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CLOSED-FORM SOLUTION FOR TWO FINITE LENGTH CRACKS MOVING IN AN ORTHOTROPIC LAYER UNDER ANTI-PLANE LOADING

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In the present paper, an integral transform method is used to obtain the solution to an elastodynamic crack problem. Exact expressions are derived for the anti-plane dynamic stress distribution around two collinear coplanar finite length cracks propagating with constant velocity in an infinitely long orthotropic layer of finite thickness. Two cases are investigated. In the first case, the lateral boundaries of the orthotropic layer, which are assumed to be clamped, are imposed equal displacements in opposite directions to produce anti-plane shear motion along the crack fronts. In the second case, the lateral boundaries of the layer are subjected to anti-plane shear loads. Using Fourier transforms, the analysis of each problem is reduced to resolving a system of triple integral equations. An analytical solution of these integral equations is obtained, leading to exact expressions for the stress intensity factors. Results are presented in graphical form corresponding to typical numerical values of the various parameters.

1. Introduction

The problem of the influence of crack propagation velocity on the stress intensity factor has been described by Sih [1]. Solutions for this complex elastodynamic problem have been sought under various assumptions. In particular, Sih and Chen [2] have discussed several problems relating to the dynamics of cracks located in a strip and obtained approximate solutions via Fredholm integral equations of the second kind. As shown by Sih [3], these solutions can be applied to practical cases of cracked composite materials.

The approach is based on the so-called Yoffé crack [4], in which crack length is kept constant in the propagation process. Justification and interpretation of this model can be found, for example, in [5] and [6]. The aim of this paper is to extend some of the solutions already available for this dynamic crack model. Indeed, in a recent paper by Singh, Moodie and Haddow [7], it has been shown, by means of integral transform techniques, that it is possible to obtain an analytical solution for the single crack moving in an elastic strip under anti-plane shear stress. Such a solution has also been obtained by Tait and Moodie [8,9] using complex variable techniques.

More recently, Danyluk and Singh [10] discussed the problem of the finite length crack propagating with constant velocity parallel to the faces of an orthotropic layer. Finally, Singh, Selvadurai and Danyluk [11] addressed the case of two cracks located at the central plane of an isotropic elastic layer embedded in bonded contact with two elastic half-space regions. The present paper extends the methods discussed in [7] and [10] to a case similar to [11], except for the important fact that the layer is orthotropic and that stress or displacement boundary conditions are imposed directly on the faces of the layer. These boundary conditions are such that they induce anti-plane fracture modes. Closed form results for this dynamic fracture problem within an orthotropic layer are believed to be of interest for the strength evaluation of some classes of composite materials as well as bonded joints.

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2. Basic equations

Consider a layer of orthotropic material occupying the region $-\infty < X < +\infty$, $-h \leqslant Y \leqslant h$, $-\infty < Z < +\infty$, where $OXYZ$ is a fixed rectangular coordinate system in which the X- and Z-axes are parallel to the principal, or natural, material directions (Fig. 1). Let us assume two identical semi-infinite cracks located in the XZ-plane with crack fronts parallel to the Z-axis. The purpose of this investigation is to determine the effect of material orthotropy and crack front velocity on the initial propagation of these cracks under anti-plane loading.

Displacement components in the X-, Y-, Z-directions are given respectively by:

$$U = V = 0, \qquad W = W(X, Y, t). \tag{1}$$

The corresponding stress field is such that

$$\sigma_X = \sigma_Y = \sigma_Z = \sigma_{XY} = 0, \qquad \sigma_{XZ} = c_{55}\frac{\partial W}{\partial X}, \qquad \sigma_{YZ} = c_{44}\frac{\partial W}{\partial Y}, \tag{2}$$

where $c_{44} = \mu_{YZ}$ and $c_{55} = \mu_{XZ}$ are the shear moduli in the YZ- and XZ-planes, respectively.

Satisfying the equations of motion leads to the classical two-dimensional wave equation:

$$\frac{\partial^2 W}{\partial X^2} + \frac{\partial^2 W}{\partial \bar{Y}^2} = \frac{1}{c_1^2}\frac{\partial^2 W}{\partial t^2}, \tag{3}$$

where

$$\bar{Y} = \frac{Y}{\sqrt{\beta}}, \qquad \beta = \frac{c_{44}}{c_{55}}, \qquad c_1 = \sqrt{\frac{c_{44}}{\rho}}. \tag{4}$$

c_1 is the shear wave speed corresponding to the Y-, Z-directions and ρ is the constant density of the material. Suppose now that a crack front is moving with constant velocity v in the X-direction. We use the Galilean transformation

$$x = X - vt, \qquad y = Y, \qquad z = Z, \qquad t' = t, \tag{5}$$

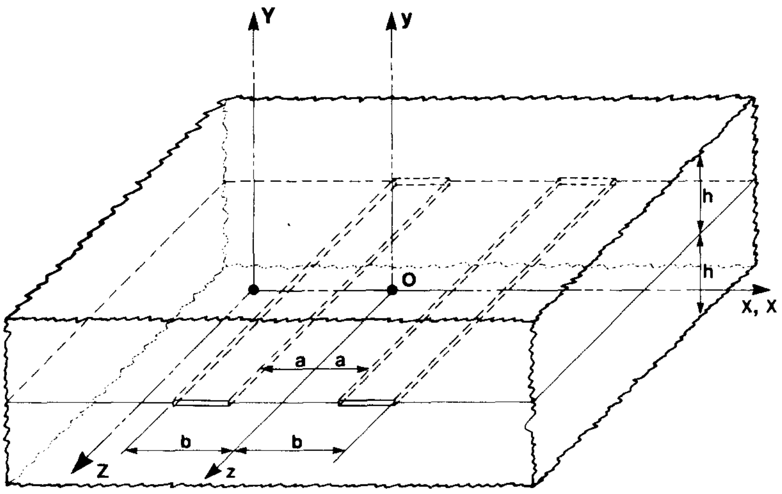

Fig. 1. Layer and cracks with coordinate systems.

where (x, y, z) represents the moving coordinate system shown in Fig. 1. With this transformation, the wave equation becomes independent of t' and eq. (3) reduces to

$$s^2\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial \bar{y}}=0, \tag{6}$$

where

$$s^2=1-\frac{v^2}{c_1^2}, \tag{7}$$

$$\bar{y}=\frac{y}{\sqrt{\beta}}. \tag{8}$$

The method of solution is illustrated by considering two types of boundary conditions.

Problem 1. Boundaries $Y=\pm h$ are displaced in opposite directions along the Z-axis by an amount $p(x)$ representing an even function, while both cracks are assumed to move in the X-direction at a constant velocity. Boundary conditions for this problem are as follows:

$$W(x, \pm h)=\pm p(x), \quad -\infty<x<+\infty, \tag{9}$$

$$\sigma_{yz}(x, 0)=0, \quad a<|x|<b, \tag{10}$$

$$W(x, 0)=0, \quad 0<|x|<a,\ b<|x|. \tag{11}$$

In order to use the integral transform technique, it is necessary to solve an alternative but equivalent problem. First, one solves the problem of the uncracked layer with clamped faces subjected to the following conditions:

$$\sigma_{yz}(x, 0)=-\frac{c_{44}}{h}p(x), \quad a<|x|<b, \tag{12}$$

$$W(x, 0)=0, \quad 0<|x|<a,\ b<|x|, \tag{13}$$

$$W(x, \pm h)=0, \quad -\infty<x<\infty. \tag{14}$$

The solution to the original problem is obtained by adding this equivalent problem solution to the following stress field:

$$\sigma_{yz}=\frac{c_{44}}{h}p(x), \qquad \sigma_{xz}=\frac{y}{h}c_{55}p'(x), \tag{15}$$

where a prime denotes differentiation with respect to x and $p''(x)=0$.

Problem 2. Boundaries $Y=\pm h$ are subjected to a shear in the Z-direction. The equivalent problem in this case is the application of a shear stress $-p(x)$ on the crack surface at $Y=0$. Boundary conditions can be written as follows:

$$\sigma_{yz}(x, 0)=-p(x), \quad a<|x|<b, \tag{16}$$

$$W(x, 0)=0, \quad 0<|x|<a,\ b<|x|, \tag{17}$$

$$\sigma_{yz}(x, \pm h)=0, \quad -\infty<x<\infty. \tag{18}$$

This second problem now consists in solving eq. (6) together with eqs. (16), (17) and (18).

A general solution to eq. (6) can be written as

$$W(x, y)=F_c\left[E_1(\xi)\,e^{-\xi ys}+E_2(\xi)\,e^{\xi ys};\ \xi\to x\right], \tag{19}$$

where s is the positive root of eq. (7) and function F_c is defined as:

$$F_c\left[E_1(\xi);\ \xi\to x\right]=\sqrt{\frac{2}{\pi}}\int_0^\infty E_1(\xi)\cos(\xi x)\,d\xi. \tag{20}$$

We also define

$$F_s[E_1(\xi);\ \xi\to x]=\sqrt{\frac{2}{\pi}}\int_0^\infty E_1(\xi)\sin(\xi x)\,d\xi. \tag{21}$$

Corresponding stresses are

$$\sigma_{yz}(x,\ y)=\frac{sc_{44}}{\sqrt{\beta}}F_c\left[\xi E_2(\xi)\,e^{\xi\bar{y}s}-\xi E_1(\xi)\,e^{-\xi\bar{y}s};\ \xi\to x\right], \tag{22}$$

$$\sigma_{xz}(x,\ y)=-c_{55}F_s\left[\xi E_1(\xi)\,e^{-\xi\bar{y}s}+\xi E_2(\xi)\,e^{\xi\bar{y}s};\ \xi\to x\right]. \tag{23}$$

Solution of Problem 1. Using eqs. (14) and (19), we find that

$$E_1(\xi)=\frac{-D(\xi)}{1-e^{-2s\xi\bar{h}}},\qquad E_2(\xi)=\frac{D(\xi)\,e^{-2\xi s\bar{h}}}{1-e^{-2\xi s\bar{h}}}, \tag{24}$$

where $\bar{h}=h/\sqrt{\beta}$. Equations (12) and (13) lead to the following triple integral equations:

$$F_c[D(\xi);\ \xi\to x]=0,\qquad 0<x<a, \tag{25}$$

$$F_c[\xi D(\xi)\coth(\xi s\bar{h});\ \xi\to x]=\frac{\sqrt{\beta}}{sh}p(x),\qquad a<x<b, \tag{26}$$

$$F_c[D(\xi);\ \xi\to x]=0,\qquad b<x, \tag{27}$$

where $D(\xi)$ is an unknown function yet to be determined. Let us assume a trial solution of the form

$$D(\xi)=\frac{1}{\xi}\sqrt{\frac{2}{\pi}}\int_a^b\phi(t)\sin(\xi t)\,dt, \tag{28}$$

where $\phi(t)$ is also an unknown function, and t henceforth represents a dummy integration variable.

It is clear that eqs. (25) and (27) will be satisfied if and only if

$$\int_a^b\phi(t)\,dt=0. \tag{29}$$

Equation (26) can be written as

$$\frac{d}{dx}F_s[D(\xi)\coth(\xi s\bar{h});\ \xi\to x]=\frac{\sqrt{\beta}}{sh}p(x),\qquad a<x<b. \tag{30}$$

Substituting $D(\xi)$ from eq. (28) into eq. (26) and using a relation from Gradshteyn and Ryzhik [12, 4.116 (3), p. 516]

$$\int_0^\infty\frac{1}{\xi}\sin(\xi x)\sin(\xi t)\coth(\xi s\bar{h})\,d\xi=\tfrac{1}{2}\log\left|\frac{\tanh(cx)+\tanh(ct)}{\tanh(cx)-\tanh(ct)}\right|, \tag{31}$$

where

$$c=\pi/2s\bar{h}, \tag{32}$$

we obtain

$$\int_a^b\frac{\phi(t)\sinh(2ct)}{\cosh(2ct)-\cosh(2cx)}\,dt=s\bar{h}R(x),\qquad a<x<b, \tag{33}$$

with

$$R(x)=\frac{\sqrt{\beta}}{sh}p(x). \tag{34}$$

Using the suitably modified Tricomi theorem proposed by Singh [13], we find that the solution of integral equation (33) is given by

$$\phi(t) = -\frac{2c}{\pi}\sqrt{\frac{\cosh(2ct)-\cosh(2ca)}{\cosh(2cb)-\cosh(2ct)}}$$
$$\times\int_a^b \sqrt{\frac{\cosh(2cb)-\cosh(2cx)}{\cosh(2cx)-\cosh(2ca)}}\,\frac{R(x)\sinh(2cx)}{\cosh(2cx)-\cosh(2ct)}\,\mathrm{d}x$$
$$+\frac{2c^2C_1}{\sqrt{(\cosh(2ct)-\cosh(2ca))(\cosh(2cb)-\cosh(2ct))}},\quad a<t<b, \tag{35}$$

where C_1 is an arbitrary constant. Using eqs. (29) and (35), we get

$$C_1 = \frac{2\sinh(cb)\cosh(ac)}{\pi F}\int_a^b \sqrt{\frac{\cosh(2ct)-\cosh(2ca)}{\cosh(2cb)-\cosh(2ct)}}\,\mathrm{d}t$$
$$\times\int_a^b \sqrt{\frac{\cosh(2cb)-\cosh(2cx)}{\cosh(2cx)-\cosh(2ca)}}\,\frac{R(x)\sinh(2cx)}{\cosh(2cx)-\cosh(2ct)}\,\mathrm{d}x, \tag{36}$$

where $F=F(\frac{1}{2}\pi,\ r)$ is an elliptic integral of the first kind and

$$r=\frac{\sqrt{\sinh^2(cb)-\sinh^2(ca)}}{\sinh(cb)\cosh(ca)}. \tag{37}$$

The following identity holds

$$\left\{\frac{[\cosh(2ct)-\cosh(2ca)][\cosh(2cb)-\cosh(2cy)]}{[\cosh(2cb)-\cosh(2ct)][\cosh(2cy)-\cosh(2ca)]}\right\}^{1/2}\left\{1+\frac{\cosh(2cy)-\cosh(2ct)}{\cosh(2ct)-\cosh(2ca)}\right\}$$
$$=\left\{\frac{[\cosh(2cb)-\cosh(2ct)][\cosh(2cy)-\cosh(2ca)]}{[\cosh(2ct)-\cosh(2ca)][\cosh(2cb)-\cosh(2cy)]}\right\}^{1/2}\left\{1-\frac{\cosh(2cy)-\cosh(2ct)}{\cosh(2cb)-\cosh(2ct)}\right\}. \tag{38}$$

Using this expression, eqs. (35) and (36) may now be written in the alternative form

$$\phi(t) = \frac{-2c}{\pi}\sqrt{\frac{\cosh(2cb)-\cosh(2ct)}{\cosh(2ct)-\cosh(2ca)}}$$
$$\times\int_a^b \sqrt{\frac{\cosh(2cx)-\cosh(2ca)}{\cosh(2cb)-\cosh(2cx)}}\,\frac{R(x)\sinh(2cx)}{\cosh(2cx)-\cosh(2ct)}\,\mathrm{d}x$$
$$+\frac{2c^2C_2}{\sqrt{(\cosh(2ct)-\cosh(2ca))(\cosh(2cb)-\cosh(2ct))}},\quad a<t<b, \tag{39}$$

where

$$C_2 = \frac{2\sinh(cb)\cosh(ca)}{\pi F}\int_a^b \sqrt{\frac{\cosh(2cb)-\cosh(2ct)}{\cosh(2ct)-\cosh(2ca)}}\,\mathrm{d}t$$
$$\times\int_a^b \sqrt{\frac{\cosh(2cx)-\cosh(2ca)}{\cosh(2cb)-\cosh(2cx)}}\,\frac{R(x)\sinh(2cx)}{\cosh(2cx)-\cosh(2ct)}\,\mathrm{d}x. \tag{40}$$

The stress component $\sigma_{yz}(x, 0)$ has to be calculated in order to obtain stress intensity factors. From eq. (30), it follows that

$$\sigma_{yz}(x, 0) = \frac{sc_{44}}{\sqrt{\beta}} \frac{\mathrm{d}}{\mathrm{d}x} F_s\left[\coth(\xi s\bar{h}) D(\xi);\ \xi \to x\right]. \tag{41}$$

And, using eq. (28), we get

$$\left[\sigma_{yz}(x, 0)\right]_{0<x<a} = \frac{-2scc_{44}}{\pi\sqrt{\beta}} \int_a^b \frac{\phi(t) \sinh(2ct)}{\cosh(2ct) - \cosh(2cx)} \mathrm{d}t, \tag{42}$$

$$\left[\sigma_{yz}(x, 0)\right]_{x>b} = \frac{2scc_{44}}{\pi\sqrt{\beta}} \int_a^b \frac{\phi(t) \sinh(2ct)}{\cosh(2cx) - \cosh(2ct)} \mathrm{d}t. \tag{43}$$

Substituting eqs. (39) and (35), respectively, into the above equations, we obtain

$$\begin{aligned}\left[\sigma_{yz}(x, 0)\right]_{0<x<a} = {}& \frac{2csc_{44}}{\pi\sqrt{\beta}} \sqrt{\frac{\cosh(2cb) - \cosh(2cx)}{\cosh(2ca) - \cosh(2cx)}} \\ & \times \int_a^b \sqrt{\frac{\cosh(2ct) - \cosh(2ca)}{\cosh(2cb) - \cosh(2ct)}} \frac{R(t) \sinh(2ct)}{\cosh(2ct) - \cosh(2cx)} \mathrm{d}t \\ & - \left(\frac{sc_{44}}{\sqrt{\beta}}\right) \frac{2c^2 C_2}{\sqrt{(\cosh(2cx) - \cosh(2ca))(\cosh(2cb) - \cosh(2cx))}},\end{aligned} \tag{44}$$

and

$$\begin{aligned}\left[\sigma_{yz}(x, 0)\right]_{x>b} = {}& \frac{2csC_{44}}{\pi\sqrt{\beta}} \sqrt{\frac{\cosh(2cx) - \cosh(2ca)}{\cosh(2cx) - \cosh(2cb)}} \\ & \times \int_a^b \sqrt{\frac{\cosh(2cb) - \cosh(2ct)}{\cosh(2ct) - \cosh(2ca)}} \frac{R(t) \sinh(2ct)}{\cosh(2cx) - \cosh(2ct)} \mathrm{d}t \\ & + \left(\frac{sc_{44}}{\sqrt{\beta}}\right) \cdot \frac{2c^2 C_1}{\sqrt{(\cosh(2cx) - \cosh(2ca))(\cosh(2cx) - \cosh(2cb))}},\end{aligned} \tag{45}$$

where we have used the following integral:

$$\begin{aligned}&\int_a^b \frac{\sinh(2cu)}{\left[\cosh(2cu) - \cosh(2ct)\right]\sqrt{(\cosh(2cu) - \cosh(2ca))(\cosh(2cb) - \cosh(2cu))}} \mathrm{d}u \\ &= \begin{cases} s\bar{h}/\sqrt{(\cosh(2ca) - \cosh(2ct))(\cosh(2cb) - \cosh(2ct))}, & 0 < t < a, \\ 0, & a < t < b, \\ -s\bar{h}/\sqrt{(\cosh(2ct) - \cosh(2ca))(\cosh(2ct) - \cosh(2cb))}. & b < t. \end{cases}\end{aligned} \tag{46}$$

Stress intensity factors for this type of loading at both crack tips are defined by

$$K_{3a} = \lim_{x \to a^-} \sqrt{2(a - x)} \left[\sigma_{yz}(x, 0)\right]_{0<x<a}, \tag{47}$$

$$K_{3b} = \lim_{x \to b^+} \sqrt{2(x - b)} \left[\sigma_{yz}(x, 0)\right]_{x>b}. \tag{48}$$

Using eqs. (44) and (45), these equations can be written as

$$K_{3a} = \frac{sc_{44}}{\sqrt{\beta}} \frac{\sqrt{2c}}{\pi} \frac{1}{\sqrt{\sinh(2ca)(\sinh^2(cb) - \sinh^2(ca))}}$$

$$\times \left\{ [\sinh^2(cb) - \sinh^2(ca)] \right.$$

$$\left. \times \int_a^b \frac{R(t)\sinh(2ct)}{\sqrt{(\sinh^2(cb) - \sinh^2(ct))(\sinh^2(ct) - \sinh^2(ca))}} \mathrm{d}t - \pi c C_2 \right\}, \tag{49}$$

$$K_{3b} = \frac{sc_{44}}{\sqrt{\beta}} \frac{\sqrt{2c}}{\pi} \frac{1}{\sqrt{\sinh(2cb)(\sinh^2(cb) - \sinh^2(ca))}}$$

$$\times \left\{ [\sinh^2(cb) - \sinh^2(ca)] \int_a^b \frac{R(t)\sinh(2ct)}{\sqrt{(\sinh^2(cb) - \sinh^2(ct))(\sinh^2(ct) - \sinh^2(ca))}} \mathrm{d}t \right.$$

$$\left. + \pi c C_1 \right\}. \tag{50}$$

We consider now the particular case where

$$p(x) \equiv \sigma_0 \quad \text{(constant)}. \tag{51}$$

From eqs. (36), (40) and (51), we find

$$C_1 = -\frac{\sigma_0\sqrt{\beta}\sinh(cb)\cosh(ca)}{shcF} \int_a^b \sqrt{\frac{\cosh(2ct) - \cosh(2ca)}{\cosh(2cb) - \cosh(2ct)}} \mathrm{d}t, \tag{52}$$

$$C_2 = \frac{\sigma_0\sqrt{\beta}\sinh(cb)\cosh(ca)}{shcF} \int_a^b \sqrt{\frac{\cosh(2cb) - \cosh(2ct)}{\cosh(2ct) - \cosh(2ca)}} \mathrm{d}t. \tag{53}$$

And, from eqs. (34), (49)–(53), it follows that

$$\frac{hK_{3a}}{\sigma_0 c_{44}} = \sqrt{2/c \sinh(2ca)(\sinh^2(cb) - \sinh^2(ca))}$$

$$\times \left[\sinh^2(cb) - \sinh^2(ca) - \frac{c\sinh(cb)\cosh(ca)}{F} \int_a^b \sqrt{\frac{\cosh(2cb) - \cosh(2ct)}{\cosh(2ct) - \cosh(2ca)}} \mathrm{d}t \right], \tag{54}$$

$$\frac{hK_{3b}}{\sigma_0 c_{44}} = \sqrt{2/c \sinh(2cb)(\sinh^2(cb) - \sinh^2(ca))}$$

$$\times \left[\sinh^2(cb) - \sinh^2(ca) - \frac{c\sinh(cb)\cosh(ca)}{F} \int_a^b \sqrt{\frac{\cosh(2ct) - \cosh(2ca)}{\cosh(2cb) - \cosh(2ct)}} \mathrm{d}t \right]. \tag{55}$$

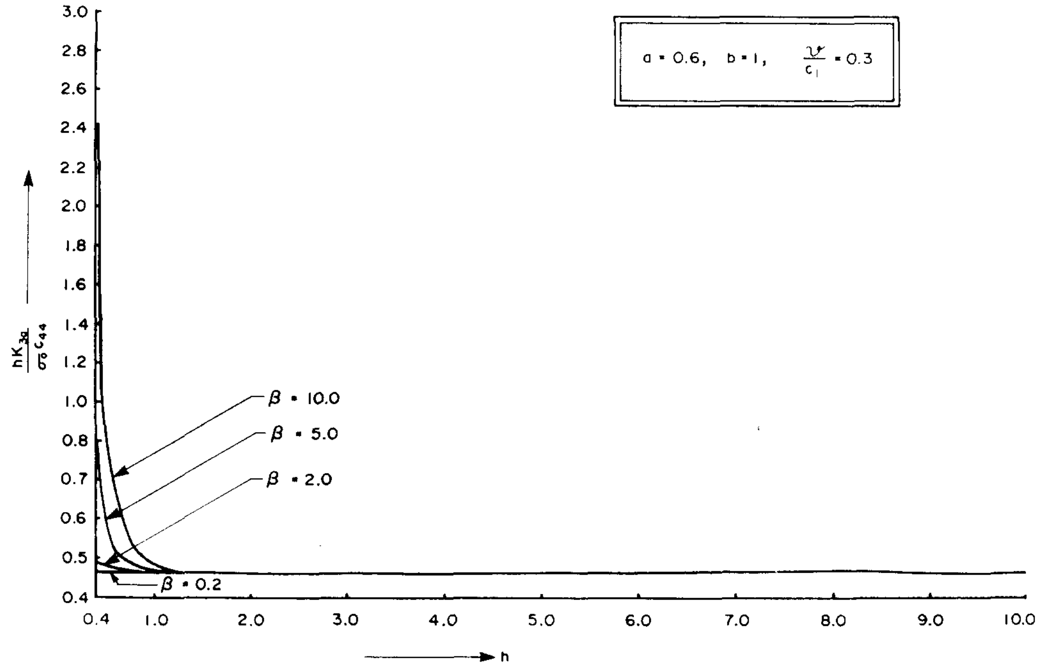

Fig. 2. Stress intensity factor versus layer thickness. Problem 1, inner crack front, $a = 0.6$.

Letting $a \to 0$, we find from eq. (55)

$$K_{3b} = \frac{c_{44}\sigma_0}{h}\sqrt{\frac{\tanh(cb)}{c}}, \tag{56}$$

which agrees with the expression given by Danyluk and Singh [10] for the stress intensity factor at the leading front of a single crack. Equations (54) and (55) have been used to calculate values of $(hK_{3a})/(\sigma_0 c_{44})$ and $(hK_{3b}/\sigma_0 c_{44})$. Some numerical results are shown in Figs. 2–5 corresponding to several values of the anisotropy ratio β and one value of crack front velocity $v/c_1 = 0.3$.

Solution of Problem 2. Using eq. (18), we find that

$$E_1(\xi) = \frac{E(\xi)}{1 + e^{-2\xi s\bar{h}}}, \tag{57}$$

$$E_2(\xi) = \frac{E(\xi)\, e^{-2\xi s\bar{h}}}{1 + e^{-2\xi s\bar{h}}}. \tag{58}$$

Equations (16) and (17) will be satisfied by $E(\xi)$, a solution of the triple integral equations

$$F_c[E(\xi);\ \xi \to x] = 0, \qquad 0 < x < a, \tag{59}$$

$$\frac{d}{dx}F_s[\tanh(\xi s\bar{h})E(\xi);\ \xi \to x] = \frac{\sqrt{\beta}}{sc_{44}}p(x), \qquad a < x < b, \tag{60}$$

$$F_c[E(\xi);\ \xi \to x] = 0, \qquad b < x. \tag{61}$$

Following the same approach as for eqs. (25), (26), (27), a solution for eqs. (59), (60), (61) can be expressed as

$$E(\xi) = \frac{1}{\xi}\sqrt{\frac{2}{\pi}}\int_a^b \phi(t)\cosh(ct)\sin(\xi t)\,dt, \tag{62}$$

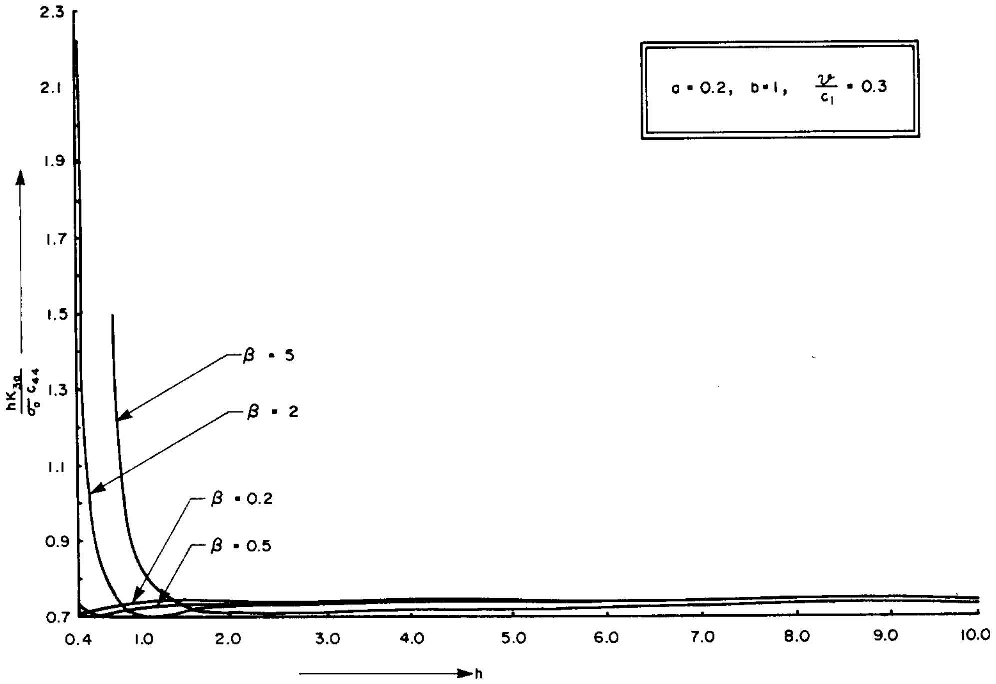

Fig. 3. Stress intensity factor versus layer thickness. Problem 1, inner crack front, $a = 0.2$.

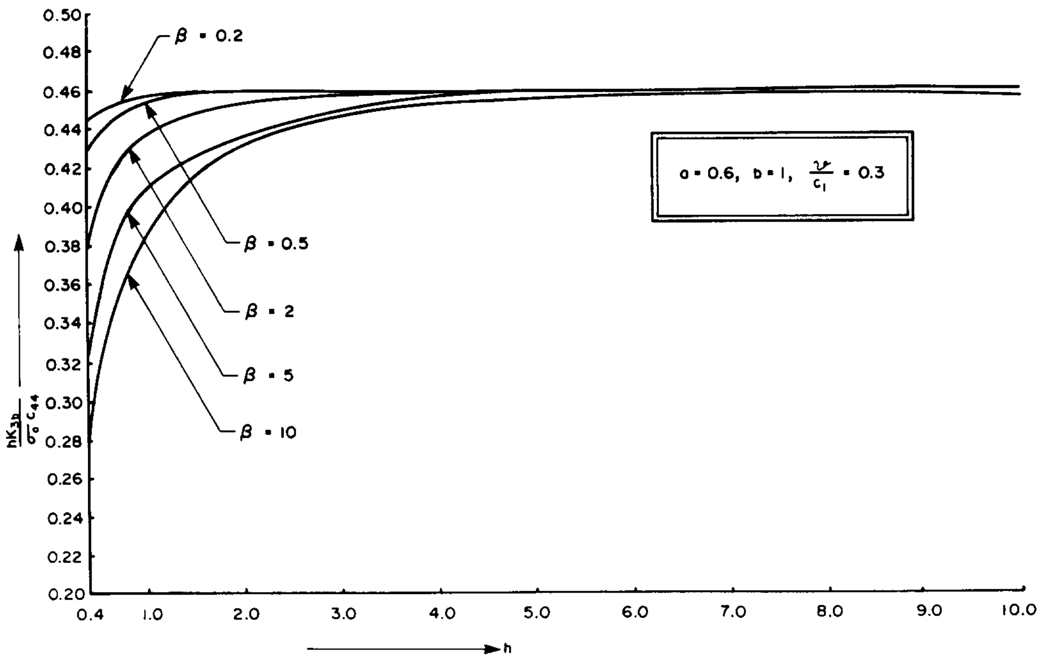

Fig. 4. Stress intensity factor versus layer thickness. Problem 1, outer crack front, $a = 0.6$.

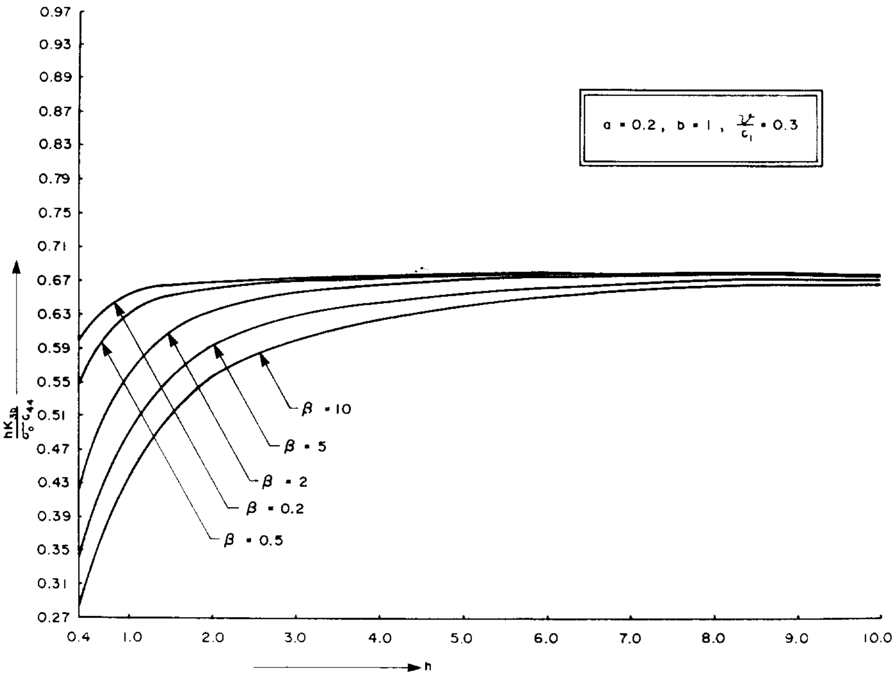

Fig. 5. Stress intensity factor versus layer thickness. Problem 1, outer crack front, $a = 0.2$.

where

$$\phi(t) = -\frac{2c\sqrt{\beta}}{\pi s c_{44}}\sqrt{\frac{\sinh^2(ct)-\sinh^2(ca)}{\sinh^2(cb)-\sinh^2(ct)}}$$

$$\times\int_a^b\sqrt{\frac{\sinh^2(cb)-\sinh^2(cx)}{\sinh^2(cx)-\sinh^2(ca)}}\,\frac{p(x)\sinh(cx)}{\sinh^2(cx)-\sinh^2(ct)}\,\mathrm{d}x$$

$$+\frac{c^2C_1}{\sqrt{(\sinh^2(ct)-\sinh^2(ca))(\sinh^2(ca)-\sinh^2(ct))}},\quad a<t<b, \tag{63}$$

$$C_1 = \frac{2\sqrt{\beta}\,\sinh(cb)}{\pi s F_1 c_{44}}\int_a^b\sqrt{\frac{\sinh^2(cb)-\sinh^2(cu)}{\sinh^2(cu)-\sinh^2(ca)}}\,p(u)\sinh(cu)\,\mathrm{d}u$$

$$\times\int_a^b\sqrt{\frac{\sinh^2(ct)-\sinh^2(ca)}{\sinh^2(cb)-\sinh^2(ct)}}\,\frac{\cosh(ct)}{\sinh^2(cu)-\sinh^2(ct)}\,\mathrm{d}t, \tag{64}$$

$$F_1 = F\left(\frac{\pi}{2}, r_1\right),\qquad r_1 = \frac{\sqrt{\sinh^2(cb)-\sinh^2(ca)}}{\sinh(cb)}, \tag{65}$$

and F_1 is an elliptic integral of the second kind. To obtain the above solution, we have used the following integral:

$$\int_0^\infty\frac{1}{\xi}\tanh(\xi s\bar{h})\sin(\xi x)\sin(\xi t)\,\mathrm{d}\xi = \tfrac{1}{2}\log\left|\frac{\sinh(cx)+\sinh(ct)}{\sinh(cx)-\sinh(ct)}\right|, \tag{66}$$

where, again, $c = \pi/2s\bar{h}$, which can be obtained from Gradshteyn and Ryzhik [12, 4.116(2), p. 516]. Using identity (38), eqs. (63) and (64) can be written as

$$\phi(t) = -\frac{2c}{\pi}\frac{\sqrt{\beta}}{sc_{44}}\sqrt{\frac{\sinh^2(cb)-\sinh^2(ct)}{\sinh^2(ct)-\sinh^2(ca)}}$$
$$\times\int_a^b\sqrt{\frac{\sinh^2(cx)-\sinh^2(ca)}{\sinh^2(cb)-\sinh^2(cx)}}\frac{p(x)\sinh(cx)}{\sinh^2(cx)-\sinh^2(ct)}\mathrm{d}x$$
$$+\frac{c^2C_2}{\sqrt{(\sinh^2(ct)-\sinh^2(ca))(\sinh^2(cb)-\sinh^2(ct))}},\quad a<t<b, \tag{67}$$

$$C_2 = \left(\frac{2}{\pi}\right)\frac{\sqrt{\beta}\sinh(cb)}{sc_{44}F_1}\int_a^b\sqrt{\frac{\sinh^2(cb)-\sinh^2(ct)}{\sinh^2(ct)-\sinh^2(ca)}}\cosh(ct)\,\mathrm{d}t$$
$$\times\int_a^b\sqrt{\frac{\sinh^2(cu)-\sinh^2(ca)}{\sinh^2(cb)-\sinh^2(cu)}}\frac{\sinh(cu)p(u)}{\sinh^2(cu)-\sinh^2(ct)}\mathrm{d}u. \tag{68}$$

The shear stress component σ_{yz} is given by

$$\sigma_{yz}(x,0) = -\frac{sc_{44}}{\sqrt{\beta}}\frac{\mathrm{d}}{\mathrm{d}x}F_s\left[\tanh(\xi s\bar{h})E(\xi);\ \xi\to x\right], \tag{69}$$

which, using eqs. (62) and (69), can be expressed as

$$\sigma_{yz}(x,0) = -\frac{csc_{44}\cosh(cx)}{\pi\sqrt{\beta}}\int_a^b\frac{\phi(t)\sinh(2ct)}{\sinh^2(ct)-\sinh^2(cx)}\mathrm{d}t,\quad 0<x<a, \tag{70}$$

$$\sigma_{yz}(x,0) = \frac{csc_{44}\cosh(cx)}{\pi\sqrt{\beta}}\int_a^b\frac{\phi(t)\sinh(2ct)}{\sinh^2(cx)-\sinh^2(ct)}\mathrm{d}t,\quad b<x. \tag{71}$$

Substituting $\phi(t)$ from eqs. (63) and (67) into eqs. (70) and (71), we get

$$\sigma_{yz}(x,0) = \frac{2c}{\pi}\cosh(cx)\sqrt{\frac{\sinh^2(cb)-\sinh^2(cx)}{\sinh^2(ca)-\sinh^2(cx)}}$$
$$\times\int_a^b\sqrt{\frac{\sinh^2(cu)-\sinh^2(ca)}{\sinh^2(cb)-\sinh^2(cu)}}\frac{\sinh(cu)p(u)}{\sinh^2(cu)-\sinh^2(cx)}\mathrm{d}u$$
$$-\frac{c^2C_2sc_{44}\cosh(cx)}{\sqrt{\beta(\sinh^2(ca)-\sinh^2(cx))(\sinh^2(cb)-\sinh^2(cx))}},\quad 0<x<a, \tag{72}$$

and

$$\sigma_{yz}(x,0) = \frac{2c}{\pi}\cosh(cx)\sqrt{\frac{\sinh^2(cx)-\sinh^2(ca)}{\sinh^2(cx)-\sinh^2(cb)}}$$
$$\times\int_a^b\sqrt{\frac{\sinh^2(cb)-\sinh^2(cu)}{\sinh^2(cu)-\sinh^2(ca)}}\frac{p(u)\sinh(cu)}{\sinh^2(cx)-\sinh^2(cu)}\mathrm{d}u$$
$$+\frac{c^2C_1sc_{44}\cosh(cx)}{\sqrt{\beta(\sinh^2(cx)-\sinh^2(ca))(\sinh^2(cx)-\sinh^2(cb))}},\quad b<x. \tag{73}$$

Here again, stress intensity factors are defined as

$$K_{3a}=\lim_{x\to a^-}\sqrt{2(a-x)}\left[\sigma_{yz}(x,0)\right]_{0<x<a}, \tag{74}$$

$$K_{3b}=\lim_{x\to b^+}\sqrt{2(x-b)}\left[\sigma_{yz}(x,0)\right]_{b<x}. \tag{75}$$

Substituting the value of $\sigma_{xz}(x, 0)$ from eqs. (72) and (73) into eqs. (74), (75) yields

$$K_{3a}=\frac{2\sqrt{2c}}{\pi}\frac{\cosh(ca)}{\sqrt{\sinh(2ca)(\sinh^2(cb)-\sinh^2(ca))}}\times\left\{(\sinh^2(cb)-\sinh^2(ca))\times\int_a^b\frac{p(u)\sinh(cu)}{\sqrt{(\sinh^2(cb)-\sinh^2(cu))(\sinh^2(cu)-\sinh^2(ca))}}\,\mathrm{d}u-\frac{\pi scc_{44}C_2}{2\sqrt{\beta}}\right\}, \tag{76}$$

$$K_{3b}=\frac{2\sqrt{2c}}{\pi}\cdot\frac{\cosh(cb)}{\sqrt{\sinh(2cb)(\sinh^2(cb)-\sinh^2(ca))}}\times\left\{(\sinh^2(cb)-\sinh^2(ca))\times\int_a^b\frac{p(u)\sinh(cu)}{\sqrt{(\sinh^2(cb)-\sinh^2(cu))(\sinh^2(cu)-\sinh^2(ca))}}\,\mathrm{d}u+\frac{\pi scc_{44}C_1}{2\sqrt{\beta}}\right\}. \tag{77}$$

Considering again the particular case in which $p(x)\equiv\sigma_0$ (constant), eqs. (76) and (77) yield

$$K_{3a}=\frac{2}{\pi}\sqrt{\frac{2}{c}}\frac{\cosh(ca)}{\sqrt{\sinh(2ca)(\sinh^2(cb)-\sinh^2(ca))}}\left\{\frac{\sinh^2(cb)-\sinh^2(ca)}{\cosh(cb)}F_2\sigma_0-\frac{\pi sc^2c_{44}C_2}{2\sqrt{\beta}}\right\}, \tag{78}$$

$$K_{3b}=\frac{2}{\pi}\sqrt{\frac{2}{c}}\frac{\cosh(cb)}{\sqrt{\sinh(2cb)(\sinh^2(cb)-\sinh^2(ca))}}\left\{\frac{\sinh^2(cb)-\sinh^2(ca)}{\cosh(cb)}F_2\sigma_0+\frac{\pi sc^2c_{44}C_1}{2\sqrt{\beta}}\right\}, \tag{79}$$

where

$$F_2=F\left(\tfrac{1}{2}\pi,\,r_2\right),\qquad r_2=\frac{\sqrt{\cosh^2(cb)-\cosh^2(ca)}}{\cosh(cb)}, \tag{80}$$

and F_2 is an elliptic integral of the second kind. Corresponding expressions for constants C_1 and C_2 are as follows:

$$C_1 = -\frac{2\sqrt{\beta}\,\sigma_0}{\pi c s c_{44} F_1} \int_a^b \sqrt{\frac{\sinh^2(cb) - \sinh^2(cu)}{\sinh^2(cu) - \sinh^2(ca)}}\,\sinh(cu)\left[F_1 + F_3\frac{\sinh^2(cu) - \sinh^2(ca)}{\sinh^2(cb) - \sinh^2(cu)}\right]\mathrm{d}u, \tag{81}$$

$$C_2 = \frac{2\sqrt{\beta}\,\sigma_0}{\pi c s c_{44} F_1}\tanh(cb) \int_a^b \sqrt{\frac{\sinh^2(cb) - \sinh^2(cu)}{\sinh^2(cu) - \sinh^2(ca)}} \times \cosh(cu)\left[F_2 + F_4\frac{\sinh^2(cu) - \sinh^2(ca)}{\sinh^2(cb) - \sinh^2(cu)}\right]\mathrm{d}u, \tag{82}$$

where

$$F_3 = F(\tfrac{1}{2}\pi,\ r_3,\ r_1), \qquad F_4 = F(\tfrac{1}{2}\pi,\ r_3,\ r_4),$$
$$r_3 = \frac{\sinh^2(cb) - \sinh^2(ca)}{\sinh^2(cb) - \sinh^2(cu)}, \qquad r_4 = \frac{\sqrt{\sinh^2(cb) - \sinh^2(ca)}}{\cosh(cb)}, \tag{83}$$

and F_3, F_4 are elliptic integrals of the third kind.

For the particular case when we let $a \to 0$, we find that $C_1 = 0$ and, from eq. (79) we get

$$K_{3b} = \frac{2\sigma_0}{\pi}\sqrt{\frac{\tanh(cb)}{c}}\,F(\tfrac{1}{2}\pi,\ \tanh(cb)), \tag{84}$$

which agrees with the result of Danyluk and Singh [10] for a single crack. Equations (78) and (79) have been used to calculate values of stress intensity ratios K_{3a}/σ_0 and K_{3b}/σ_0. Some results are shown

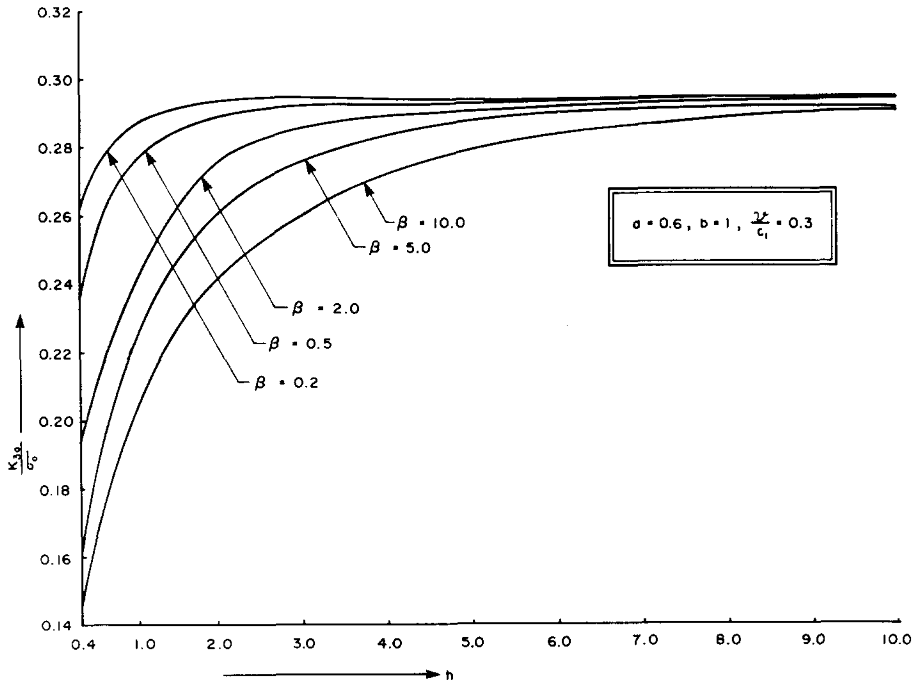

Fig. 6. Stress intensity factor versus layer thickness. Problem 2, inner crack front, $a = 0.6$.

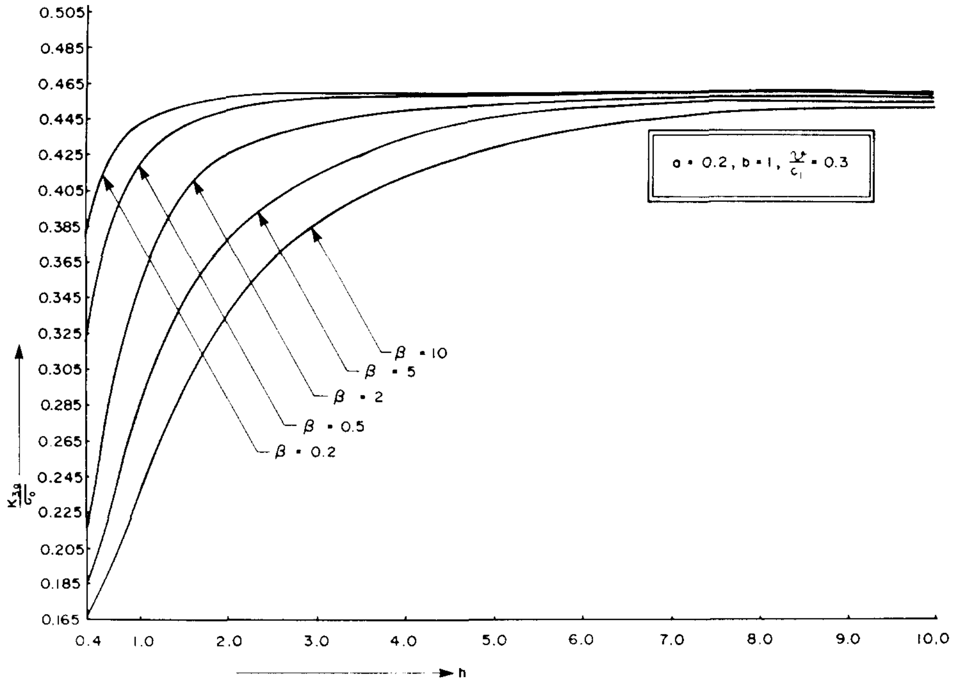

Fig. 7. Stress intensity factor versus layer thickness. Problem 2, inner crack front, $a = 0.2$.

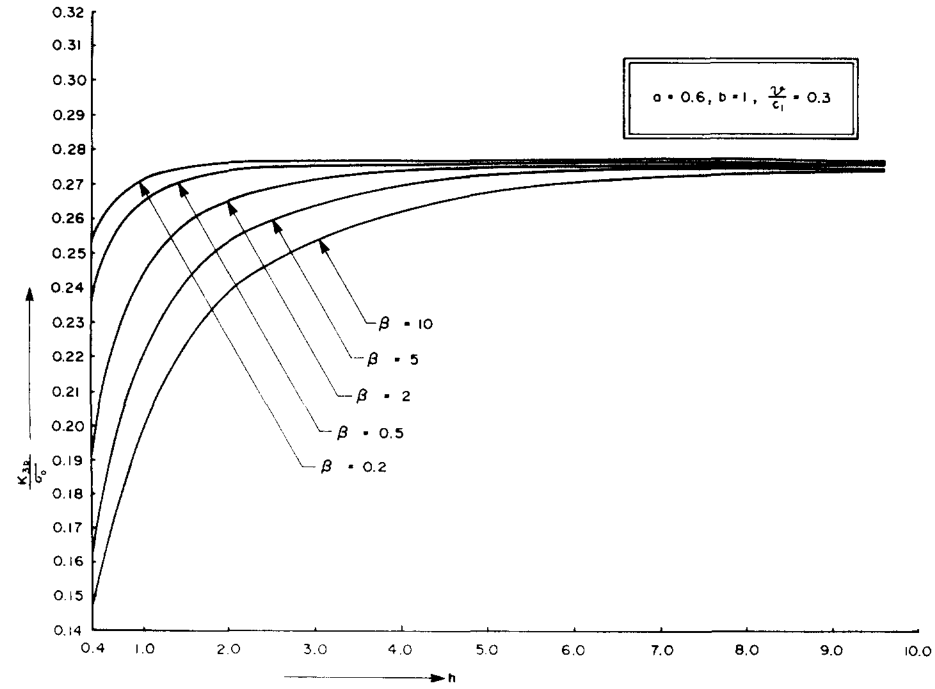

Fig. 8. Stress intensity factor versus layer thickness. Problem 2, outer crack front, $a = 0.6$.

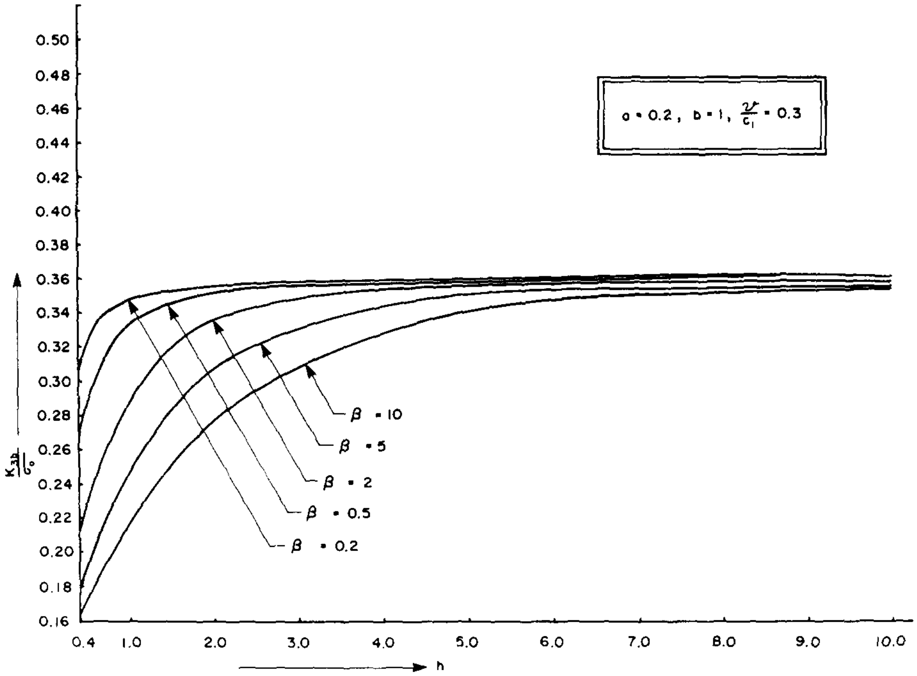

Fig. 9. Stress intensity factor versus layer thickness. Problem 2, outer crack front, $a = 0.2$.

graphically in Figs. 6–9 for several values of the anisotropy ratio β and for one typical value of crack front velocity ratio $v/c_1 = 0.3$.

3. Discussion

The analytical solutions which have been found for the present problems are supposed to apply to both cracks, either the leading one or the trailing one, when one considers a given direction of propagation. However, only crack fronts corresponding to material separation are physically meaningful. Thus, our results only apply to the outer front of the leading crack and to the inner front, $|x| = a$, of the trailing crack. Another hypothesis which has been made is that inner and outer fronts move at same velocity, which is unlikely in real cases. Thus, in a given situation, one should use proper velocities to calculate either the outer or the inner dynamic stress intensity factors. In addition, the results will be generally only applicable to the initial stage of crack propagation since this process will modify the distance between the two inner crack fronts and branching is also likely to appear, whereby cracks tend to emerge at one or the other of the layer surfaces. As examples, just a few numerical results have been calculated and illustrated graphically. However, it is a straightforward matter to study the influence of the various parameters (crack size and distance, layer thickness, propagation velocity, orthotropy ratio) through the closed form equations which have been derived for the dynamic stress intensity factors.

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