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Boussinesq's problem for a debonded boundary

By A. P. S. Selvadurai, B. M. Singh and M. C. Au, Dept. of Civil Engineering, Carleton University, Ottawa, Ontario, Canada, K1S 5B6

1. Introduction

The classical contact problem which deals with the indentation of an elastic half space region by a rigid circular punch was first examined by Boussinesq [1]. A subsequent treatment of this problem by Harding and Sneddon [2] makes use of Hankel transform techniques to reduce the mixed boundary value problem to a system of dual integral equations. Contact problems in classical elasticity have received considerable attention and extensive accounts of the subject, dealing with rigid indentors with arbitrary profiles possessing either smooth, adhesive or frictional interfaces are given by Galin [3], Lur'e [4], Sneddon [5], de Pater and Kalker [6] and Gladwell [7]. A related class of contact problem focusses on the internal indentation of cracks by rigid punches or rigid inclusions. This particular category of indentation problem has some relevance to the stress analysis of multiphase composites and to geomechanics. In the context of multiphase composites, such internal indentations can occur during thermally induced cracking of the multiphase composite [8]. A mismatch of thermal expansion coefficients between the inclusion and the crack can lead to the initiation and propagation of a fracture. In the context of geomechanical applications the indenting disc inclusion can be used to model flat anchoring regions which are formed by hydraulic fracture of a geological stratum by expansive cementaceous fluids.

This paper focusses on a specific problem related to the axial loading of a rigid circular punch which is located at the debonded region of an elastic medium-rigid boundary interface. The elastic-rigid interface approximates the case where one material possesses higher elastic stiffness characteristics. The contact between the punch and the debonded elastic medium is assumed to be smooth. The punch is subjected to a central force (P) which is directed along the z-axis. The analysis of the problem concentrates primarily on the evaluation of the elastic stiffness of the punch in the delaminated region. For this reason, and in view of the axisymmetric nature of the problem, it is convenient to adopt a mathematical analysis which is based on a Hankel transform development of the governing equations. The Hankel transform development of the mixed boundary value problem yields a set of integral equations which can effectively be reduced to a single Fredholm integral equation of the second kind. This integral equation is solved in a numerical fashion. Numerical results of primary interest to the paper, illustrate the manner in which the stiffness of the punch in the delaminated region is influenced by the Poisson's ratio of the elastic material and the ratio of the radius of the rigid punch to the radius of the debonded region. Analogous numerical results are also given for the flaw opening mode stress intensity factor at the boundary of the delaminated region.

2. Fundamental equations

For the analysis of the axisymmetric state of deformation induced by the indenting punch it is convenient to employ the formulation based on the strain potential approach of Love [9]. In the absence of body forces, the solution of the displacement equations of equilibrium can be represented in terms of a bi-harmonic function $\phi(r, z)$; i.e.

$$\nabla^2\nabla^2\phi(r, z) = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

The components of the displacement vector $\mathbf{u}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ referred to the cylindrical polar coordinate system can be expressed in terms of the derivatives of ϕ. We have

$$2Gu_r = -\frac{\partial^2\phi}{\partial r\,\partial z} \tag{3}$$

$$2Gu_z = 2(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2} \tag{4}$$

where G and ν are the linear elastic shear modulus and Poisson's ratio respectively. Similarly, the components of the stress tensor are given by

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{\partial^2\phi}{\partial r^2}\right) \tag{5}$$

$$\sigma_{\Theta\Theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2\phi - \frac{1}{r}\frac{\partial\phi}{\partial r}\right) \tag{6}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right\} \tag{7}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2\phi - \frac{\partial^2\phi}{\partial z^2}\right\} \tag{8}$$

3. The indentation problem

We consider the problem of a rigid circular inclusion of radius a which is located at a debonded region on an elastic medium-rigid boundary interface. The debonded region has a circular plan form with radius b. The rigid circular punch is subjected to a central force P which acts in the $+ve$ z-direction (Fig. 1). The punch experiences a rigid body translation Δ along the $+ve$ z-direction. The indentation between the rigid punch and the elastic medium is assumed to be smooth and bi-lateral. The relevant mixed boundary conditions associated with the indentation problem are as follows:

$$u_z(r, 0) = \Delta; \qquad 0 \le r \le a \tag{9}$$

$$\sigma_{rz}(r, 0) = 0; \qquad 0 < r < b \tag{10}$$

$$\sigma_{zz}(r, 0) = 0; \qquad a < r < b \tag{11}$$

$$u_r(r, 0) = 0; \qquad b \le r < \infty \tag{12}$$

$$u_z(r, 0) = 0; \qquad b \le r < \infty \tag{13}$$

For the integral equation formulation of the mixed boundary value problem posed by equations (9) to (13) we seek solutions of (1) which can be obtained by Hankel transform development of the governing differential equation. Furthermore, the displacement and stress fields derived from $\phi(r, z)$ should reduce to zero as $(r^2 + z^2)^{1/2} \to \infty$. Following Sneddon [5], the

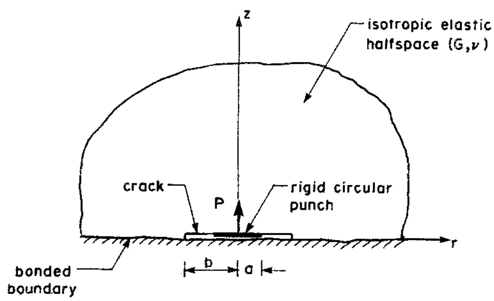

Figure 1
Indentation of a debonded interface.

relevant solution is given by

$$\phi(r, z) = \int_0^\infty \xi\{A(\xi) + zB(\xi)\}\, e^{-\xi z} J_0(\xi r)\, d\xi \tag{14}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions which are to be determined by satisfying the boundary conditions (9) to (13) on the plane $z = 0$. Using the integral representation for $\phi(r, z)$ given by (14) in the expressions for **u** and **σ**, it can be shown that the mixed boundary conditions (9) to (13) reduce to the following system of integral equations:

$$H_0[\xi\{\xi A(\xi) + 2(1-2\nu)B(\xi)\}; r] = -2G\Delta; \qquad 0 \le r \le a \tag{15}$$

$$H_1[\xi^2\{\xi A(\xi) - 2\nu B(\xi)\}; r] = 0; \qquad 0 < r < b \tag{16}$$

$$H_0[\xi^2\{\xi A(\xi) + (1-2\nu)B(\xi)\}; r] = 0; \qquad a < r < b \tag{17}$$

$$H_1[\xi\{-\xi A(\xi) + B(\xi)\}; r] = 0; \qquad b \le r < \infty \tag{18}$$

$$H_0[\xi\{\xi A(\xi) + 2(1-2\nu)B(\xi)\}; r] = 0; \qquad b < r < \infty \tag{19}$$

where H_n is the Hankel transform of order n, defined by

$$H_n[A^*(\xi); r] = \int_0^\infty \xi A^*(\xi) J_n(\xi r)\, d\xi \tag{20}$$

We now make the assumption that as $b \to \infty$, we should recover from the developed solution the appropriate result for the surface loading of a halfspace region by a smooth circular punch. To achieve this we use the substitutions

$$N(\xi) = \xi^3 A(\xi) + (1-2\nu)\xi^2 B(\xi) \tag{21}$$

$$M(\xi) = \xi^3 A(\xi) - 2\nu\xi^2 B(\xi) \tag{22}$$

The equations (15) to (19) can now be written in the form:

$$H_0\left[\xi^{-1}\left\{N(\xi) - \frac{(1-2\nu)}{2(1-\nu)} M(\xi)\right\}; r\right] = -\frac{G\Delta}{(1-\nu)}; \qquad 0 \le r \le a \tag{23}$$

$$H_1[M(\xi); r] = 0; \qquad 0 < r < b \tag{24}$$

$$H_0[N(\xi); r] = 0; \qquad a < r < b \tag{25}$$

$$H_1\left[\xi^{-1}\left\{M(\xi) - \frac{(1-2\nu)}{2(1-\nu)} N(\xi)\right\}; r\right] = 0; \qquad b \le r < \infty \tag{26}$$

$$H_0\left[\xi^{-1}\left\{N(\xi) - \frac{(1-2\nu)}{2(1-\nu)} M(\xi)\right\}; r\right] = 0; \qquad b \le r < \infty \tag{27}$$

The equations (23), (25) and (27) can also be written in the following form:

$$H_0[\xi^{-1}N(\xi); r] = \frac{(1-2\nu)}{2(1-\nu)} H_0[\xi^{-1}M(\xi); r] - \frac{G\Delta}{(1-\nu)} = G_1(r); \quad 0 \le r \le a \tag{28}$$

$$H_0[N(\xi); r] = 0; \qquad a < r < b \tag{29}$$

$$H_0[\xi^{-1}N(\xi); r] = \frac{(1-2\nu)}{2(1-\nu)} H_0[\xi^{-1}M(\xi); r] = G_2(r); \qquad b \le r < \infty \tag{30}$$

We assume that (29) admits a representation of the form

$$H_0[N(\xi); r] = \begin{cases} f_1(r); & 0 < r < a \\ f_2(r); & b < r < \infty \end{cases} \tag{31}$$

Using properties of the Hankel inversion theorem it can be shown that

$$N(\xi) = \left\{ \int_0^a r f_1(r) J_0(\xi r)\, dr + \int_b^\infty r f_2(r) J_0(\xi r)\, dr \right\}. \tag{32}$$

Selvadurai and Singh [10] have outlined a method of solution of the system of triple integral equations defined by (28), (29) and (30). Using the techniques outlined in [10] we can reduce the triple system to the following system of coupled integral equations

$$F_1(r) + \frac{2}{\pi} \int_b^\infty \frac{s F_2(s)\, ds}{(s^2 - r^2)} = g_1(r); \qquad 0 < r < a \tag{33}$$

$$F_2(r) + \frac{2r}{\pi} \int_0^a \frac{F_1(s)\, ds}{(r^2 - s^2)} = g_2(r); \qquad b < r < \infty \tag{34}$$

where

$$g_1(r) = \frac{d}{dr} \int_0^r \frac{t G_1(t)\, dt}{(r^2 - t^2)^{1/2}}; \qquad 0 < r < a \tag{35}$$

$$g_2(r) = -\frac{d}{dr} \int_r^\infty \frac{t G_2(t)\, dt}{(t^2 - r^2)^{1/2}}; \qquad b < r < \infty \tag{36}$$

$$F_1(s) = \int_s^a \frac{r f_1(r)\, dr}{(r^2 - s^2)^{1/2}}; \qquad 0 < s < a \tag{37}$$

$$F_2(s) = \int_b^s \frac{r f_2(r)\, dr}{(s^2 - r^2)^{1/2}}; \qquad b < s < \infty \tag{38}$$

Using the results (see e.g. Erdelyi et al. [11])

$$\int_0^r \frac{t J_0(\xi t)\, dt}{(r^2 - t^2)^{1/2}} = \frac{\sin(\xi r)}{\xi}; \qquad \int_r^\infty \frac{t J_0(\xi t)\, dt}{(t^2 - r^2)^{1/2}} = \frac{\cos(\xi r)}{\xi} \tag{39}$$

the expressions (35) and (36) can be reduced to the forms:

$$g_1(r) = -\frac{G\Delta}{(1-\nu)} + \frac{(1-2\nu)}{2(1-\nu)}\int_0^\infty M(\xi)\cos(\xi r)\,d\xi; \qquad 0 < r < a \tag{40}$$

$$g_2(r) = \frac{(1-2\nu)}{2(1-\nu)}\int_0^\infty M(\xi)\sin(\xi r)\,d\xi; \qquad b < r < \infty \tag{41}$$

The integral equations (24) and (26) can be written as

$$H_1[M(\xi); r] = 0; \qquad 0 < r < b \tag{42}$$

$$H_1[\xi^{-1}M(\xi); r] = \frac{(1-2\nu)}{2(1-\nu)} H_1[\xi^{-1}N(\xi); r] = L(r); \qquad b \le r < \infty \tag{43}$$

Assuming that

$$H_1[M(\xi); r] = g(r); \qquad b < r < \infty \tag{44}$$

and using the Hankel inversion theorem, the equation (43) can be reduced to the form

$$\int_b^\infty ug(u)\,du \int_0^\infty J_1(\xi u)J_1(\xi r)\,d\xi = \frac{(1-2\nu)}{2(1-\nu)}\int_0^\infty N(\xi)J_1(\xi r)\,d\xi \tag{45}$$

Using the result

$$\int_0^\infty J_1(\xi u)J_1(\xi r)\,d\xi = \frac{2ru}{\pi}\int_{\max(r,u)}^\infty \frac{d\omega}{\omega^2[(\omega^2-r^2)(\omega^2-u^2)]^{1/2}} \tag{46}$$

the equation (45) can be reduced to an integral equation of the Abel type, i.e.

$$\int_r^\infty \frac{S(\omega)\,d\omega}{\omega^2(\omega^2-r^2)^{1/2}} = \frac{\pi(1-2\nu)}{4r(1-\nu)}\int_0^\infty N(\xi)J_1(\xi r)\,d\xi; \qquad b < r < \infty \tag{47}$$

where

$$S(\omega) = \int_b^\omega \frac{u^2 g(u)\,du}{(\omega^2-u^2)^{1/2}}; \qquad b < \omega < \infty \tag{48}$$

The solution of (47) can be written as

$$S(r) = -\frac{(1-2\nu)r^2}{2(1-\nu)}\frac{d}{dr}\int_r^\infty \frac{d\omega}{(\omega^2-r^2)^{1/2}}\int_0^\infty N(\xi)J_1(\xi\omega)\,d\xi; \qquad b < r < \infty \tag{49}$$

Considering the results (37) and (38), the integral expression (32) for $N(\xi)$ can be written as

$$N(\xi) = \frac{2}{\pi}\left[\int_0^a F_1(s)\cos(\xi s)\,ds + \int_b^\infty F_2(s)\sin(\xi s)\,ds\right] \tag{50}$$

Substituting (50) in (49) and performing the integrations we obtain the following result:

$$S(r) = \frac{(1-2\nu)}{\pi(1-\nu)}\left[\frac{\pi}{2}\int_0^a F_1(s)\,ds - \int_b^\infty \left\{\frac{sr}{(s^2-r^2)} - \tfrac{1}{2}\ln\left|\frac{s+r}{s-r}\right|\right\} F_2(s)\,ds\right]; \quad b < r < \infty \quad (51)$$

The solution of the Abel integral equation (48) is given by

$$g(u) = \frac{2}{\pi u^2}\frac{d}{du}\int_b^u \frac{\omega S(\omega)\,d\omega}{(u^2-\omega^2)^{1/2}}; \qquad b < u < \infty \quad (52)$$

Considering (40), (41) and the inverse Hankel transform of (44) it can be shown that

$$g_1(r) = -\frac{G\Delta}{(1-\nu)} + \frac{(1-2\nu)}{2(1-\nu)}\int_b^\infty g(u)\,du; \qquad 0 < r < a \quad (53)$$

$$g_2(r) = \frac{r(1-2\nu)}{2(1-\nu)}\int_r^\infty \frac{g(u)\,du}{(u^2-r^2)^{1/2}}; \qquad b < r < \infty \quad (54)$$

Considering (51), (52) and the substitutions

$$F_1(r) = E\Delta T_1(r); \qquad F_2(r) = E\Delta T_2(r) \quad (55)$$

(where $E[=2G(1+\nu)]$; is Young's modulus) the integral equations (33) and (34) can be reduced to the following system of coupled Fredholm integral equations of the second-kind:

$$T_1(r) + \frac{2}{\pi}\int_b^\infty \frac{sT_2(s)\,ds}{(s^2-r^2)} - \frac{1}{4b}\left(\frac{1-2\nu}{1-\nu}\right)^2\int_0^a T_1(s)\,ds + \frac{1}{2\pi}\left(\frac{1-2\nu}{1-\nu}\right)^2\int_b^\infty T_2(s)\,ds\int_b^\infty\left\{\frac{s\omega}{(s^2-\omega^2)} - \tfrac{1}{2}\ln\left|\frac{s+\omega}{s-\omega}\right|\right\}\frac{d\omega}{\omega^2} = -\frac{1}{2(1-\nu^2)}; \qquad 0 < r < a \quad (56)$$

$$T_2(r) + \frac{2r}{\pi}\int_0^a \frac{T_1(s)\,ds}{(r^2-s^2)} - \frac{1}{4\pi b}\left(\frac{1-2\nu}{1-\nu}\right)^2\ln\left|\frac{b+r}{b-r}\right|\int_0^a T_1(s)\,ds + \frac{1}{2\pi^2 b}\left(\frac{1-2\nu}{1-\nu}\right)^2\ln\left|\frac{b+r}{b-r}\right|\int_b^\infty\left\{\frac{sb}{(s^2-b^2)} - \tfrac{1}{2}\ln\left|\frac{s+b}{s-b}\right|\right\}T_2(s)\,ds - \frac{1}{2\pi^2}\left(\frac{1-2\nu}{1-\nu}\right)^2\int_b^\infty sT_2(s)\left\{\frac{1}{(s^2-b^2)}\ln\left|\frac{b+r}{b-r}\right| + \frac{r}{(r^2-s^2)}\left[\frac{1}{s}\ln\left|\frac{b+s}{b-s}\right| - \frac{1}{r}\ln\left|\frac{b+r}{b-r}\right|\right]\right\}ds = 0; \qquad b < r < \infty \quad (57)$$

Integrating (56) with respect to r over the region $0 < r < a$, we obtain

$$\int_0^a T_1(t)\,dt = -\left[1 - \tfrac{1}{4}\left(\frac{1-2\nu}{1-\nu}\right)^2 \frac{a}{b}\right]^{-1} \left\{ \frac{a}{2(1-\nu^2)} + \frac{1}{\pi}\int_b^\infty \left[\ln\left|\frac{s+a}{s-a}\right| - \tfrac{1}{4}\left(\frac{1-2\nu}{1-\nu}\right)^2 \frac{a}{b} \ln\left|\frac{s+b}{s-b}\right|\right] T_2(s)\,ds \right. \tag{58}$$

Re-substituting (58) into (56), $T_1(r)$ can be expressed in terms of $T_2(s)$ in the form

$$T_1(r) = -\left[1 - \tfrac{1}{4}\left(\frac{1-2\nu}{1-\nu}\right)^2 \frac{a}{b}\right]^{-1} \left\{ \frac{1}{2(1-\nu^2)} + \frac{1}{4\pi b}\left(\frac{1-2\nu}{1-\nu}\right)^2 \int_b^\infty \left[\ln\left|\frac{s+a}{s-a}\right| - \ln\left|\frac{s+b}{s-b}\right|\right] T_2(s)\,ds \right\} \tag{59}$$

Substituting (58) and (59) into (57) and performing the integrations, we obtain the following single Fredholm integral equation of the second kind for $T_2(r)$:

$$\begin{aligned} T_2(r) &+ \frac{1}{\pi^2}\int_b^\infty \frac{2rs}{(r^2-s^2)} \left\{ \left[\frac{1}{r}\ln\left|\frac{r+a}{r-a}\right| - \frac{1}{s}\ln\left|\frac{s+a}{s-a}\right|\right] \right. \\ &\left. - \tfrac{1}{4}\left(\frac{1-2\nu}{1-\nu}\right)^2 \left[\frac{1}{r}\ln\left|\frac{r+b}{r-b}\right| - \frac{1}{s}\ln\left|\frac{s+b}{s-b}\right|\right] \right\} T_2(s)\,ds \\ &- \frac{1}{4\pi^2 b}\left(\frac{1-2\nu}{1-\nu}\right)^2 \left[1 - \tfrac{1}{4}\left(\frac{1-2\nu}{1-\nu}\right)^2 \frac{a}{b}\right]^{-1} \left[\ln\left|\frac{r+a}{r-a}\right| - \ln\left|\frac{r+b}{r-b}\right|\right] \cdot \\ &\cdot \int_b^\infty \left[\ln\left|\frac{s+a}{s-a}\right| - \ln\left|\frac{s+b}{s-b}\right|\right] T_2(s)\,ds \\ &= \frac{1}{2(1-\nu^2)}\left[1 - \tfrac{1}{4}\left(\frac{1-2\nu}{1-\nu}\right)^2 \frac{a}{b}\right]^{-1} \left\{ \ln\left|\frac{r+a}{r-a}\right| \right. \\ &\left. - \tfrac{1}{4}\left(\frac{1-2\nu}{1-\nu}\right)^2 \frac{a}{b} \ln\left|\frac{r+b}{r-b}\right| \right\} \end{aligned} \tag{60}$$

This formally completes the analysis of the problem related to the indentation of a debonded interface by a rigid circular punch. The solution of the integral equation (60) in turn will yield through (59) the solution for the function $T_1(r)$. By using the substitutions (55) and the integral equations (37) and (38) it is possible to obtain the solutions for $f_1(r)$ and $f_2(r)$; consequently the expressions $A(\xi)$ and $B(\xi)$ can be determined via equations (21) to (32). The Fredholm-type integral equation (60) can be solved, in a numerical fashion, to obtain results of engineering interest. As indicated previously, results of primary interest are the elastic load-displacement relationship for the rigid circular punch and the flaw opening mode stress intensity factor at the boundary of the debonded region. These results can be evaluated, in compact form, in terms of the functions $T_1(r)$ and $T_2(r)$.

Considering (25) we observe that

$$\sigma_{zz}(r, 0) = \int_0^\infty \xi N(\xi) J_0(\xi r)\, d\xi = \frac{1}{r}\frac{d}{dr}\left\{r \int_0^\infty N(\xi) J_1(\xi r)\, d\xi\right\} \tag{61}$$

Substituting the value of $N(\xi)$ defined by (32) in (61) we obtain

$$\sigma_{zz}(r, 0) = f_1(r); \qquad 0 < r < a \tag{62}$$

The force-displacement relationship for the rigid circular punch can be obtained by considering the equilibrium equation

$$P = -2\pi \int_0^a r f_1(r)\, dr \tag{63}$$

Considering the Abel-type integral equation (37) and the substitutions (55), the equation (63) can be reduced to the form

$$\frac{P(1-\nu)}{4Ga\Delta} = -\frac{2(1-\nu^2)}{a} \int_0^a T_1(s)\, ds \tag{64}$$

The analytical procedure based on the integral transform technique does not take into account the oscillatory singular stresses that can occur at the boundary of the delaminated region (see e.g. Williams [12], England [13], Erdogan [14] and Atkinson [15]). Such oscillatory singular behaviour is restricted to very narrow zones of the delaminated boundary and consequently they are expected to have virtually no influence on the evaluated stiffness of the smoothly indenting punch when $a/b < 1$. The influences of such oscillatory stress singularities may be important when dealing with the indentation problem where the punch exhibits an adhesive contact. As has been observed by Erdogan [14] the integral transform based solution has a regular square-root type non-oscillatory singularity at the boundary of the delaminated region. The physical limitations of overlapping material contact in the regions of oscillatory behaviour have, however, been open to criticism [16]. Purely for purposes of illustration we shall record here the flaw opening mode stress intensity factor as derived from the stress field which exhibits a "smeared" non-oscillatory singular behaviour at the boundary of the delaminated region. It can be shown that in the bonded region

$$\sigma_{zz}(r, 0) = f_2(r); \qquad b < r < \infty \tag{65}$$

Considering the Abel-type integral equation (38) we can reduce (65) to the form

$$\sigma_{zz}(r, 0) = \frac{2}{\pi}\left[\frac{F_2(b)}{(r^2-b^2)^{1/2}} + \int_b^r \frac{F_2(s)\, ds}{(r^2-s^2)^{1/2}}\right]; \qquad b < r < \infty \tag{66}$$

The flaw opening mode stress intensity factor at the boundary of the debonded region is given by

$$K_I^b = \lim_{r \to b+} [2(r-b)]^{1/2} \sigma_{zz}(r, 0) \tag{67}$$

Considering (55), (66) and (67) we have

$$\frac{K_I^b \sqrt{a}}{2G\Delta(1+\nu)} = \frac{2}{\pi} \left(\frac{a}{b}\right)^{1/2} T_2(b) \tag{68}$$

4. Numerical solution of the integral equation

We shall present here a brief outline of the numerical procedures which are used to solve the Fredholm integral equation of the second-kind defined by (60). By making use of the substitutions

$$T_2(r) = -\frac{\pi\eta^2\psi_2(\eta)}{2(1-\nu^2)} \tag{69}$$

$$r = \frac{b}{\eta}; \qquad s = \frac{b}{\xi}; \qquad c = \frac{a}{b} \tag{70}$$

$$C_1 = -\tfrac{1}{4}\left(\frac{1-2\nu}{1-\nu}\right)^2 \tag{71}$$

we can write (60) in the form

$$\pi^2\eta^2\psi_2(n) + \int_0^1 K(\eta, \xi)\psi_2(\xi)\, d\xi = R(\eta); \qquad 0 < \eta < 1 \tag{72}$$

where

$$R(\eta) = -\frac{1}{(1+C_1c)}\left[\ln\left|\frac{1+c\eta}{1-c\eta}\right| + C_1 c \ln\left|\frac{1+\eta}{1-\eta}\right|\right] \tag{73}$$

and

$$\begin{aligned} K(\eta, \xi) = {} & \frac{C_1}{(1+C_1c)}\left[\ln\left|\frac{1+\eta}{1-\eta}\right| - \ln\left|\frac{1+c\eta}{1-c\eta}\right|\right]\left[\ln\left|\frac{1+\xi}{1-\xi}\right| - \ln\left|\frac{1+c\xi}{1-c\xi}\right|\right] \\ & - \frac{2\xi\eta}{(\xi^2-\eta^2)}\left\{\left[\xi \ln\left|\frac{1+c\xi}{1-c\xi}\right| - \eta \ln\left|\frac{1+c\eta}{1-c\eta}\right|\right]\right. \\ & \left. - C_1\left[\xi \ln\left|\frac{1+\xi}{1-\xi}\right| - \eta \ln\left|\frac{1+\eta}{1-\eta}\right|\right]\right\} \end{aligned} \tag{74}$$

In order to develop a numerical solution for the integral equation (68), the integral is replaced by a quadrature formula and the resulting matrix equation for (68) with N points is

$$\mathbf{K}_{ij}\psi_2(\eta_j) = \mathbf{R}(\eta_i) \tag{75}$$

with $i, j = 1, 2, 3, \ldots, N$,

$$\eta_i = \tfrac{1}{2}(1 + g_i)$$

$$\mathbf{K}_{ij} = \begin{cases} \pi^2\eta_i^2 + L(\eta_i)\lambda_i/2; & i = j \\ K(\eta_i, \eta_j)\lambda_j/2; & i \neq j \end{cases} \tag{76}$$

where g_i and λ_i are the points and the weights of the quadrature scheme. The function $L(\eta_i)$ is obtained from $K(\eta_i, \eta_j)$ as a limiting case when $\eta_j \to \eta_i$, i.e.,

$$L(x) = C_1\left[x \ln\left|\frac{1+x}{1-x}\right| + \frac{2x}{(1-x^2)}\right] - \left[x \ln\left|\frac{1+cx}{1-cx}\right| + \frac{2cx^2}{(1-c^2x^2)}\right]$$
$$+ \frac{C_1}{(1+C_1c)}\left[\ln\left|\frac{1+x}{1-x}\right| - \ln\left|\frac{1+cx}{1-cx}\right|\right]^2 \tag{77}$$

Upon solution of the matrix equation (71), and using (58) and (64), the load on the rigid circular punch can be expressed in the form

$$\frac{P(1-\nu)}{4Ga\Delta} = \frac{1}{(1+C_1c)}\left\{1 - \sum_{i=1}^{N}\left[\frac{1}{c}\ln\left|\frac{1+c\eta_i}{1-c\eta_i}\right| + C_1\left|\frac{1+\eta_i}{1-\eta_i}\right|\right]\psi_2(\eta_i)\frac{\lambda_i}{2}\right\} \tag{78}$$

Similarly the stress intensity factor given by (67) can be approximated by

$$\frac{K_I^b\sqrt{a}}{2G\Delta(1+\nu)} = -\frac{\sqrt{c}\eta_l^2\psi_2(\eta_l)}{(1-\nu^2)} \tag{79}$$

where η_l is the point nearest to $\eta = 1$. It may be noted that a number of other schemes are available for the numerical solution of Fredholm integral equations of the second kind. These procedures are fully documented by Atkinson [17], Baker [18] and Delves and Walsh [19].

5. Numerical results and conclusions

The numerical procedure outlined in the preceding section is used to evaluate the load-displacement relationship for the rigid circular punch indenting the debonded boundary of the elastic medium and the flaw opening mode stress intensity factor at the boundary of the debonded region. The number, N, in the numerical procedure was varied until convergence of the numerical results was established. The Figure 2 (with $N = 16$) illustrates the manner in which the load-displacement response of

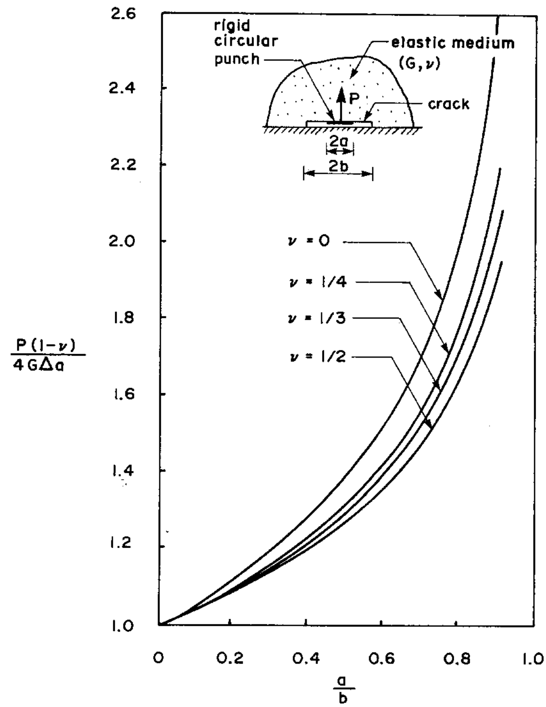

Figure 2
The axial stiffness of the rigid circular punch indenting a debonded interface.

the rigid circular punch is influenced by the Poisson's ratio of the elastic material and the ratio of the radius of the rigid circular punch to that of the debonded region. In the numerical evaluations this aspect ratio (a/b) is limited to the range $(0, 0.9)$. As is evident, when $(a/b) \to 1$, the displacement boundary conditions in the bonded region dominate and this contributes to a relatively large increase in the stiffness of the indenting punch.

The paper also records (Figure 3) the manner in which the flaw opening mode stress intensity factor at the boundary of the delaminated region is influenced by the aspect ratio a/b and Poisson's ratio ν. As has been pointed out, in these evaluations, the stress singularity at the boundary of the delaminated region exhibits a regular non-oscillatory behaviour. It may, however, be noted that in the limit of material incompressibility, the oscillatory behaviour in the stress field at the boundary of the delaminated region disappears and the results presented here for K_I will be the exact results.

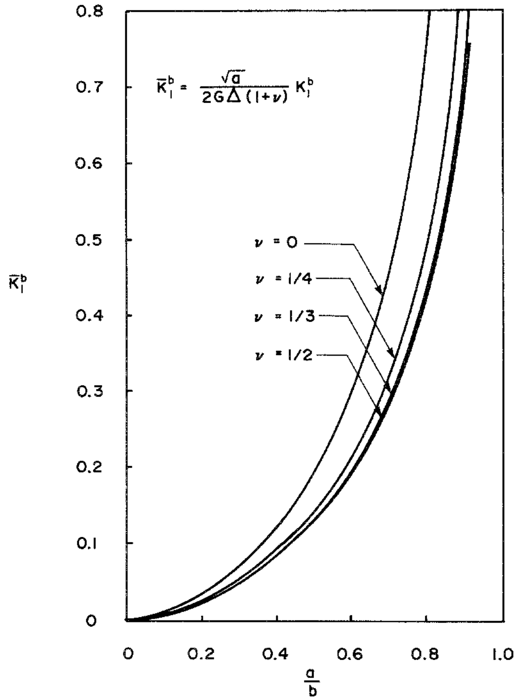

Figure 3
The stress intensity factor at the boundary of the debonded region.

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Abstract

The present paper examines the problem of the axial loading of a rigid circular punch which is located at the debonded boundary of an elastic medium-rigid boundary interface. The mathematical analysis of the problem focusses primarily on the evaluation of the load-displacement response for the punch boundary of the debonded region. The numerical results presented in the paper illustrate the manner in which the axial stiffness of the punch is influenced by the Poisson's ratio of the elastic medium and the relative dimensions of the debonded region.

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