters non-homogeneous soil profiles whose upper part has been exposed to desiccation and weathering. The stiffness and thickness of the weathered crust depend on a variety of factors such as geologic history, climatic conditions, soil permeability and vegetation. Experimental studies related to properties and behaviour of soils with weathered crusts have been reported by several investigators (Bjerrum 1954, 1973; Simon, Christian & Ladd 1974; Dascal & Tournier 1975; Lo and Becker 1979; Graham 1979; Ng and Lo 1985; Graham & Shields 1985). These studies indicate that a bilinear variation of shear modulus with depth as depicted in Fig. 1 represents a realistic idealization for many soils actually encountered. A review of literature indicates, however, that the influence of weathering on the response of embedded foundations has not been investigated previously.

In this Paper, the Authors consider the torsional interaction between a rigid circular or cylindrical foundation embedded in a weathered non-homogeneous elastic soil which exhibits a bilinear variation of shear modulus with depth. Structural foundations are subjected to torsional loads during wind storms and earthquakes. In the case of machine foundations unbalanced masses of reciprocating engines result in torsional loads. Another common example is the pier-type foundations of power transmission towers which can be subjected to significant torsional loads as a result of non-uniform tension in the transmission cables, created by wind or ice action. The assessment of the torsional stiffness of embedded struc-

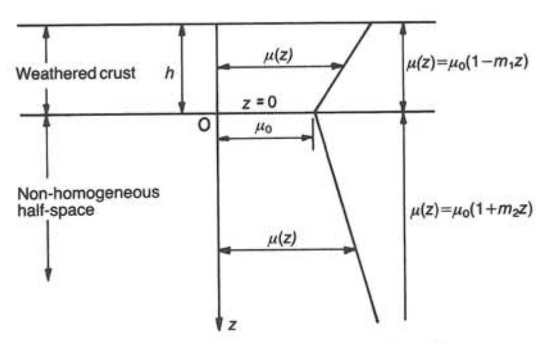

Fig. 1. Shear modulus profile of elastic soil medium

tural foundations is therefore important to the static and dynamic analysis of structure–foundation system.

The study of torsional behaviour of foundations embedded in elastic soils has received some attention. Existing studies are mainly based on homogeneous elastic half-space models (Poulos, 1975; Selvadurai, 1984) or a layered elastic half-space model where both the layer and the underlying half-space are assumed to be homogeneous (Luco, 1976; Karasudhi, Rajapakse & Hwang, 1984; Rajapakse & Selvadurai, 1985; Selvadurai & Rajapakse, 1987). Randolph (1979) presented an approximate analytical solution based on a simplified stress field in the soil to evaluate the torsional stiffness of an elastic pile embedded in a homogeneous soil or a linearly non-homogeneous soil with zero shear modulus at the surface level. The main objective of this study is to investigate rigorously the influence of the weathered crust and the non-homogeneous character of the underlying soil on the torsional response of embedded rigid foundations.

The geometry of the foundation–soil system considered in the present study is depicted in Fig. 2. The depth of the foundation is assumed to be such that the base of the cylindrical foundation terminates at any position within the weathered crust or within the underlying non-homogeneous half-space region. A boundary integral equation method based on exact displacement and traction Green's functions corresponding to a weathered non-homogeneous medium is used to determine the torsional stiffness. An approximate closed form solution is also derived to estimate the torsional stiffness of a rigid cylindrical foundation.

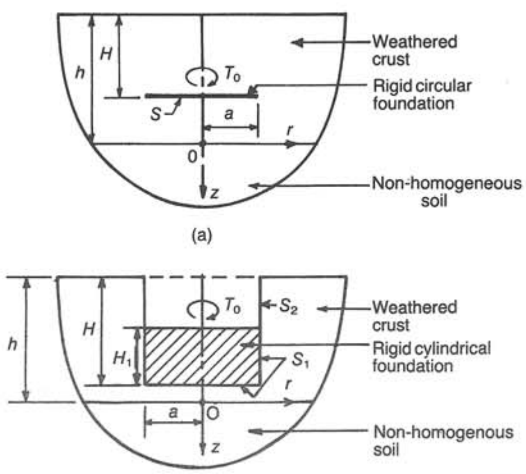

Fig. 2. Geometry of rigid foundation–soil interaction problem: (a) embedded circular foundation; (b) rigid cylindrical foundation

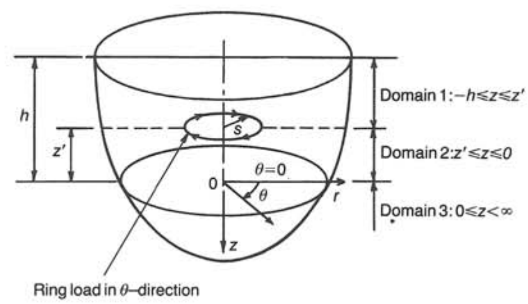

Fig. 3. Loading configuration for Green's functions

The solutions for torsional stiffness obtained from the approximate analysis are compared with the rigorous boundary integral equation solutions to establish the accuracy and range of validity of the approximate analysis.

FUNDAMENTAL SOLUTIONS

Figure 3 illustrates an isotropic non-homogeneous elastic half-space with a weathered crust. A cylindrical polar co-ordinate system (r, θ, z) is chosen such that z-axis is normal to the free surface of the half-space. It is convenient to introduce a non-dimensional length parameter and to perform analysis in terms of dimensionless quantities. The radius a of the foundation (Fig. 2) is selected as the unit length. The class of problems under consideration possesses a state of symmetry about the z-axis and the stresses and displacements are independent of circumferential co-ordinate θ. Owing to the state of symmetry imposed by axisymmetric torsion, the displacements u and w in the r and z directions respectively vanish. In the absence of body forces, the non-zero displacement component $v(r, z)$ in the θ-direction is governed by the following displacement equation of equilibrium

$$\frac{\partial^2 v}{\partial z^2} + \frac{\mu'(z)}{\mu(z)}\frac{\partial v}{\partial z} + \left[\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right] = 0 \quad (1)$$

where $\mu(z)$ is the shear modulus of the medium and $\mu'(z)$ is the first derivative of $\mu(z)$ with respect to z.

The non-zero stress components $\sigma_{\theta z}$ and $\sigma_{\theta r}$ referred to the cylindrical polar co-ordinate system can be expressed in the form

$$\sigma_{\theta z} = \mu(z)\frac{\partial v}{\partial z}\,;\ \sigma_{\theta r} = \mu(z)\left[\frac{\partial v}{\partial r} - \frac{v}{r}\right] \quad (2)$$

Consider an isotropic non-homogeneous medium where shear modulus varies according to the following form

$$\mu(z) = \mu_0(1 \pm mz), \quad m > 0 \quad (3)$$

The general solution for displacement $v(r, z)$ governed by equation (1) of an elastic medium with shear modulus variation given by equation (3) can be obtained by employing Hankel integral transforms (Sneddon, 1951). It can be shown that the general solution for displacement $v(r, z)$ can be expressed in the form

$$v(r, z) = \int_0^\infty \{A(\xi)I_0[(m^{-1} \pm z)\xi] + B(\xi)K_0[\xi(m^{-1} \pm z)]\}J_1(\xi r)\, d\xi \quad (4)$$

In equation (4), $A(\xi)$ and $B(\xi)$ are arbitrary functions which should be determined by invoking appropriate boundary and continuity conditions; J_n is the Bessel function of the first kind of order n; I_n and K_n denote modified Bessel functions of the first and the second kind of order n respectively. The solutions for stress components $\sigma_{\theta r}(r, z)$ and $\sigma_{\theta z}(r, z)$ can be obtained by substituting equation (4) in equation (2).

In the ensuing section dealing with the formulation of the interaction problem it is necessary to derive the solution for displacement Green's functions of a weathered non-homogeneous soil in the absence of the rigid foundation. The loading configuration required to be considered in the derivation of Green's functions is shown in Fig. 3. The shear moduli variation within the weathered crust $(-h < z < 0)$ is expressed as

$$\mu(z) = \mu_0(1 - m_1 z), \quad m_1 > 0 \quad (5)$$

and the shear modulus variation within the underlying half space $(0 < z < \infty)$ is expressed as

$$\mu(z) = \mu_0(1 + m_2 z), \quad m_2 > 0 \quad (6)$$

The boundary-value problem shown in Fig. 3 can be solved by defining a fictitious plane at $z = z'$ and considering it as a three-domain problem. The displacement field in each domain denoted by $v_i(r, z)$, $(i = 1, 2, 3)$ for the case where the ring load is applied within the weathered crust $(-h < z' < 0)$ can be expressed as

$$v_1(r, z) = \int_0^\infty \{A_1(\xi)I_0[\lambda_1(z)] + B_1(\xi)K_0[\lambda_1(z)]\} \times J_1(\xi r)\, d\xi \quad (-h \leqq z \leqq z') \quad (7)$$

$$v_2(r, z) = \int_0^\infty \{A_2(\xi)I_0[\lambda_1(z)] + B_2(\xi)K_0[\lambda_1(z)]\} \times J_1(\xi r)\, d\xi \quad (z' \leqq z \leqq 0) \quad (8)$$

$$v_3(r, z) = \int_0^\infty B_3(\xi)K_0[\lambda_2(z)] \times J_1(\xi r)\, d\xi \quad (0 \leqq z < \infty) \quad (9)$$

where

$$\lambda_1(z) = \xi(m_1^{-1} - z);\ \lambda_2(z) = \xi(m_2^{-1} + z) \quad (10)$$

The arbitrary functions $A_i(\xi)$ (i = 1, 2) and $B_i(\xi)$ (i = 1, 2, 3) can be determined from boundary conditions at $z = -h$ and continuity conditions at $z = z'$ and $z = 0$. The procedure to be followed is identical to that used by Selvadurai & Rajapakse (1987) for the case of a layered elastic half-space. The explicit solution for the five arbitrary functions are presented in Appendix 1.

ANALYSIS FOR RIGID CIRCULAR FOUNDATION

Figure 2(a) shows a rigid circular foundation of radius a buried in a weathered non-homogeneous soil. The foundation is subjected to a torque T_0 about the z-axis and experiences a rotation ϕ_0 about the z-axis. The depth of embedment H of the foundation is arbitrary such that it can be located either within the weathered crust or within the underlying non-homogeneous half-space. The resultant contact traction in the θ-direction denoted by $T(s)$, acting on surface S $(0 \leqslant s \leqslant a,\ z = H - h)$ is governed by the following integral equation.

$$\int_0^a G(r, H - h; s, H - h)T(s)\, ds = \phi_0 r, \quad 0 \leqq r \leqq a \quad (11)$$

In equation (11), $G(r, z; s, z')$ is the displacement Green's function of the soil medium in the absence of the foundation. $G(r, z; s, z')$ denotes displacement in the θ-direction at point (r, z) due to a unit ring load through point (s, z'). The explicit solution for $G(r, z; s, z')$ is discussed in the preceding section.

A convenient way to solve the integral equation (11) is to discretize the contact surface S into a set of N concentric regions as shown in Fig. 4 and to assume that within each of the concentric regions the traction $T_j(s)$ (j = 1, 2, ..., N) varies in

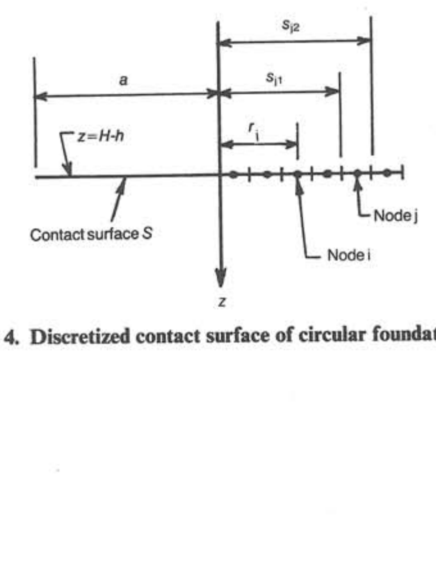

Fig. 4. Discretized contact surface of circular foundation

the following linear form

$$T_j(s) = s\bar{T}_j, \quad s_{j1} < s < s_{j2} \tag{12}$$

where s_{j1} and s_{j2} are inner and outer radii of the jth concentric region (Fig. 4) and $\bar{T}_j$ is the unknown intensity of traction corresponding to jth element. In view of equation (12) a discrete version of equation (11) can be written as

$$\sum_{j=1}^{n} \bar{G}(r_i, H-h; s_{j1}, s_{j2})\bar{T}_j = \phi_0 r_i, \quad i = 1, 2, \ldots, N \tag{13}$$

where

$$\bar{G}(r_i, H-h; s_{j1}\ s_{j2}) = \int_{s_{j1}}^{s_{j2}} G(r_i, H-h; s, H-h)s\ \mathrm{d}s \tag{14}$$

and r_i denotes the radial coordinate of ith node (Fig. 4).

It can be shown that $\bar{G}$ is given by equations (7)–(10) and (48)–(63) with the term $sJ_1(\xi s)$ in equations (49), (58)–(60) and (62) replaced by $[s_{j2}{}^2 J_2(\xi s_{j2}) - s_{j1}{}^2 J_2(\xi s_{j1})]/\xi$.

Numerical solution of equation (13) results in $\bar{T}_j$ corresponding to each concentric element. The resultant torque T_0 acting on the foundation is then given by

$$T_0 = \sum_{j=1}^{n} \bar{T}_j \frac{\pi}{2}[s_{j2}{}^4 - s_{j1}{}^4] \tag{15}$$

ANALYSIS FOR RIGID CYLINDRICAL FOUNDATION

Boundary integral solution

Figure 2(b) shows a rigid cylindrical foundation in a weathered non-homogeneous medium. The foundation is subjected to a torque T_0 and experiences a rotation ϕ_0 about the z-axis. It is assumed that the foundation is perfectly bonded to the surrounding elastic soil along the contact surface S_1. The boundary conditions relevant to the interaction problem shown in Fig. 2(b) can be expressed as

$$\sigma_{\theta z}(r, -h) = 0, \qquad r > a \tag{16}$$

$$v(r, z) = \phi_0 r, \quad (r, z)\varepsilon S_1 \tag{17}$$

$$T(r, z) = 0, \qquad (r, z)\varepsilon S_2 \tag{18}$$

The solution of the elastic soil medium subjected to boundary conditions given by equations (16)–(18) yields traction $T(r, z)$ in the θ-direction on S_1 and displacement $v(r, z)$ on S_2. The torque T_0 producing the rotation ϕ_0 is given by

$$T_0 = \int_{S_1} rT(r, z)\ \mathrm{d}S_1, \quad (r, z)\varepsilon S_1 \tag{19}$$

In the present study the interaction problem is solved by using the indirect boundary integral equation method (Ohsaki, 1973). Consider a weathered non-homogeneous half-space V^* (Fig. 5) in the absence of the foundation. Fictitious surfaces $S_1{}^*$ and $S_2{}^*$ which are identical to surface S_1 and S_2 respectively are defined in V^* as shown in Fig. 5. A traction field in the θ-direction denoted by $\tau(r, z)$ is applied along the axi-symmetric surface $\bar{S}$ defined interior to $S_1{}^*$ and $S_2{}^*$ (Fig. 5). The magnitude of $\tau(r, z)$ is such that displacement $v(r, z)$ in $S_1{}^*$ is equal to $\phi_0 r$ and traction $T(r, z)$ on $S_2{}^*$ is equal to zero. Under these conditions $\tau(r, z)$ is governed by the following dual integral equations.

$$\int_{\bar{S}} G(r, z; s, z)\tau(s, z')s\ \mathrm{d}\bar{S} = \phi_0 r, \qquad (r, z)\varepsilon S_1{}^*, (s, z')\varepsilon \bar{S} \tag{20}$$

$$\int_{\bar{S}} \psi(r, z; s, z)\tau(s, z')s\ d\bar{S} = 0, \qquad (r, z)\varepsilon S_2{}^*, (s, z')\varepsilon \bar{S} \tag{21}$$

where $\psi(r, z; s, z')$ is the traction Green's function of the soil medium which can be expressed in terms of $G(r, z; s, z')$ by using equation (2). In addition, $\bar{S}$ in equations (20) and (21) implies the generating curve of surface $\bar{S}$.

The unknown traction $T(r, z)$ on $S_1{}^*$ and unknown displacement $v(r, z)$ on $S_2{}^*$ can be expressed in terms of $\tau(r, z)$ through the following integral equations.

$$T(r, z) = \int_{\bar{S}} \psi(r, z; s, z')\tau(s, z)s\ \mathrm{d}\bar{S} \tag{22}$$

$$v(r, z) = \int_{\bar{S}} G(r, z; s, z')\tau(s, z)s\ \mathrm{d}\bar{S} \tag{23}$$

A numerical solution of equations (20) and (21) can be obtained by discretizing $\bar{S}$, $S_1{}^*$ and $S_2{}^*$ by

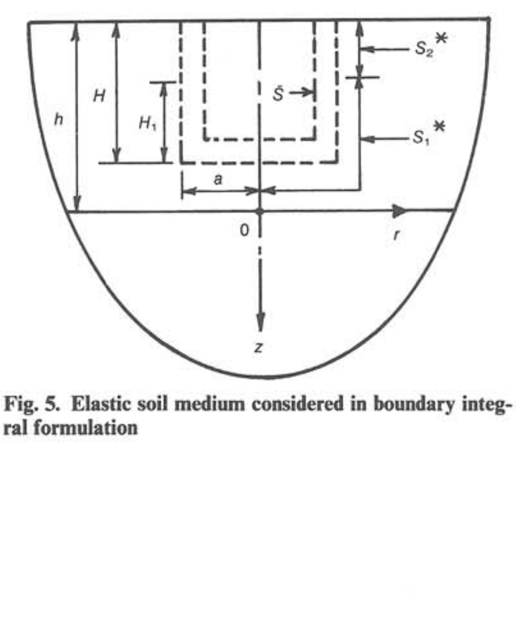

Fig. 5. Elastic soil medium considered in boundary integral formulation

using $\bar{N}$, N_1 and N_2 nodal points respectively. A discrete version of equations (20) and (21) can be expressed as

$$[A]\{\tau\} = \{R\} \quad (24)$$

where

$$[A] = \begin{bmatrix} [G(r_i, z_i; s_j, z_j')s\Delta\bar{S}_j]_{N_1 \times \bar{N}} \\ \hline [\psi(\bar{r}_i, \bar{z}_i; s_j, z_j')s\Delta\bar{S}_j]_{N_2 \times \bar{N}} \end{bmatrix} \quad (25)$$

$$\{R\} = \begin{bmatrix} \{\phi_0 r_i\} \\ \hline \{0\} \end{bmatrix} \quad (26)$$

$$\{\tau\} = \{\tau(s_j, z_j')\} \quad (27)$$

In equations (25)–(27), $(r_i, z_i) \in S_1^*$, $(\bar{r}_i, \bar{z}_i) \in S_2^*$ and $(s_j, z_j') \in \bar{S}$. In addition $\Delta\bar{S}_j$ is the tributary length corresponding to node j on $\bar{S}$.

A least-square solution of equation (24) yields

$$\{\tau\} = [[A]^T[A]]^{-1}[A]^T\{R\} \quad (28)$$

The equations (22) and (23) corresponding to unknown traction and displacement can be expressed as

$$T(r, z) = \langle \psi(r, z; s_j, z_j')s\Delta\bar{S}_j \rangle \{\tau\}, \qquad (r, z) \in S_1^* \quad (29)$$

$$v(r, z) = \langle G(r, z; s_j, z_j')s\Delta\bar{S}_j \rangle \{\tau\}, \qquad (r, z) \in S_2^* \quad (30)$$

The resultant torque T_0 is obtained by numerical integration of equation (19) with $T(r, z)$ evaluated by using equation (29).

Approximate closed form solution

It is evident from the preceding analysis that a considerable computational effort is required in the boundary integral solution. From a practical point of view, it is useful to investigate the possibility of developing a simple closed form solution for torsional stiffness of a rigid cylindrical foundation based on a simplified representation of stress field in the non-homogeneous soil. Approximate closed form solutions for torsional stiffness of rigid foundations embedded in homogeneous elastic media have been presented by Luco (1976), Randolph (1981), Rajapakse & Selvadurai (1985) and Selvadurai & Rajapakse (1987). The approximate solution developed in this Paper is based on two basic assumptions. It is assumed that the elastic soil surrounding the mantle of the foundation can be modelled as a series of thin elastic layers in smooth contact and the deformations of the mantle and the base can be uncoupled. The torsional stiffness of the rigid cylindrical foundation is obtained by adding the torque required to produce a unit rotation of the soil surrounding the mantle derived on the basis of the simplified stress field to the torque corresponding to a rigid circular foundation resting on a non-homogeneous soil half-space.

In the absence of body forces, axisymmetric pure torsion problems are governed by the equilibrium equation

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = 0 \quad (31)$$

The assumption that the surrounding soil is modelled as a series of thin elastic layers in smooth contact simplifies equation (31) to

$$\frac{d\sigma_{r\theta}}{dr} + \frac{2\sigma_{r\theta}}{r} = 0 \quad (32)$$

The solution of equation (32) is given by

$$\sigma_{r\theta} = Ar^{-2} \quad (33)$$

where A is an integration constant.

The substitution of equation (33) in equation (2) results in

$$\frac{dv}{dr} - \frac{v}{r} = \frac{Ar^{-2}}{\mu} \quad (34)$$

where μ is the shear modulus of a thin elastic layer.

The general solution of equation (34) is given by

$$v = Br - \frac{A}{2\mu r} \quad (35)$$

where B is an integration constant. Noting that the solution for a thin layer has to be bounded as $r \to \infty$, $B \equiv 0$.

Consider a rigid cylindrical foundation of radius a which is subjected to rotation ϕ_0, about z-axis. The displacement in the θ-direction along the mantle ($r = a$) is given by

$$(v)_{r=a} = \phi_0 a = \frac{-A}{2\mu a} \quad (36)$$

The substitution of equation (36) in equation (35) yields the following expression for circumferential displacement of a thin layer

$$v = \phi_0 a^2 r^{-1} \quad (37)$$

The stress component $\sigma_{r\theta}$ is given by

$$\sigma_{r\theta} = -2\mu\phi_0 a^2 r^{-2} \quad (38)$$

The torque T_s required to rotate a system of thin elastic layers of total height H_1 by an angle ϕ_0 along the shaft of a cylindrical cavity of radius

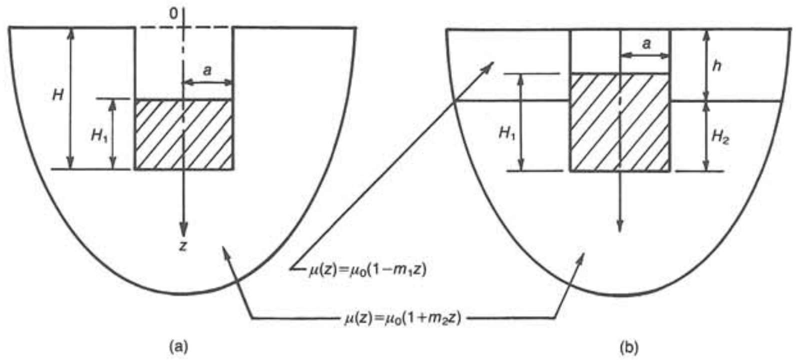

Fig. 6. Geometry of interaction system considered in derivation of approximate solution

a is given by

$$T_s = \int_{(\bar{H}-\bar{H}_1)}^{\bar{H}} 2\pi a^3[\sigma_{r\theta}]_{r=a}\,dz \qquad (39)$$

where $\bar{H} = H/a$ and $\bar{H}_1 = H_1/a$.

The origin of z-co-ordinate in equation (39) is as shown in Fig. 6(a). The substitution of equation (38) and $\mu(z) = \mu_0(1 + m_2 z)$ in equation (39) results in

$$T_s = 4\pi a^3 \mu_0 \phi_0[\bar{H}_1 + 0{\cdot}5m_2 \bar{H}_1(2\bar{H} - \bar{H}_1)] \qquad (40)$$

The torque T_b corresponding to a rotation ϕ_0 of a rigid circular plate of radius a bonded to the surface of a linearly non-homogeneous half-space with surface shear modulus $\mu(H)$ $[=\mu_0(1 + m_2\bar{H}_1)]$ can be expressed as

$$T_b = \frac{16a^3\mu_0\phi_0}{3}(1 + m_2\bar{H}_1)\eta \qquad (41)$$

where η is a correction factor to account for non-homogeneous nature of the soil half-space which is determined from the analysis of a rigid plate.

The total torque T_a required to produce rotation ϕ_0 of the embedded cylindrical foundation shown in Fig. 6(a) can be expressed as

$$\begin{aligned} T_a &= T_s + T_b \\ &= \frac{16\mu_0 a^3\phi_0}{3}\{\eta(1 + m_2\bar{H}_1) \\ &\quad + 0{\cdot}75\pi\bar{H}_1[1 + 0{\cdot}5m_2(2\bar{H} - \bar{H}_1)]\} \end{aligned} \qquad (42)$$

The preceding analysis can be extended to evaluate the torque T_a required to rotate the cylindrical foundation shown in Fig. 6(b) without any fundamental difficulty. The solution for T_a corresponding to the foundation shown in Fig. 6(b) can be expressed as

$$\begin{aligned} T_a = \frac{16\mu_0 a^3\phi_0}{3}&\{\eta(1 + m_2\bar{H}_2) \\ &+ 0{\cdot}75\pi[\bar{H}_2(1 + 0{\cdot}5m_2\bar{H}_2) \\ &+ \bar{H}_3(1 + 0{\cdot}5m_1\bar{H}_3)]\} \end{aligned} \qquad (43)$$

where $\bar{H}_3 = \bar{H}_1 - \bar{H}_2$.

NUMERICAL RESULTS AND DISCUSSION

A computer code based on the solution procedure described in preceding sections has been developed to investigate numerically the torsional response of rigid foundations embedded in weathered non-homogeneous elastic soils. The Green's functions required in the analysis are evaluated by using numerical quadrature methods. For the purpose of convenient presentation, the following non-dimensional quantities are defined

$$\text{depth of weathered crust } (\bar{h}) = h/a \qquad (44)$$

$$\text{depth of embedment of foundation } (\bar{H}) = H/a \qquad (45)$$

$$\text{height of foundation } (\bar{H}_1) = H_1/a \qquad (46)$$

$$\text{torsional stiffness } (\bar{T}_0) = 3T_0/16\bar{\mu}a^3\phi_0 \qquad (47)$$

where $\bar{\mu}$ is a reference shear modulus.

In the ensuing sections the dependence of torsional stiffness $\bar{T}_0$ on non-dimensional parameters $\bar{h}$, $\bar{H}$, $\bar{H}_1$ and gradients m_1 and m_2 of shear modulus profile are investigated.

Rigid circular foundation (Fig. 2(a))

Figure 7(a) shows the variation of $\bar{T}_0$ of a rigid circular foundation embedded in an isotropic non-homogeneous elastic soil medium in the absence of weathered crust ($\bar{h} = 0$). In this case the gradient of the shear moduli profile m_2 (Fig. 1) can be considered as a total measure of the degree of non-homogeneity of the soil medium. The reference shear modulus $\bar{\mu}$ used in equation (47) is taken as equal to $\mu(\bar{H})$. Solutions for $\bar{T}_0$ are presented in Fig. 7(a) for $0 < \bar{H} < 2{\cdot}0$ and for $0{\cdot}0 < m_2 < 2{\cdot}0$. The behaviour of the solution is interesting and indicates that $\bar{T}_0$ is independent of both $\bar{H}$ and m_2 for $\bar{H} > 2{\cdot}0$. It is important to realize that the behaviour of $\bar{T}_0$ observed in Fig. 7(a) is a consequence of the use of $\bar{\mu} = \mu(\bar{H})$ in equation (47). If $\bar{\mu} = \mu(0)$ is used in equation (47) $\bar{T}_0$ monotonically increases with increasing $\bar{H}$ and m_2. It is also noted from Fig. 7(a) that $\bar{T}_0$ for a deeply buried ($\bar{H} > 2{\cdot}0$) rigid circular foundation in a non-homogeneous soil is identical to that of a foundation deeply buried in an isotropic homogeneous soil with shear modulus equal to $\mu(\bar{H})$.

The Figs 7(b)–(d) show the variation of torsional stiffness of a rigid circular foundation embedded in a non-homogeneous elastic soil medium with a weathered crust ($\bar{h} \neq 0$). The dependence of $\bar{T}_0$ on m_1 and depth of embedment $\bar{H}$ is investigated by considering three different depths of weathered crust represented by $\bar{h} = 0{\cdot}5$, $1{\cdot}0$ and $2{\cdot}0$. The gradient of the shear moduli profile of the underlying non-homogeneous soil half-space m_2 is taken as equal to $1{\cdot}0$. The location of the embedded foundation is represented by the depth ratio $\bar{H}/\bar{h}$ with $\bar{H}/\bar{h} = 0{\cdot}0$ and $1{\cdot}0$ representing a foundation located at the top and bottom level of the weathered crust respectively. The reference shear modulus $\bar{\mu}$ used in the equation (47) is taken as equal to μ_0 where μ_0 is the shear modulus at the bottom level of the weathered crust (Fig. 1).

As expected, $\bar{T}_0$ increases gradually with m_1 for a foundation located at a given depth. It is noted

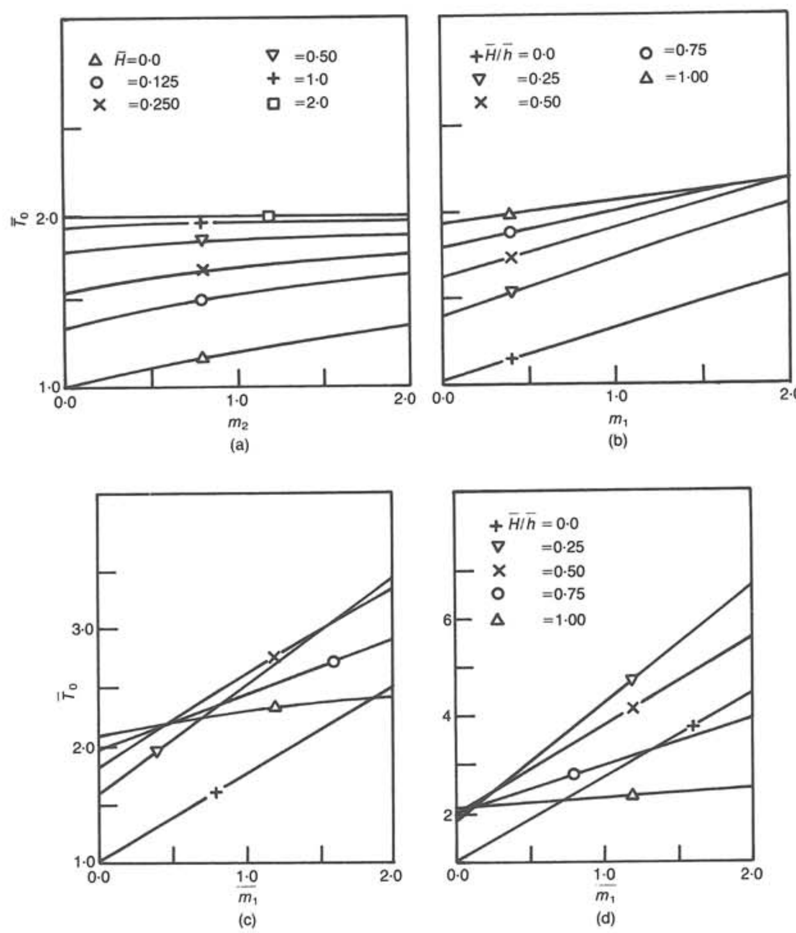

Fig. 7. Normalized torsional stiffness $\bar{T}_0$ of rigid circular foundation: (a) $\bar{h} = 0$; (b) $\bar{h} = 0{\cdot}50$, $m_2 = 1{\cdot}0$; (c) $\bar{h} = 1{\cdot}0$, $m_2 = 1{\cdot}0$; (d) $\bar{h} = 2{\cdot}0$, $m_2 = 1{\cdot}0$

that the rate of increase of $\bar{T}_0$ with m_1 decreases with increasing $\bar{H}/\bar{h}$. This is due to the fact that as depth of embedment $\bar{H}$ increases the influence due to the presence of weathered crust gradually diminishes. It is also found that if $(\bar{H}\text{-}\bar{h}) > 1$, $\bar{T}_0$ is nearly independent of m_1. Solutions presented by Figs 7(a)–(d) also indicate that for shallow depths of weathered crust ($\bar{h} = 0{\cdot}0$–$0{\cdot}50$) $\bar{T}_0$ increases gradually with depth of embedment, reflecting that the increase in stiffness due to embedment is dominant over the reduction in stiffness due to weathering of the soil. However, for foundations located near the bottom of a deep weathered crust ($\bar{h} > 1{\cdot}0$) the reduction of stiffness due to weathering is dominant over the increase in stiffness due to embedment as indicated in Fig. 7(d). In general solutions for $\bar{T}_0$ presented in Fig. 7 show that parameters $\bar{H}$, $\bar{h}$, m_1 and m_2 have a significant influence on the torsional response of a rigid circular foundation.

Cylindrical foundations in non-homogeneous soil

Figure 8 shows the variation of torsional stiffness $\bar{T}_0$ of a rigid cylindrical foundation embedded in a non-homogeneous elastic soil medium without a weathered crust ($\bar{h} = 0$). The depth of excavation $\bar{H}$ (Fig. 2(b)) is assumed to be equal to the height of the foundation $\bar{H}_1$. The reference shear modulus $\bar{\mu}$ used in equation (47) is taken equal to $\mu(0) = \mu_0$. The solutions presented in Fig. 8 corresponds to foundations with non-dimensional height $\bar{H}_1$ varying from 0·25 to 2·0 and $0 < m_2 < 2{\cdot}0$. With increasing $\bar{H}_1$ the torsional stiffness shows significant dependence on the gradient m_2 of the shear modulus profile. For $\bar{H}_1 = 0{\cdot}25$, torsional stiffness is increased by nearly 1·7 times when m_2 is changed from 0·0 to 2·0. However, $\bar{T}_0$ increases by nearly 3·5 times for an identical increase of m_2 for a foundation with $H_1 = 2{\cdot}0$.

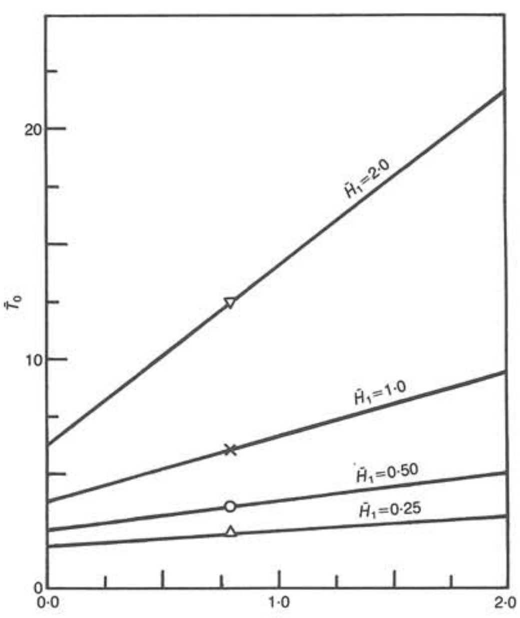

Fig. 8. Normalized torsional stiffness $\bar{T}$ of rigid cylindrical foundations ($\bar{h} = 0{\cdot}0°$, $\bar{H} = \bar{H}_1$)

Table 1. Ratio of T_a and T_0 for rigid cylindrical foundation embedded in non-homogeneous elastic soil

m_2	T_a/T_0		
	$\bar{H} = \bar{H}_1 =$ 0·25	$\bar{H} = \bar{H}_1 =$ 2·0	$\bar{H} = \bar{H}_1 =$ 4·0
0·0	0·87	0·93	0·96
0·4	0·85	0·92	0·95
0·8	0·85	0·92	0·95
1·2	0·84	0·92	0·95
1·6	0·84	0·92	0·95
2·0	0·83	0·92	0·95

Comparison of approximate and boundary integral solutions

Table 1 presents the ratio T_a/T_0 of a rigid cylindrical foundation as shown in Fig. 6(a). T_0 is the torque obtained from the boundary integral equation solution scheme and T_a is the approximate torque given by equation (42). The case to $T_a/T_0 = 1{\cdot}0$ corresponds to a perfect agreement in the two solutions. These results (Table 1) indicate that equation (42) represents a good engineering approximation for torsional stiffness of a rigid cylindrical foundation. Note that by virtue of the assumptions employed in the derivation of equation (42), the ratio $T_a/T_0 < 1{\cdot}0$. The accuracy of the approximate solution is found to increase with increasing height of the foundation and decreases slightly with increasing gradient of the shear modulus profile. Additional numerical solutions corresponding to foundations where $\bar{H} \neq \bar{H}_1$ also indicate a behaviour similar to that observed in Table 1. In addition the equation (42) implies that torsional stiffness of cylindrical foundations varies linearly with gradient m_2 and depth $\bar{H}$ and quadratically with foundation height $\bar{H}_1$. For rigid cylindrical foundations with $\bar{H}_1 > 2{\cdot}0$ the error associated with equation (42) is found to be less than 10%.

Table 2 presents the values of ratio T_a/T_0 of a rigid cylindrical foundation embedded in a non-homogeneous soil with a weathered crust (Fig.

Table 2. Ratio of T_a and T_0 for rigid cylindrical foundations embedded in non-homogeneous elastic soil with a weathered crust ($m_2 = 1{\cdot}0$, $\bar{h} = 2{\cdot}0$)

m_1	T_a/T_0	
	$\bar{H}_1 = 1{\cdot}5$, $\bar{H}_2 = 1{\cdot}0$	$\bar{H}_1 = 2{\cdot}0$, $\bar{H}_2 = 1{\cdot}0$
0·0	0·93	0·96
0·5	0·93	0·95
1·0	0·92	0·95
2·0	0·91	0·94

6(b)). It is evident from these solutions that equation (43) provides a close engineering approximation for torsional stiffness of cylindrical foundation embedded in a non-homogeneous elastic soil with a weathered crust. The equation (43) also indicates that the torsional stiffness varies linearly with m_1 and m_2 and quadratically with depths $\bar{H}_1$ and $\bar{H}_2$. Both equations (42) and (43) have a tremendous computational advantage over solutions based on the boundary integral equation method. This together with solutions presented in Tables 1 and 2 confirms that closed form solutions given by equations (42) and (43) could serve as a very good approximation for the torsional stiffness of rigid cylindrical foundations embedded in a non-homogeneous soil with a weathered crust.

CONCLUSIONS

The torsional response of rigid axisymmetric foundations embedded in non-homogeneous elastic soil with a weathered crust is examined rigorously. Based on the numerical study the following conclusions are drawn.

The torsional stiffness of rigid circular foundations increases with increasing gradients of shear modulus profile. The rate of increase in stiffness with gradients m_1 and m_2 is influenced by both the depth of embedment $\bar{H}$ and depth of weathered crust $\bar{h}$. If $(\bar{H}\text{-}\bar{h}) > 1{\cdot}0$ then stiffness is governed only by m_2. For foundations embedded within a shallow weathered crust ($\bar{h} < 0{\cdot}5$) the increase in stiffness due to embedment is dominant over reduction in stiffness due to weathering and this leads to a gradual increase of stiffness with depth. For deep weathered crusts ($\bar{h} > 1{\cdot}0$) an increase in stiffness is observed with embedment depth initially indicating the stiffening effect due to the embedment and followed by a decrease in stiffness due to the influence of weathering.

Rigorous solutions for torsional stiffness of cylindrical foundations indicate that the height of the foundation $\bar{H}_1$ has a significant influence on the torsional stiffness. The influence of the degree of non-homogeneity of soil (m_1 and m_2) on the torsional stiffness is found to increase with increasing height of the foundation. It is found that a closed form solution based on a simplified representation of the stress field in the surrounding soil can be derived to estimate the torsional stiffness of rigid cylindrical foundations embedded in a weathered non-homogeneous soil. The approximate closed form solution yields solutions that are within 10% of rigorous solutions for foundations with height–radius ratio $\bar{H}$ greater than 2·0. For short foundations ($\bar{H}_1 < 2{\cdot}0$) the error varies between 10–20%. The closed form solution indicates that torsional stiffness varies linearly with gradients of shear modulus profile and depth of weathered crust and quadratically with the height of the foundation.

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APPENDIX 1

The explicit solutions for the five arbitrary functions appearing equations (7)–(9) are given by

$$\langle A_1 \ A_2 \ B_1 \ B_2 \rangle = B_3 \langle \beta_1 \ \beta_2 \ \beta_3 \ \beta_4 \rangle \tag{48}$$

where

$$B_3 = \frac{sJ_1(\xi s)[\alpha_{11}(-h) + \alpha_{01}(z')][\alpha_{01}(0) + \alpha_{11}(0)]}{\mu(z')I_1[\lambda_1(z')][\alpha_{01}(z') + \alpha_{11}(z')]D} \tag{49}$$

$$D = \gamma_0(0)[\alpha_{11}(-h) - \alpha_{11}(0)] - \gamma_1(0)[\alpha_{11}(-h) + \alpha_{01}(0)] \tag{50}$$

$$\beta_1 = \alpha_{11}(-h)\beta_3 \tag{51}$$

$$\beta_2 = \frac{[\gamma_0(0)\alpha_{11}(0) + \gamma_1(0)\alpha_{01}(0)]}{[\alpha_{01}(0) + \alpha_{11}(0)]} \tag{52}$$

$$\beta_3 = \frac{\{\gamma_0(0)[\alpha_{11}(0) + \alpha_{01}(z') + \gamma_1(0)][\alpha_{01}(0) - \alpha_{01}(z')]\}}{[\alpha_{11}(-h) + \alpha_{01}(z')][\alpha_{01}(0) + \alpha_{11}(0)]} \tag{53}$$

$$\beta_4 = [\gamma_0(0) - \gamma_1(0)]/[\alpha_{01}(0) + \alpha_{11}(0)] \tag{54}$$

$$\alpha_{i1}(z) = K_i[\lambda_1(z)]/I_i[\lambda_1(z)] \qquad i = 0, 1 \tag{55}$$

and

$$\gamma_i(z) = K_i[\gamma_2(z)]/I_i[\gamma_1(z)] \quad i = 0, 1 \tag{56}$$

For the case where the ring load is applied within the underlying non-homogeneous half-space ($0 \leqq z' < \infty$), the domains of definition of equation (7)–(9) are changed to $(-h < z < 0)$, $(0 < z < z')$ and $(z' < z < \infty)$ respectively. The explicit solution for arbitrary functions are given by

$$A_1 = B_1\alpha_{11}(-h) \tag{57}$$

$$B_1 = \frac{sJ_1(\xi s)\alpha_{02}(z')[\bar{\gamma}_0(0) + \bar{\gamma}_1(0)] + B_2[\gamma_0(0) + \gamma_1(0)][\alpha_{02}(z') + \alpha_{12}(z')]}{\mu(z')I_1[\lambda_2(z')][\alpha_{01}(0) + \alpha_{11}(0)][\alpha_{02}(z') + \alpha_{12}(z')]} \tag{58}$$

$$A_2 = sJ_1(\xi s)\alpha_{02}(z')/[\alpha_{02}(z') + \alpha_{12}(z')] \tag{59}$$

$$B_2 = \frac{sJ_1(\xi s)\alpha_{02}(z')\{\bar{\gamma}_0(0)[\alpha_{11}(0) - \alpha_{11}(-h)] - \bar{\gamma}_1(0)[\alpha_{11}(-h) + \alpha_{01}(0)]\}}{\mu(z')I_1[\lambda_2(z')][\alpha_{02}(z') + \alpha_{12}(z')]_{\bar{D}}} \tag{60}$$

$$\bar{D} = \alpha_{11}(-h)[\gamma_0(0) + \gamma_1(0)] - \gamma_0(0)[\alpha_{11}(0) - \alpha_{01}(0)] \tag{61}$$

$$B_3 = B_2 + \frac{sJ_1(\xi s)}{\mu(z')I_1[\lambda_2(z')][\alpha_{02}(z') + \alpha_{12}(z')]} \tag{62}$$

where

$$\bar{\gamma}_n(z) = I_n[\lambda_2(z)]/I_n[\lambda_1(z)] \quad n = 0, 1 \tag{63}$$

and α_{ij} and γ_i are defined by equations (55) and (56) respectively.

The equations (7)–(10), together with equations (48)–(63) yield explicit solutions for displacement at any point within the weathered non-homogeneous soil due to a circumferential ring load (Fig. 3). Solutions for stress components are obtained by using equation (2).

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