SELVADURAI, A. P. S. (1976). *Géotechnique* **26**, No. 4, 603–612.

The load–deflexion characteristics of a deep rigid anchor in an elastic medium

A. P. S. SELVADURAI*

The axially symmetric problem of an infinite elastic solid containing a bonded symmetrically loaded spheroidal rigid inclusion is considered. This problem is of interest in connexion with the geotechnical study of the time-independent, load–deflexion characteristics of deep rigid anchors embedded in cohesive soil or rock media. The solution to the deep rigid anchor problem is obtained by making use of Boussinesq's three-function approach applicable for the rotationally symmetric problem in the classical theory of elasticity. Load–deflexion relationships for rigid anchors of both prolate and oblate spheroidal shapes are presented in exact closed form. By treating the single anchor as a solid of revolution it is possible to investigate the influence of various geometric aspect ratios. From the exact load–displacement relationships presented here it is possible to recover solutions to anchors with spherical, circular or elongated shapes simply as limiting cases.

On examine le problème à symétrie axiale d'un solide élastique infini contenant une inclusion rigide sphéroïdale chargée symétriquement. Ce problème présente un intérêt pour l'étude géotechnique des caractéristiques indépendantes du temps de la déflexion–charge, d'ancrages rigides profonds dans des sols cohérents ou rocheux. La solution du problème de l'ancrage rigide profond est obtenue en utilisant les trois-fonctions de Boussinesq, applicables pour le problème circulaire symétrique dans la théorie classique de l'élasticité. Les relations déflexion–charge pour les ancrages rigides de forme sphéroïdale allongée aussi bien qu'aplatie sont présentées sous une forme exacte. En traitant l'ancrage seul, comme un solide de révolution, il est possible d'examiner l'influence de divers paramètres géométriques. A partir des relations exactes entre deplacement et charge presentées ici, il est possible de retrouver des solutions pour les ancrages de formes sphériques, circulaires ou allongées, simplement comme cas limites.

In recent years ground anchors have been extensively used in both temporary and permanent geotechnical structures such as retaining walls, foundations and earth slopes which require support to resist either lateral, uplift or gravitational loads. A variety of anchors including the grouted rod type, which were developed predominantly for use as rock anchors, embedded plates and driven deep screw anchors have found efficient use in the aforementioned areas of study (Girault, 1969; Hanna, 1972; Adams and Klym, 1972; Jaeger, 1972; McRostie *et al.*, 1972; Johnston and Ladanyi, 1974). Most investigations to date have largely concentrated on the analysis and experimental verification of the ultimate bearing capacity of both shallow and deep anchors, taking into account either their individual or group action. In order for the analytical treatment of the anchor problem to be complete it is, of course, advantageous to establish the load–deflexion characteristics of anchors especially at working loads. A complete analysis of these effects requires a knowledge of the mechanical response of both the anchor or the anchor system and the mechanical response of the surrounding soil medium. However, owing to the complexity of stress–strain characteristics of both the soil and the anchors, and their group action it becomes expedient to introduce certain plausible simplifications of the anchor problem. Thus, to a first approximation the soil can be considered to be a linearly deformable elastic solid and the anchors are regarded as being rigid.

Discussion on this Paper closes 1 March, 1977. For further details see page 648.

* Associate Professor, Department of Civil Engineering, Carleton University, Ottawa.

This Paper is primarily concerned with the analytical treatment of the load–deflexion characteristics of a deep rigid anchor embedded in an isotropic infinite elastic medium. This type of anchor is usually constructed by first drilling a small diameter shaft to the necessary depth; subsequently an expandable reaming device is used to develop the required anchor shape; finally the enlarged cavity is filled with concrete and this serves as the anchorage. Alternatively, the anchor could be regarded as being composed of a power driven deep screw pile. It is assumed that (*a*) the resulting anchor has the shape of a solid of revolution; or more precisely, it is approximated by a rigid spheroidal region, (*b*) the anchor is in bonded contact with the surrounding soil medium, and (*c*) the depth of embedment of the anchor is large in comparison with the largest dimension of the anchor such that the presence of external boundaries do not in any way influence its mechanical behaviour. The spheroidal rigid anchor is subjected to a resultant load directed along its axis of symmetry. Any frictional restraint that may be offered by the tie rod or other device used for the purposes of applying this load will also be neglected.

The deep anchor problem as formulated above constitutes an axisymmetric problem in the classical theory of elasticity which can be analysed by means of stress functions in a variety of ways (Sadowsky and Sternberg, 1947; Sternberg, 1960; Truesdell, 1960). These include an adaptation of the general three harmonic function approaches of Boussinesq (1885) and Timpe (1924) and the more widely used single biharmonic function approach of Love (1944). The technique used here is that of Boussinesq for which case the general solution of the displacement equations of equilibrium, in the absence of body forces, is representable as the sum of the displacement fields derived from two harmonic functions. The governing equations are essentially those derived by Sternberg *et al.* (1951) for generalized axisymmetric curvilinear co-ordinates. It is found that the load–deflexion characteristics for anchors of both prolate and oblate spheroidal shapes can be obtained in exact closed form in terms of elementary functions. From these solutions it is possible to recover solutions to anchors with spherical, circular or elongated shapes simply as limiting cases. From the spatial symmetry of the deep anchor problem it also follows that the load–deflexion relationships thus developed are also valid for situations in which the surrounding soil medium experiences moderately large elastic deformations. The stress analysis of elastic media subjected to moderately large deformations is carried out by taking into consideration the effects of both the linear and quadratic terms in the displacement gradients (Selvadurai and Spencer, 1972; Selvadurai, 1975).

GOVERNING EQUATIONS

The solution of the axisymmetric problem in three-dimensional linear elasticity, in the absence of body forces, is facilitated by the use of the three-function approach proposed by Boussinesq (1885). According to this formulation the general solution of the displacement equations can be represented as the sum of the displacement fields generated by two harmonic stress functions Φ and Ψ. Sadowsky and Sternberg (1947, 1949) and Sternberg *et al.* (1951) have presented the general expressions for the displacement and stress fields corresponding to Boussinesq's method referred to a general system of axisymmetric curvilinear co-ordinates. For the purposes of analysis of the spheroidal anchor problem, and for future reference, the results referred to a prolate spheroidal co-ordinate system (α, β, γ) will be summarized. The prolate spheroidal co-ordinate system is defined here by the transformation

$$\left.\begin{aligned} x &= c_p \sinh\alpha \sin\beta \cos\gamma \\ y &= c_p \sinh\alpha \sin\beta \sin\gamma \\ z &= c_p \cosh\alpha \cos\beta \end{aligned}\right\} \qquad (1)$$

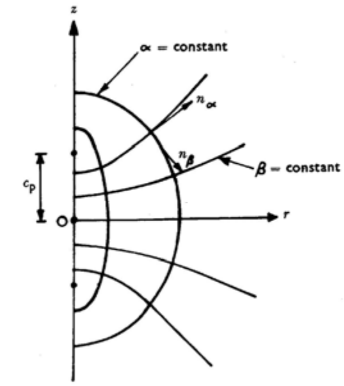

Fig. 1. Prolate spheroidal co-ordinates in a meridian plane

in which c_p is a positive constant. The parametric surfaces α=constant, say α_0, $\beta=\beta_0$, $\gamma=\gamma_0$ form a triple orthogonal confocal family of prolate spheroids, hyperboloids of two sheets and meridional half-planes respectively (Fig. 1). By considering the expression for a differential arc length (ds) given by

$$(\mathrm{d}s)^2 = \left(\frac{\mathrm{d}\alpha}{h_1}\right)^2 + \left(\frac{\mathrm{d}\beta}{h_2}\right)^2 + \left(\frac{\mathrm{d}\gamma}{h_3}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

it can be shown that the metric, or local, scale coefficients are given by

$$\left.\begin{aligned} h_1 &= h_2 = [c_p{}^2 (\sinh^2 \alpha + \sin^2 \beta)]^{-1/2} = h \\ h_3 &= (c_p \sinh \alpha \sin \beta)^{-1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

For deformations which are symmetric about the z-axis the displacements and stresses in the elastic medium are independent of the longitude γ. The Boussinesq potentials $\Phi(\alpha, \beta)$ and $\Psi(\alpha, \beta)$ referred to the spheroidal co-ordinate system satisfy the differential equations

$$\nabla^2 \Phi(\alpha, \beta) = 0; \qquad \nabla^2 \Psi(\alpha, \beta) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4a)$$

where

$$\nabla^2 = h^2 \left(\frac{\partial^2}{\partial \alpha^2} + \frac{\partial^2}{\partial \beta^2} + \coth \alpha \frac{\partial}{\partial \alpha} + \cot \beta \frac{\partial}{\partial \beta}\right) \quad . \quad . \quad . \quad . \quad . \quad (4b)$$

is Laplace's operator in prolate spheroidal co-ordinates. We denote the curvilinear components of the displacement vector by u_α and u_β and the curvilinear components of the Cauchy stress tensor $\boldsymbol{\sigma}$ are given by

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{\alpha\alpha} & 0 & \sigma_{\alpha\beta} \\ 0 & \sigma_{\gamma\gamma} & 0 \\ \sigma_{\alpha\beta} & 0 & \sigma_{\beta\beta} \end{bmatrix} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

The displacement fields of the Boussinesq solutions in curvilinear co-ordinates are given by

$$(u_\alpha; u_\beta) = \frac{h}{2G} (\Phi_\alpha; \Phi_\beta) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6a)$$

$$\{u_\alpha; u_\beta\} = \frac{h}{2G} \{[g\Psi_\alpha - (3-4\nu) g_\alpha \Psi]; \quad [g\Psi_\beta - (3-4\nu) g_\beta \Psi]\} \quad . \quad . \quad . \quad . \quad (6b)$$

where G, ν are respectively the linear elastic shear modulus and Poisson's ratio of the elastic medium and $g = c_p \cosh\alpha\cos\beta$. We shall also note that the subscripts attached to functions such as Φ and Ψ which originally bear no subscripts denote partial differentiation. The stress components derived from $\Phi(\alpha, \beta)$ and $\Psi(\alpha, \beta)$ take the form

$$\left.\begin{aligned}\sigma_{\alpha\alpha} &= h^2\left(\Phi_{\alpha\alpha}+\frac{h_\alpha}{h}\Phi_\alpha-\frac{h_\beta}{h}\Phi_\beta\right)\\ \sigma_{\beta\beta} &= h^2\left(\Phi_{\beta\beta}+\frac{h_\beta}{h}\Phi_\beta-\frac{h_\alpha}{h}\Phi_\alpha\right)\\ \sigma_{\gamma\gamma} &= h^2\left(\frac{f_\alpha}{f}\Phi_\alpha+\frac{f_\beta}{f}\Phi_\beta\right)\\ \sigma_{\alpha\beta} &= h^2\left(\Phi_{\alpha\beta}+\frac{h_\beta}{h}\Phi_\alpha+\frac{h_\alpha}{h}\Phi_\beta\right)\end{aligned}\right\} \quad (7a)$$

and

$$\left.\begin{aligned}\sigma_{\alpha\alpha} &= h^2\left[g\Psi_{\alpha\alpha}+\left(\frac{g}{h}h_\alpha-2g_\alpha\right)\Psi_\alpha-\frac{g}{h}h_\beta\Psi_\beta+2\nu(g_\alpha\Psi_\alpha-g_\beta\Psi_\beta)\right]\\ \sigma_{\beta\beta} &= h^2\left[g\Psi_{\beta\beta}+\left(\frac{g}{h}h_\beta-2g_\beta\right)\Psi_\beta-\frac{g}{h}h_\alpha\Psi_\alpha+2\nu(g_\beta\Psi_\beta-g_\alpha\Psi_\alpha)\right]\\ \sigma_{\gamma\gamma} &= h^2\left[\frac{g}{f}(f_\alpha\Psi_\alpha+f_\beta\Psi_\beta)-2\nu(g_\alpha\Psi_\alpha+g_\beta\Psi_\beta)\right]\\ \sigma_{\alpha\beta} &= h^2\left[g\Psi_{\alpha\beta}+\frac{g}{h}(h_\alpha\Psi_\beta+h_\beta\Psi_\alpha)-(1-2\nu)(g_\alpha\Psi_\beta+g_\beta\Psi_\alpha)\right]\end{aligned}\right\} \quad (7b)$$

respectively, where $f = c_p \sinh\alpha\sin\beta$.

THE PROLATE SPHEROIDAL ANCHOR PROBLEM

We consider an isotropic elastic infinite medium which contains a rigid anchor of a prolate spheroidal shape. The anchor is assumed to be in bonded contact with the surrounding elastic medium at its boundary $\alpha = \alpha_0$. It is subjected to a resultant force P directed along its axis of symmetry (Fig. 2); this causes a rigid body translation of the anchor, δ, in the z direction. The displacement boundary conditions at the anchor/soil interface are

$$u_\alpha(\alpha_0, \beta) = \frac{\delta\cos\beta}{\Omega_0}; \qquad u_\beta(\alpha_0, \beta) = \frac{-\delta\coth\alpha_0\sin\beta}{\Omega_0} \quad (8a)$$

where

$$\Omega = [1+\sin^2\beta\,\mathrm{cosech}^2\,\alpha]^{1/2} \quad (8b)$$

and

$$\Omega_0 = \Omega(\alpha_0)$$

Since the elastic medium is of infinite extent, in addition to these boundary conditions, all the displacement and stress components should tend to zero as $\alpha \to \infty$. For the solution of this displacement boundary value problem we require two independent solutions of the governing differential equations (4a). The harmonic functions which will be employed here are the Lamé products associated with the spheroidal co-ordinate system; the general expression for Φ is given by

$$\left.\begin{aligned}\Phi(\alpha, \beta) = [P_n^{(m)}(\cos\beta) \text{ or } Q_n^{(m)}(\cos\beta)][P_n^{(m)}(\cosh\alpha) \text{ or } Q_n^{(m)}(\cosh\alpha)]\\ (m, n = 0, 1, 2, 3, \ldots)\end{aligned}\right\} \quad (9)$$

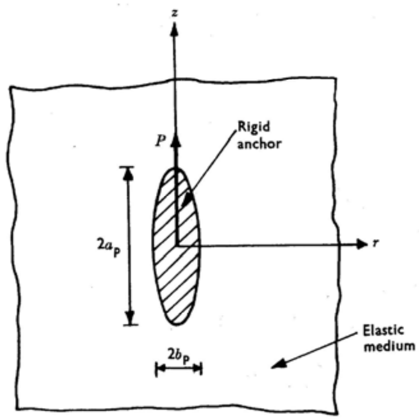

Fig. 2. Prolate spheroidal anchor

where $P_n^{(m)}$ and $Q_n^{(m)}$ are associated Legendre functions of the first and second kind (e.g. Hobson, 1931). Considering the form of the boundary conditions (8a) and the conditions at infinity it can be shown that the appropriate solutions of eqns (4a) are

$$\Phi(\alpha, \beta) = C_1 \left[1 - \frac{\cosh \alpha}{2} \log \left(\frac{\cosh \alpha + 1}{\cosh \alpha - 1}\right)\right] \cos \beta \quad . \quad . \quad . \quad . \quad (10a)$$

and

$$\Psi(\alpha, \beta) = \tfrac{1}{2} C_2 \log \left(\frac{\cosh \alpha + 1}{\cosh \alpha - 1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10b)$$

where C_1 and C_2 are arbitrary constants. These harmonic functions when substituted in the sets of eqns (6) and (7) give displacement and stress components which are single-valued in the domain $\alpha_0 \leqslant \alpha < \infty$; $0 \leqslant \beta \leqslant \pi$, and which tend to zero as $\alpha \to \infty$. The arbitrary constants C_1 and C_2 can be determined by making use of the boundary conditions (8a) and the complete expressions for the displacement components

$$\left.\begin{aligned} u_\alpha &= \frac{1}{4Gc_p\Omega}\{C_1[2 \operatorname{cosech} \alpha \coth \alpha - \log \xi] \\ &\qquad + C_2 c_p[-2 \coth \alpha \operatorname{cosech} \alpha - (3-4\nu) \log \xi]\} \cos\beta \\ u_\beta &= \frac{1}{4Gc_p\Omega}\{C_1[-2 \operatorname{cosech} \alpha + \coth \alpha \log \xi] + C_2 c_p[(3-4\nu) \coth \alpha \log \xi]\} \sin \beta \end{aligned}\right\} \quad (11)$$

where

$$\xi = \frac{\cosh \alpha + 1}{\cosh \alpha - 1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

The expressions (11) together with the boundary conditions (8a) give

$$(C_1; C_2) = \frac{2\delta c_p G}{\chi_0} (\coth \alpha_0; \operatorname{cosech} \alpha_0 \operatorname{sech} \alpha_0) \quad . \quad . \quad . \quad . \quad . \quad (13a)$$

where

$$\chi_0 = -\operatorname{cosech} \alpha_0 \operatorname{sech} \alpha_0 \{\tfrac{1}{2}[(3-4\nu) + \cosh^2 \alpha_0] \log \xi_0 - \cosh \alpha_0\} \quad . \quad . \quad (13b)$$

and

$$\xi_0 = \xi(\alpha_0)$$

The force–displacement relationship for the prolate spheroidal anchor can be obtained by considering the resultant of tractions in the z direction acting on any closed surface α=constant. It is, however, convenient to compute this resultant force by choosing the closed surface to be the anchor/elastic medium boundary $\alpha=\alpha_0$. The resultant force in the z direction is given by

$$P = 2\pi c_p{}^2 \int_0^\pi \theta_0{}^{1/2}(\sigma_{\alpha\alpha}n_z - \sigma_{\alpha\beta}n_r)_{\alpha=\alpha_0} \sinh\alpha_0 \sin\beta \,d\beta \qquad (14)$$

where

$$\theta(\alpha, \beta) = (\sinh^2\alpha + \sin^2\beta); \qquad \theta_0 = \theta(\alpha_0) \qquad (15a)$$

and n_r and n_z are the components of the unit outward normal to $\alpha=\alpha_0$ in the positive r and z directions, evaluated from the expressions

$$[n_r;\, n_z] = \frac{1}{\theta^{1/2}}(\cosh\alpha\sin\beta;\, \sinh\alpha\cos\beta) \qquad (15b)$$

The complete stress components $\sigma_{\alpha\alpha}$ and $\sigma_{\alpha\beta}$ derived from the harmonic functions (10) and the relationships (7), can be written as

$$\left.\begin{aligned} \sigma_{\alpha\alpha} &= \frac{C_1}{c_p{}^2}\left\{-\frac{1}{\theta^2} - \frac{\operatorname{cosech}^2\alpha}{\theta} + \frac{1}{\cosh^2\alpha_0}\left[\frac{\coth^2\alpha}{\theta} + \frac{\cosh^2\alpha}{\theta^2} + \frac{2(1-\nu)}{\theta}\right]\right\}\cos\beta \\ \sigma_{\alpha\beta} &= \frac{C_1}{c_p{}^2}\coth\alpha\left\{-\frac{1}{\theta^2} + \frac{1}{\cosh^2\alpha_0}\left[\frac{\cos^2\beta}{\theta^2} - \frac{(1-2\nu)}{\theta}\right]\right\}\sin\beta \end{aligned}\right\} \qquad (16)$$

By substituting the above expressions in (14) and performing the integrations we obtain the relationship for the applied force as

$$P = -\frac{8\pi C_1(1-\nu)}{\cosh^2\alpha_0} \qquad (17)$$

A combination of eqns (13a) and (17) thus yields the load–deflexion relationship for the prolate spheroidal anchor:

$$P = 16\pi(1-\nu)\,\delta c_p G\{\tfrac{1}{2}[(3-4\nu)+\cosh^2\alpha_0]\log\xi_0 - \cosh\alpha_0\}^{-1} \qquad (18)$$

Further, by considering the geometry of the prolate spheroidal inclusion the focal distance c_p (Fig. 1) can be related to the dimensions of the major axis and the equatorial radius of the inclusion is

$$a_p = c_p\cosh\alpha_0; \qquad b_p = c_p\sinh\alpha_0 \qquad (19)$$

Thus eqn (18) can be re-written as

$$P = 16\pi(1-\nu)\,\delta G a_p(1-\lambda^2)^{1/2}\left\{\tfrac{1}{2}\left[(3-4\nu)+\frac{1}{(1-\lambda^2)}\right]\log\left[\frac{1+\sqrt{(1-\lambda^2)}}{1-\sqrt{(1-\lambda^2)}}\right] - \frac{1}{(1-\lambda^2)^{1/2}}\right\}^{-1} \qquad (20)$$

where

$$\lambda(=b_p/a_p) < 1$$

Limiting cases

Incompressible elastic medium. In the particular case of an incompressible elastic medium $\nu=\frac{1}{2}$ and the load–deflexion relationship reduces to

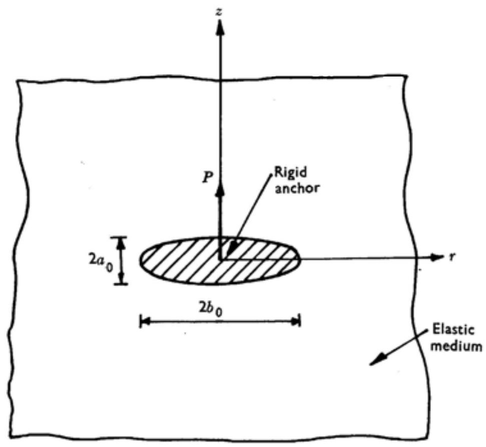

Fig. 3. Oblate spheroidal anchor

$$P = 8\pi\ \delta G a_p (1-\lambda^2)^{1/2}\left\{\frac{1}{2}\left(\frac{2-\lambda^2}{1-\lambda^2}\right)\log\left[\frac{1+\sqrt{(1-\lambda^2)}}{1-\sqrt{(1-\lambda^2)}}\right]-\frac{1}{(1-\lambda^2)^{1/2}}\right\}^{-1} \quad . \quad (21a)$$

This result is consistent with the force–displacement relationship obtained for the undrained elastic problem (Selvadurai, 1976) by a consideration of the mathematical equivalence between the incompressible elastic problem and the analogous slow viscous flow problem.

Rigid spherical anchor. In the particular case when $\lambda \to 1$ we obtain from eqn (20) the load–deflexion relationship to the problem of a spherical rigid anchor embedded in an isotropic elastic medium. Taking the limit of (20) as $\lambda \to 1$ we obtain

$$P = \frac{24\pi\ \delta G a_p(1-\nu)}{(5-6\nu)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21b)$$

where a_p is the radius of the spherical anchor. This result is identical with the load–deflexion relationship obtained by Josselin de Jong (1957) in connexion with the deformations of a consolidating medium loaded by a rigid sphere.

Rigid anchor with an elongated shape. In the special case where the major axis, a_p, of the prolate spheroid is much greater than its equatorial radius b_p the spheroid resembles a long thin rod (very similar to rock anchors of the grouted rod type (e.g. Obert and Duvall, 1967). For this limiting case, the load–deflexion relationship takes the form

$$P = 8\pi\ \delta G a_p(1-\nu)\left\{2(1-\nu)\left[\log\left(\frac{a_p}{b_p}\right)+\log 2\right]-\frac{1}{2}\right\}^{-1} \quad . \quad . \quad . \quad (21c)$$

where the third and higher order terms in (b_p/a_p) have been neglected.

The analysis of the prolate spheroidal anchor problem presented here is primarily concerned with the determination of the load–deflexion characteristics of the rigid anchor. The state of stress induced in the surrounding elastic material as a result of the loaded anchor can be obtained in a straightforward manner by considering the expressions for the stresses, (7) and the harmonic functions (10).

THE OBLATE SPHEROIDAL ANCHOR PROBLEM

The foregoing solutions were established on the assumption that the spheroidal inclusion is prolate, i.e. $(a_p/b_p) > 1$. The solution to the associated problem concerning the deep anchor with the shape of an oblate spheroid (Fig. 3) can be obtained by methods very similar to those

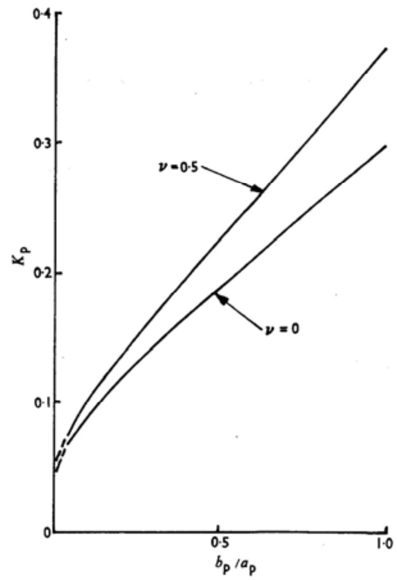

Fig. 4. Load–deflexion characteristics of a prolate spheroidal anchor: $P = 16\pi\delta G a_p K_p$ where K_p is defined by eqn (20)

employed in the previous sections. Omitting details of calculation it can be shown that the load–deflexion relationship for the analogous problem of the oblate spheroidal anchor is given by

$$P = 16\pi(1-\nu)\,\delta G c_0\{\sinh\alpha_0 - [\sinh^2\alpha_0 - (3-4\nu)]\cot^{-1}(\sinh\alpha_0)\}^{-1} \quad . \quad (22)$$

where

$$c_0{}^2 = (b_0{}^2 - a_0{}^2);\ \sinh\alpha_0 = \frac{a_0}{(b_0{}^2 - a_0{}^2)^{1/2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad (23)$$

and a_0 and b_0 are, respectively, the half length of the minor axis and the equatorial radius of the oblate spheroidal anchor.

Limiting cases

Incompressible elastic medium. Again, in the particular case of an incompressible elastic medium $\nu = \frac{1}{2}$ and the load–deflexion relationship becomes

$$P = 8\pi\,\delta G a_0(\mu^2-1)^{1/2}\left\{\frac{1}{(\mu^2-1)^{1/2}} - \left(\frac{2-\mu^2}{\mu^2-1}\right)\cot^{-1}\left[\frac{1}{(\mu^2-1)^{1/2}}\right]\right\}^{-1} \quad . \quad (24a)$$

where $\mu\ (=b_0/a_0) > 1$ and $\sinh\alpha_0 = (\mu^2-1)^{-1/2}$. This result is in agreement with the equivalent expressions derived from the viscous flow analogy.

Rigid circular disc anchor. As $\sinh\alpha_0 \to 0$, the oblate spheroid degenerates to a flat circular rigid disc of infinitesimal thickness, which is in bonded contact with the surrounding infinite elastic medium. Taking the appropriate limit of eqn (22) we obtain the load–deflexion relationship for this particular case as

$$P = \frac{32(1-\nu)\,\delta G b_0}{(3-4\nu)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24b)$$

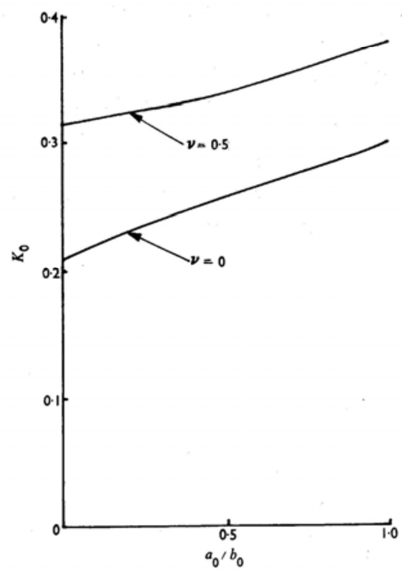

Fig. 5. Load–deflexion characteristics of an oblate spheroidal anchor: $P = 16\pi\delta G b_0 K_0$ where K_0 is defined by eqn (22)

where b_0 is the radius of the disc. Eqn (24b) is identical with the result obtained by Collins (1962) in connexion with the problem of the penny-shaped inclusion embedded in an infinite elastic solid. This particular result was obtained by a procedure which reduces the axi-symmetric problem in linear elasticity to the solution of Fredholm integral equations of the second kind (Tricomi, 1957).

Rigid spherical anchor. The load–deflexion relationship for the rigid spherical anchor embedded in an infinite elastic medium can also be obtained as a limiting case of eqn (22), as $\sinh \alpha_0 \to \infty$. Considering a series representation for $\cot^{-1}(\sinh \alpha_0)$ we have

$$P = 16\pi(1-\nu)\,\delta G a_0 \left\{ \sinh^2 \alpha_0 - [\sinh^3 \alpha_0 - (3-4\nu)\sinh \alpha_0] \times \left[\frac{1}{\sinh \alpha_0} - \frac{1}{3\sinh^3 \alpha_0} + \frac{1}{5\sinh^5 \alpha_0} - \cdots \right] \right\}^{-1} \quad (24c)$$

Taking the limit of (24c) as $\sinh \alpha_0 \to \infty$, the relationship (21b) is recovered directly.

CONCLUSIONS

This Paper presents closed form analytical results for a rigid spheroidal deep anchor embedded in bonded contact with an isotropic infinite elastic medium. The influence of the geometric aspect ratio of the anchor (i.e. a_p/b_p or b_0/a_0) on the load–deflexion characteristics is illustrated in Figs 4 and 5. These results indicate that the load–deflexion characteristics of the bonded anchors are significantly influenced by their geometric shape and that these effects appear to be more pronounced in the case where the rigid anchor has the shape of a prolate spheroid.

The deep anchor problem examined here inherently has certain limitations; factors such as interaction effects of neighbouring anchors, the influence of finite depth of embedment, the presence of non-contiguous boundary conditions at the anchor/soil interface and so forth are clearly excluded from the analysis. The first two factors are generally of peripheral interest with regard to the mechanical behaviour of single anchors located at a large depth and spacing. However, the analysis as presented in this Paper could be further extended to include boundary conditions associated with a completely smooth interface or an interface which exhibits Coulomb friction, provided it is explicitly assumed that the interface is capable of sustaining tensile surface tractions. This would seem to be a reasonable assumption for the case of deep anchors in which sufficient compressional stresses may exist, due to the self weight of the overburden to prevent any loss of contact at the interface. The class of anchor problems which take into account influences of neighbouring boundaries, the effects of partial separation or partial slip at the anchor/soil interface, admittedly, requires a much more rigorous mathematical analysis.

ACKNOWLEDGEMENTS

The work described in this Paper was in part supported by a Research Grant awarded by the National Research Council of Canada.

REFERENCES

Adams, J. I. & Klym, T. W. (1972). A study of anchorages for transmission tower foundations. *Can. Geotech. Jnl* **9**, No. 1, 89–104.

Boussinesq, J. (1885). *Applications des Potentiels.* Paris: Gauthier-Villars.

Collins, W. D. (1962). Some axially symmetric stress distributions in elastic solids containing penny-shaped cracks. I. Cracks in an infinite solid and a thick plate. *Proc. R. Soc. Series A* **203**, 359–386.

Girault, P. (1969). Discussion on Anchorages especially in soft ground. *Proc. Seventh Int. Conf. Soil Mech. Fdn Engng, Mexico City*, 214–215.

Hanna, T. H. (1972). Studies on anchors and anchor-supported retaining walls in sand. In Stress–strain behaviour of soils. *Proc. Roscoe Mem. Symp., Cambridge*, 450–458. Henley-on-Thames: Foulis.

Hobson, E. W. (1931). *The theory of spherical and ellipsoidal harmonics.* London: Cambridge University Press.

Jaeger, C. (1972). *Rock mechanics and engineering.* London: Cambridge University Press.

Johnston, G. H. & Ladanyi, B. (1974). Field tests of deep power installed screw anchors in permafrost. *Can. Geotech. Jnl* **11**, No. 3, 348–358.

Josselin de Jong, G. de (1957). Application of stress functions to consolidation problems. *Proc. Fourth Int. Conf. Soil Mech. Fdn Engng, London* **3**, 320–323.

Love, A. E. H. (1944). *A treatise on the mathematical theory of elasticity.* New York: Dover Publications.

McRostie, G. C., Burn, K. N. & Mitchell, R. J. (1972). The performance of a tied-back sheet piling in clay. *Can. Geotech. Jnl* **9**, No. 2, 206–218.

Obert, L. & Duvall, W. I. (1967). *Rock mechanics and the design of structures in rock.* New York: John Wiley.

Sadowsky, M. A. & Sternberg, E. (1947). Stress concentration around an ellipsoidal cavity in an infinite body under arbitrary plane stress perpendicular to the axis of revolution of cavity. *Jnl Appl. Mech.* **69**, A191–A201.

Sadowsky, M. A. & Sternberg, E. (1949). Stress concentration in a triaxial ellipsoidal cavity. *Jnl Appl. Mech.* **16**, 149–157.

Selvadurai, A. P. S. (1975). The distribution of stress in a rubber-like elastic material bounded internally by a spherical inclusion. *Mechanics Res. Comm.* **2**, 99–106.

Selvadurai, A. P. S. (1976). The response of a deep rigid anchor due to undrained elastic deformations of the surrounding soil medium. Unpublished work.

Selvadurai, A. P. S. & Spencer, A. J. M. (1972). Second-order elasticity with axial symmetry. I. General theory. *Int. Jnl Engng Sci.* **10**, 97–114.

Sternberg, E. (1960). On some recent developments in the linear theory of elasticity. In Structural mechanics. *Proc. First Symp. Naval Structural Mechanics*, 48–73. Oxford: Pergamon Press.

Sternberg, E., Eubanks, R. A. & Sadowsky, M. A. (1951). On the stress function approaches of Boussinesq and Timpe to the axisymmetric problem of elasticity theory. *Jnl Appl. Phys.* **22**, No. 9, 1121–1124.

Timpe, A. (1924). Achsensymmetrische Deformation von Umdrehungskörpern. *Z. angew. Math. Mech.* **4**, 361–376.

Tricomi, F. G. (1957). *Integral equations.* New York: Interscience.

Truesdell, C. (1960). Invariant and complete stress functions for general continua. *Archs Ration. Mech. Analysis* **4**, 1–29.