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In-plane loading of a cracked elastic solid by a disc inclusion with a Mindlin-type constraint

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1. Introduction

The stress analysis of elastic media reinforced with either elastic or rigid inclusions is of interest to the study of composite materials. The boundary value problems pertaining to these inclusion problems also occupy a prominent position in the mathematical theory of three-dimensional elastostatics. Comprehensive accounts of inclusion problems in classical elasticity are given by Mura [1], Willis [2] and Walpole [3]. Flat disc shaped inclusions are a particular limiting case of the general class of three-dimensional ellipsoidal or spheroidal inclusions. A number of investigators have examined the disc inclusion problem related to an elastic medium to study effects such as transverse elastic isotropy of the medium, annular and elliptical configuration of the inclusion, flexural behaviour of the inclusion, influence of the externally applied loads, traction free boundaries, bi-material regions, and delaminations at the inclusion-elastic medium interface. A comprehensive account of the disc inclusion problem in classical elasticity will be given in a forthcoming article by Selvadurai [4].

An examination of the literature on inclusion problems indicates that the category of problems which investigate the interaction between cracks and inclusions located in elastic media has received only limited attention. The paper by Keer [5] examines the partial adhesive contact between a disc inclusion and an elastic medium. Selvadurai and Singh [6] have examined the problem of the internal indentation of a penny-shaped crack by a smoothly embedded rigid circular disc inclusion. This category of problems is of some interest to the study of thermally or environmentally induced fracture and degradation of multiphase composites. In this paper, we examine the problem of the asymmetric loading of an external circular crack by a rigid circular disc inclusion located at the centre of the intact region (Fig. 1). The contact at the inclusion-elastic medium interface is characterized by a partially relaxed displacement constraint. Within the inclusion region a displacement boundary condition is prescribed in the direction of translation of the inclusion and a traction free boundary condition is prescribed in the in-plane direction normal to the direction of movement. This type of

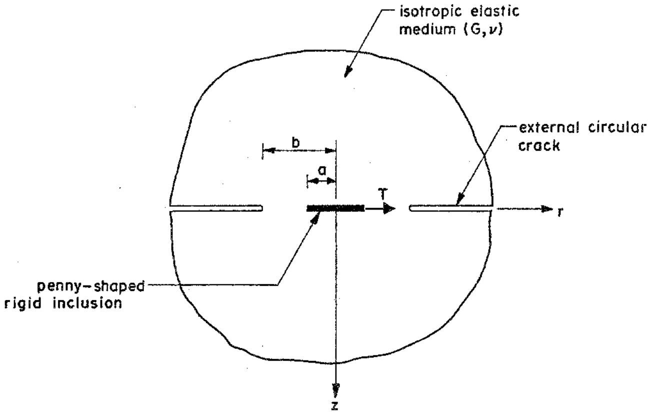

Figure 1
Geometry of the penny-shaped inclusion and the external circular crack.

imperfect contact constraint was first proposed by Mindlin [7] in connection with the translation of a plane "frictionless" punch in contact with an elastic halfspace. The inclusion problem is examined in relation to a set of mixed boundary conditions associated with a halfspace region. The mixed boundary value problem is formulated via solutions derived from a Hankel-transform development of the governing field equations. The integral equations associated with the mixed boundary conditions are reduced to a system of Abel integral equations. These equations are solved in an approximate fashion by employing power series expansions of the governing functions. The ratio of the radius of the penny-shaped inclusion to the radius of the external circular crack is used as the small non-dimensional parameter of the power series expansions. Numerical results are derived for the translational stiffness of the disc inclusion and the stress intensity factor at the boundary of the external circular crack. The results for the lateral stiffness are compared with exact analytical results derived for the inclusion embedded in an uncracked elastic solid and a completely cracked solid. The limiting result for the stress intensity factor at the boundary of the external circular crack (which is recovered as the inclusion radius reduces to zero) are also compared with the exact analytical results for the stress intensity factor at the boundary of an external circular crack subjected to an in-plane central force.

2. Basic equations

For the analysis of the asymmetric problem related to the loading of the external circular crack by a central rigid circular disc inclusion located in its plane we employ the displacement functions $\varphi(r, \theta, z)$ and $\psi(r, \theta, z)$ proposed by Muki [8]. These functions are special reductions of the more general class of Neuber-Papkovich functions [9], and satisfy the differential equations

$$\nabla^2 \nabla^2 \varphi(r, \theta, z) = 0 \qquad (1)$$

and

$$\nabla^2 \psi(r, \theta, z) = 0 \tag{2}$$

where ∇^2 is Laplace's operator referred to the generalized cylindrical polar coordinate system. The displacement and stress components referred to the cylindrical polar coordinate system can be expressed in terms of $\varphi(r, \theta, z)$ and $\psi(r, \theta, z)$ in the following forms:

$$2Gu_r = -\frac{\partial^2 \varphi}{\partial r \, \partial z} + \frac{2}{r}\frac{\partial \psi}{\partial \theta} \tag{3}$$

$$2Gu_\theta = -\frac{1}{r}\frac{\partial^2 \varphi}{\partial \theta \, \partial z} - 2\frac{\partial \psi}{\partial r} \tag{4}$$

$$2Gu_z = 2(1-\nu)\nabla^2 \varphi - \frac{\partial^2 \varphi}{\partial z^2} \tag{5}$$

and

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu \nabla^2 - \frac{\partial^2}{\partial r^2}\right)\varphi + \frac{\partial}{\partial \theta}\left(\frac{2}{r}\frac{\partial}{\partial r} - \frac{2}{r^2}\right)\psi \tag{6}$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu \nabla^2 - \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\varphi - \frac{1}{r}\frac{\partial}{\partial \theta}\left(2\frac{\partial}{\partial r} - \frac{2}{r}\right)\psi \tag{7}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\varphi \tag{8}$$

$$\sigma_{\theta z} = \frac{1}{r}\frac{\partial}{\partial \theta}\left\{(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\varphi - \frac{\partial^2 \psi}{\partial r \, \partial z} \tag{9}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2 - \frac{\partial^2}{\partial z^2}\right\}\varphi + \frac{1}{r}\frac{\partial^2 \psi}{\partial \theta \, \partial z} \tag{10}$$

$$\sigma_{r\theta} = \frac{1}{r}\frac{\partial^2}{\partial \theta \, \partial z}\left(\frac{1}{r} - \frac{\partial}{\partial r}\right)\varphi - \left(2\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial z^2}\right)\psi \tag{11}$$

where G and ν are the linear elastic shear modulus and Poisson's ratio respectively.

3. The external circular crack-disc inclusion interaction

We consider the problem of a penny-shaped rigid inclusion of radius a which is embedded in bonded contact with an isotropic elastic medium. The plane containing the disc inclusion is weakened by an external circular crack of radius b ($b > a$) (Fig. 1). The disc inclusion is subjected to a central in-plane force T which acts in the $+ve\,x$ direction. The in-plane displacement of the rigid circular inclusion is denoted by δ. From an examination of the problem, it is evident that the lateral translation induces a state symmetry about the plane $z = 0$. Therefore, we may restrict the analysis to the examination of a single halfspace region occupying $z \geqq 0$. The relevant displacement and traction boundary conditions associated with the inclusion problem are as follows:

$$u_r(r, \theta, 0) = \delta \cos\theta; \quad 0 \leqq r \leqq a \tag{12}$$

$$u_\theta(r, \theta, 0) = -\delta \sin\theta; \quad 0 \leqq r \leqq a \tag{13}$$

$$u_z(r, \theta, 0) = 0; \quad 0 \leqq r \leqq b \tag{14}$$

and

$$\sigma_{rz}(r, \theta, 0) \sin\theta + \sigma_{\theta z}(r, \theta, 0) \cos\theta = 0; \quad 0 < r < \infty \tag{15}$$

$$\sigma_{rz}(r, \theta, 0) \cos\theta + \sigma_{\theta z}(r, \theta, 0) \sin\theta = 0; \quad a < r < \infty \tag{16}$$

$$\sigma_{zz}(r, \theta, 0) = 0; \quad b < r < \infty \tag{17}$$

The specification of boundary condition (15) in the entire region $z = 0$; $0 < r < \infty$ characterizes the partial Mindlin-type constraint within the disc inclusion region.

For the integral equation formulation of the mixed boundary value problem posed by (12)–(17), we seek solutions of (1) and (2) which can be obtained by a Hankel-transform development of the same. Also, the displacement and stress fields derived from $\varphi(r, \theta, z)$ and $\psi(r, \theta, z)$ should reduce to zero as $(r^2 + z^2)^{1/2} \to \infty$. Following Muki [8] and Sneddon [10], the relevant solutions take the forms

$$\varphi(r, \theta, z) = \left[\int_0^\infty \xi \{A(\xi) + z B(\xi)\} e^{-\xi z} J_1(\xi r)\, \mathrm{d}\xi\right] \cos\theta \tag{18}$$

and

$$\psi(r, \theta, z) = \left[\int_0^\infty \xi A^*(\xi) e^{-\xi z} J_1(\xi r)\right] \sin\theta \tag{19}$$

where $A(\xi)$, $B(\xi)$ and $A^*(\xi)$ are arbitrary functions which are to be determined by satisfying the boundary conditions (12)–(17) on the plane $z = 0$. Using the integral representations for $\varphi(r, \theta, z)$ and $\psi(r, \theta, z)$ given by (18) and (19) in the expressions (3)–(11) for the displacements and stresses, it can be shown that the mixed boundary conditions (12)–(17) yield the following system of integral equations.

$$H_0[\xi\{3\xi A(\xi) - (1 + 4\nu) B(\xi)\}; r] = 4G\delta; \quad 0 \leqq r \leqq a \tag{20}$$

$$H_1[\xi\{-\xi A(\xi) - 2(1 - 2\nu) B(\xi)\}; r] = 0; \quad 0 \leqq r \leqq a \tag{21}$$

$$H_0[\xi^2\{-\xi A(\xi) + 2\nu B(\xi)\}; r] = 0; \quad a \leqq r \leqq \infty \tag{22}$$

$$H_1[\xi^2\{\xi A(\xi) + (1 - 2\nu) B(\xi)\}; r] = 0; \quad b \leqq r \leqq \infty \tag{23}$$

where $H_n[f(\xi); r]$ is the Hankel transform of order n which is defined by

$$H_n[f(\xi); r] = \int_0^\infty \xi f(\xi) J_n(\xi r) \, d\xi . \tag{24}$$

We now make the assumption that as $b \to \infty$, we should recover, from the solution developed, the appropriate results for the problem of the lateral translation of a penny-shaped rigid inclusion which is embedded in an uncracked elastic solid. It is convenient to introduce functions $C(\xi)$ and $D(\xi)$ such that

$$A(\xi) = \frac{1}{\xi^3} \{C(\xi) + 2D(\xi)\} \tag{25}$$

$$B(\xi) = -\frac{1}{\xi^2(1-2\nu)} \{C(\xi) + D(\xi)\} . \tag{26}$$

The integral Eqs. (20)–(23) can now be written as

$$H_0\left[\xi^{-1}\left\{D(\xi) + \frac{2(2-\nu)}{(7-8\nu)} C(\xi)\right\}; r\right] = \frac{4G\delta(1-2\nu)}{(7-8\nu)}; \quad 0 \leqq r \leqq a \tag{27}$$

$$H_1[\xi^{-1} C(\xi); r] = 0; \quad 0 \leqq r < a \tag{28}$$

$$H_0[\{C(\xi) + 2(1-\nu) D(\xi)\}; r] = 0; \quad a < r < \infty \tag{29}$$

$$H_1[D(\xi); r] = 0; \quad b < r < \infty . \tag{30}$$

Using the following integral

$$\int_t^\infty \frac{r J_0(\xi r) \, dr}{(r^2 - t^2)^{1/2}} = \frac{\cos(\xi t)}{\xi} \tag{31}$$

the integral Eq. (29) can be written in the following form:

$$\int_0^\infty [C(\xi) + 2(1-\nu) D(\xi)] \cos(\xi t) \, d\xi = 0; \quad a < t < \infty . \tag{32}$$

We assume that

$$\int_0^\infty [C(\xi) + 2(1-\nu) D(\xi)] \cos(\xi t) \, d\xi = \frac{\pi}{2} f_1(t); \quad 0 < t < a . \tag{33}$$

Using (32) and (33) we obtain

$$C(\xi) + 2(1-\nu) D(\xi) = \int_0^a f_1(t) \cos(\xi t) \, dt . \tag{34}$$

The Eq. (28) is satisfied automatically if we assume that

$$C(\xi) = \int_b^\infty f_2(t) \sin(\xi t) \, dt \tag{35}$$

where $f_2(\infty) = 0$. With the aid of (34) and (35) we find that

$$2(1-\nu)D(\xi) = \int_0^a f_1(t)\cos(\xi t)\,dt - \int_b^\infty f_2(t)\sin(\xi t)\,dt. \tag{36}$$

From Eq. (30) we find that

$$\frac{\partial}{\partial r}\int_0^\infty D(\xi)J_0(\xi r)\,d\xi = 0; \quad b < r < \infty. \tag{37}$$

By substituting the value of $D(\xi)$ from (36) into (37) we obtain an integral equation of the Abel type for $f_2(t)$; i.e.,

$$\frac{\partial}{\partial r}\int_r^\infty \frac{f_2(t)\,dt}{(t^2-r^2)^{1/2}} = -r\int_0^a \frac{f_1(\xi)\,d\xi}{(r^2-\xi^2)^{3/2}}; \quad b < r < \infty. \tag{38}$$

The solution of (38) gives

$$f_2(t) = \frac{2t}{\pi}\int_t^\infty \frac{r\,dr}{(r^2-t^2)^{1/2}}\int_0^a \frac{f_1(\xi)\,d\xi}{(r^2-\xi^2)^{3/2}}; \quad b < t < \infty. \tag{39}$$

Changing the order of the integrations we find that

$$f_2(t) = \frac{2t}{\pi}\int_0^a \frac{f_1(\xi)\,d\xi}{(t^2-\xi^2)}; \quad b < t. \tag{40}$$

Also using (35) and (36), the integral Eq. (27) can be reduced to the following Abel type integral equation for $f_1(t)$;

$$\int_0^r \frac{f_1(t)\,dt}{(r^2-t^2)^{1/2}} = \frac{8G\delta(1-\nu)(1-2\nu)}{(7-8\nu)} - \frac{(1-2\nu)^2}{(7-8\nu)\pi}\int_b^\infty \frac{f_2(\xi)\,d\xi}{(\xi^2-r^2)^{1/2}};$$
$$0 \leqq r \leqq a. \tag{41}$$

Again, the solution of (41) can be written in the form

$$f_1(t) = \frac{16G\delta(1-\nu)(1-2\nu)}{(7-8\nu)\pi} - \frac{2(1-2\nu)^2}{(7-8\nu)\pi}\int_b^\infty \frac{\xi f_2(\xi)\,d\xi}{(\xi^2-t^2)}; \quad 0 \leqq t \leqq a. \tag{42}$$

By combining (40) and (42) we obtain the following integral equation for $f_1(t)$;

$$f_1(t) = \frac{16G\delta(1-\nu)(1-2\nu)}{(7-8\nu)\pi} + \frac{2(1-2\nu)^2}{(7-8\nu)\pi^2}\int_0^a$$
$$\cdot\left[\frac{1}{(t^2-s^2)}\left\{t\ln\left|\frac{b-t}{b+t}\right| - s\ln\left|\frac{b-s}{b+s}\right|\right\}sf_1(s)\right]ds; \quad 0 < t < a. \tag{43}$$

Introducing the substitutions

$$t = at_1; \quad s = as_1; \quad f_1(at_1) = \frac{16G\delta(1-\nu)(1-2\nu)}{(7-8\nu)\pi}F_1(t_1) \tag{44}$$

the integral Eq. (43) can be written in the following form

$$F_1(t_1) = 1 + \frac{2(1-2\nu)^2}{(7-8\nu)\pi^2} \int_0^1 \left[\frac{1}{(t_1^2 - s_1^2)} \left\{ t_1 \ln \left| \frac{1 - c t_1}{1 + c t_1} \right| - s_1 \ln \left| \frac{1 - c s_1}{1 + c s_1} \right| \right\} \cdot s_1 F_1(s_1) \right] \mathrm{d}s_1; \quad 0 < t_1 < 1 \tag{45}$$

where $c = a/b$.

In the ensuing, we develop a series solution of the integral Eq. (45) by assuming that $c < 1$. We assume that $F_1(t_1)$ admits a power series representation of the form

$$F_1(t_1) = \sum_{i=0}^{N} c^i m_i(t_1). \tag{46}$$

By substituting (46) into (45) and expanding the kernel function in a power series in c, we obtain the following system of equations:

$$\sum_{i=0}^{N} c^i m_i(t_1) = 1 - \frac{4(1-2\nu)^2}{(7-8\nu)\pi^2} \int_0^1 s_1 \sum_{i=0}^{N} c^i m_i(s_1) \cdot \left\{ c + \frac{c^3}{3}(t_1^2 + s_1^2) + \frac{c^5}{5}(t_1^2 + s_1^4 + t_1^2 s_1^2) + 0(c^7) \right\} \mathrm{d}s_1; \quad 0 < t_1 < 1. \tag{47}$$

By comparing like terms of order c^i $(i = 0, 1, \ldots, N)$ we obtain either explicit or integral expressions for $m_i(t_1)$. These expressions are given in Appendix A. The evaluation of $m_i(t_1)$ formally completes the analysis of the mixed boundary value problem defined by (12)–(17). The expression for $f_2(t_1)$ can be determined from (40) and the functions $f_1(t_1)$ and $f_2(t_1)$ can be used to determined $C(\xi)$ and $D(\xi)$. Avoiding details of the calculations it can be shown that

$$\begin{aligned} F_1(t_1) = {} & 1 - 2c\zeta + 4c^2\zeta^2 - 4c^3\zeta \left\{ 2\zeta^2 + \frac{1}{6}\left(\frac{2}{3} + t_1^2\right) \right\} \\ & + 4c^4\zeta^2 \left\{ \frac{1}{3}\left(\frac{1}{2} + t_1^2\right) + 4\left(\frac{1}{12} + 2\zeta^2\right) \right\} \\ & - 4c^5\zeta \left\{ \frac{1}{10}\left(t_1^4 + \frac{t_1^2}{2} + \frac{1}{3}\right) + 4\zeta^2\left(\frac{1}{3} + 2\zeta^2\right) + \frac{2}{3}\zeta^2\left(\frac{1}{2} + t_1^2\right) \right\} \\ & + 0(c^6) \end{aligned} \tag{48}$$

where

$$\zeta = \frac{(1-2\nu)^2}{(7-8\nu)\pi^2}. \tag{49}$$

4. Load-displacement relationship for the penny-shaped rigid inclusion

The load-displacement relationship for the in-plane translation of the disc inclusion located at the weakened plane can be obtained by evaluating the integral

$$T = -2 \int_0^{2\pi} \int_0^a \{\sigma_{rz}(r,\theta,0)\cos\theta - \sigma_{\theta z}(r,\theta,0)\sin\theta\}\, r\, dr\, d\theta . \tag{50}$$

In terms $C(\xi)$ and $D(\xi)$, (50) can be expressed in the form

$$T = \frac{4\pi}{(1-2\nu)} \int_0^a \left[\int_0^\infty \xi \{C(\xi) + 2(1-\nu) D(\xi)\} J_0(\xi r)\, d\xi \right] r\, dr . \tag{51}$$

Using the derivations given in the preceding section, (51) can be reduced to the result

$$T = \frac{64\, G\, \delta\, a(1-\nu)}{(7-8\nu)} \int_0^1 F_1(u_1)\, du_1 . \tag{52}$$

Evaluating (52) we obtain

$$\begin{aligned} T = \frac{64\, G\, \delta\, a(1-\nu)}{(7-8\nu)} \Bigg[& 1 - 2c\zeta + 4c^2\zeta^2 - 4c^3\zeta\left(\frac{1}{6} + 2\zeta^2\right) \\ & + 4c^4\zeta^2\left(\frac{11}{18} + 8\zeta^2\right) - 4c^5\zeta \\ & \cdot \left\{\frac{7}{700} + 4\zeta^2\left(\frac{1}{3} + 2\zeta^2\right) + \frac{5}{9}\zeta^2\right\} + 0(c^6) \Bigg] . \end{aligned} \tag{53}$$

5. Stress intensity factor at the boundary of the externally cracked region

From the symmetry of the state of deformation about the plane $z = 0$ and the traction free nature of the cracked region ($b < r < \infty$), it is evident that the non-zero stress intensity factor is of the K_1 type, which is induced by the normal stress σ_{zz}. The expression (30) for the normal stress is

$$\sigma_{zz}(r,\theta,0) = \left[\int_0^\infty \xi D(\xi) J_1(\xi r)\, d\xi \right] \cos\theta . \tag{54}$$

Using (36) we have

$$D(\xi) = \frac{1}{2(1-\nu)} \left[\frac{f_1(a)\sin(a\xi)}{\xi} - \frac{f_2(b)\cos(b\xi)}{\xi} - \frac{1}{\xi}\int_b^\infty f_2'(t)\cos(\xi t)\, dt \right] \tag{55}$$

where $f_2(\infty) = 0$ and the prime denotes derivative of the function with respect to its argument. The integral Eq. (40) can be written in the form

$$f_2(u_1 b) = \frac{2c}{\pi u_1} \int_0^1 f_1(a t_1) \left\{ 1 + \frac{c^2 t_1^2}{u_1^2} + \frac{c^4 t_1^4}{u_1^4} + \frac{c^6 t_1^6}{u_1^6} + \frac{c^8 t_1^8}{u_1^8} + 0(c^{10}) \right\} dt_1; \quad u_1 > 1. \tag{56}$$

Using the third equation of (44) and the expression for $F_1(t_1)$ given by (48) in (56) we obtain the following result for $f_2(u_1 b)$:

$$f_2(u_1 b) = \frac{32 G \delta (1-\nu)(1-2\nu)}{(7-8\nu)\pi^2 u_1} \cdot \left[c - 2c^2 \zeta + c^3 \left\{ 4\zeta^2 + \frac{1}{3u_1^2} \right\} - 2c^4 \zeta \left\{ \frac{5}{18} + 4\zeta^2 + \frac{1}{3u_1^2} \right\} + c^5 \left\{ \zeta^2 \left(\frac{1}{3u_1^2} + \frac{11}{18} + 8\zeta^2 \right) + \frac{2}{5u_1^4} \right\} + 0(c^6) \right]. \tag{57}$$

The stress intensity factor K_1 for the flaw opening modes is defined by

$$K_1 = \lim_{r \to b^-} [2(b-r)]^{1/2} \sigma_{zz}(r, \theta, 0). \tag{58}$$

Using (54) and (55) in (58) we can evaluate the stress intensity factor in the following form

$$K_1 = \frac{f_2(b) \cos\theta}{2(1-\nu)\sqrt{b}}. \tag{59}$$

From (57) and (59) we obtain the following series expansion, in terms of c, for the stress intensity factor at the boundary of the external circular crack:

$$K_1 = \frac{16 G \delta (1-2\nu)\cos\theta}{(7-8\nu)\pi^2 \sqrt{b}} \left[c - 2c^2\zeta + c^3 \left\{ 4\zeta^2 + \frac{1}{3} \right\} - 2c^4 \zeta \left\{ \frac{11}{18} + 4\zeta^2 \right\} + c^5 \left\{ \zeta^2 \left(\frac{17}{18} + 8\zeta^2 \right) + \frac{2}{5} \right\} + 0(c^6) \right]. \tag{60}$$

6. Limiting cases and numerical results

In this section we shall present a numerical evaluation of the expressions (53) and (60) respectively for the in-plane translational stiffness of the inclusion at the cracked plane and the stress intensity factor at the boundary of the external circular crack. Before performing any numerical evaluations it is useful to record here the accuracy of the series estimates in predicting exact solution to certain limiting cases.

In the limiting case when the radius of the externally cracked region becomes infinite (i.e., $c \to 0$), the result (53) reduces to

$$T = \frac{64\,G\,\delta\,a(1-\nu)}{(7-8\,\nu)}. \tag{61}$$

This expression is in agreement with the results obtained, independently by Keer [11], Kassir and Sih [12] and Selvadurai [13] for the in-plane translational

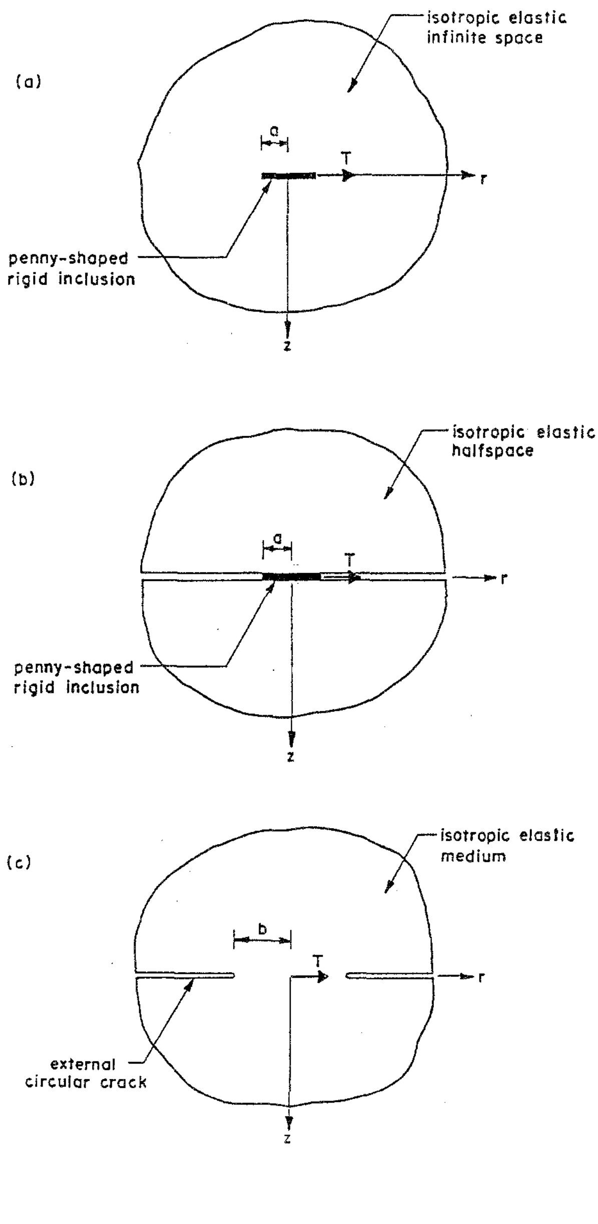

Figure 2
Translational loading of a penny-shaped rigid inclusion located at a cracked plane: limiting cases.

stiffness of a penny-shaped rigid inclusion embedded in an isotropic elastic solid (Fig. 2a) by making use of potential function methods, ellipsoidal harmonic function techniques and dual integral equation formulations respectively. For future reference, we note that in the particular instance, when the externally cracked region extends to the boundary of the penny-shaped rigid inclusion (i.e., $c = 1$), the problem effectively reduces to the in-plane translation of an inclusion which is bonded to two non-interacting halfspace regions (Fig. 2b). By making use of the results given by Gladwell [14] it can be shown that the *exact* load-displacement relationship for the inclusion is given by

$$T = \frac{16 G \delta a}{\left[1 + \frac{(1-2\nu)}{\ln(3-4\nu)}\right]}. \tag{62}$$

It is evident that owing to the symmetry of the displacement component u_z about $z = 0$ the inclusion does not experience any rotation about the y-axis. In the limiting case of material incompressibility both (61) and (62) reduce to the single result

$$T = \frac{32 G \delta a}{3}. \tag{63}$$

Since in the limit of material incompressibility the in-plane translational stiffness of the inclusion embedded in an intact solid coincides with equivalent results for the completely cracked solid it may be concluded that for $\nu = \frac{1}{2}$, the extent of cracking has no influence on the translational stiffness. From (53), it may be observed that as $\nu \to \frac{1}{2}$, $\zeta \to 0$ and the result (53) also reduces to (63). The Fig. 3 illustrates the manner in which the non-dimensional in-plane

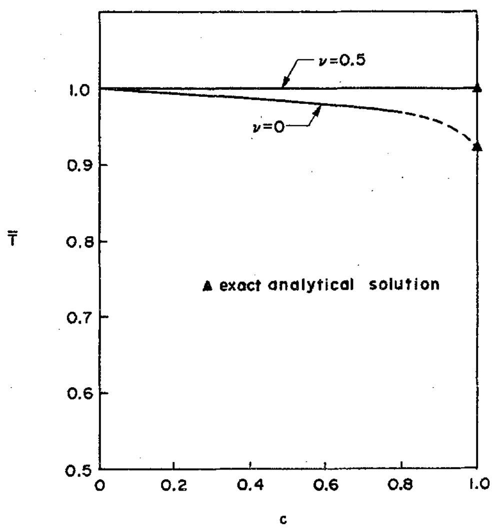

Figure 3
Non-dimensional in-plane translational stiffness of a penny-shaped inclusion located at a cracked plane.

translational stiffness $\bar{T}$ given by

$$\frac{T(7-8\nu)}{64\,G\,\delta\,a(1-\nu)} = \bar{T}(c, \nu) \tag{64}$$

is influenced by the extent of the cracked region and Poisson's ratio of the elastic material.

The accuracy of the result (60) for the stress intensity factor at the boundary of the external circular crack can be established by invoking the following limiting procedure. In the special case when $a \to 0$, the geometry of the inclusion approaches that of a concentrated in-plane force with magnitude $T = 64\,G\,\delta\,a(1-\nu)/(7-8\nu)$, which acts at the origin (Fig. 2c). It is also evident that as $a \to 0$, terms of order c^2 and higher can be neglected in (60). Using these reductions (60) simplifies to the result

$$K_1 = \frac{(1-2\nu)\,T\cos\theta}{4\pi^2(1-\nu)\,b^{3/2}}\,. \tag{65}$$

The result (65) is in agreement with the expression given by Kassir and Sih [15] for the flaw opening mode stress intensity factor at the boundary of an external circular crack loaded by an in-plane central force T. It may also be observed that in the limit of material incompressibility, the boundary condition (17) pertaining to σ_{zz} is satisfied over the entire region $r \geqq 0$; consequently $K_1 \equiv 0$. The Fig. 4

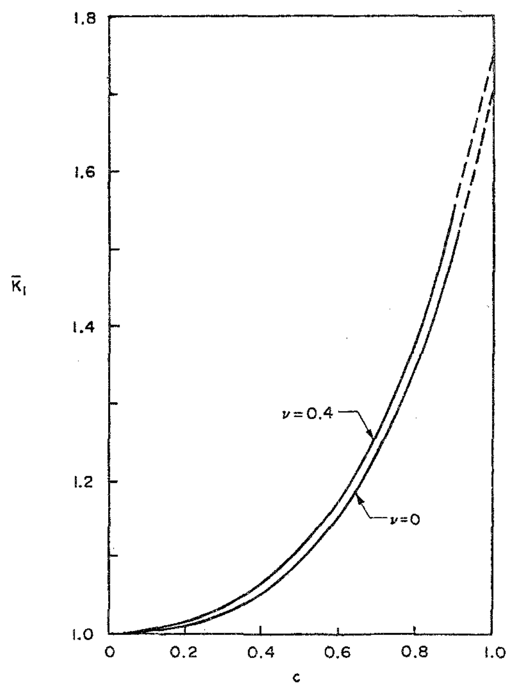

Figure 4
Normalized flaw opening mode stress intensity factor at the boundary of the external circular crack due to in-plane loading of the embedded inclusion.

illustrates the manner in which the normalized stress intensity factor $\bar{K}_1$ given by

$$\frac{4 K_1 (1 - \nu) b^{3/2} \pi^2}{(1 - 2\nu)\cos\theta} \left\{ \frac{(7 - 8\nu)}{64 G \delta a (1 - \nu)} \right\} = \bar{K}_1 (c, \nu) \tag{66}$$

is influenced by the extent of the cracked region and Poisson's ratio of the elastic material. In both cases the results for $\bar{T}$ and $\bar{K}_1$ are presented for $\nu \in (0, 0.5)$ and $c \in (0, 0.9)$. These results indicate that the solutions derived from the series expansion schemes compare very favourably with known exact solutions for limiting cases.

7. Conclusions

The paper develops certain analytical results for the problem of the interaction between an external circular crack and a penny-shaped rigid inclusion located at the plane of the crack and subjected to an in-plane central load. The mixed boundary value problem associated with this asymmetric three-dimensional elastostatic problem can be reduced to the solution of two coupled Abel-type integral equations. These can be further reduced to a single Fredholm integral equation of the second-kind, which can be solved by adopting a power series representation of the function involved. The series expansion parameter corresponds to the ratio of the radius of the inclusion to the radius of the external circular crack. Results for the in-plane translational stiffness of the penny-shaped rigid inclusion and the stress intensity factor at the boundary of the external circular crack are presented in power series form. Several limiting results of a closed form nature can be recovered from these solutions. Due to the nature of the power series approximation scheme, the results presented are valid for $c \ll 1$. The numerical results, however, yield reasonable trends for $c \in (0, 0.9)$. It is also important to note that the existence of the external circular crack does not lead to any appreciable reductions in the in-plane translational stiffness of the embedded inclusion.

It may also be noted that the particular loading examined in this paper corresponds to the situation where the resultant force at infinity corresponds to a force of magnitude T which acts in the negative x-direction. As has been observed by Stallybrass [16], the nature of the regularity conditions on the resultant force prescribed at infinity influences the magnitude of the stress intensity factor at the boundary of the exterior crack. If a solution to the problem of the externally cracked solid with a zero force resultant at infinity is required then certain auxiliary solutions need to be added to the results presented in this study, to recover the relevant stress intensity factor at the boundary of the external circular crack.

Appendix A

The general expressions for $m_i(t)$ $(i = 0, 1, 2, \ldots, 5)$ take the following forms:

$$m_0(t_1) = 1$$

$$m_1(t_1) = -2\zeta$$

$$m_2(t_1) = 4\zeta^2$$

$$m_3(t_1) = -4\zeta\left[\int_0^1 s\,m_2(s)\,ds + \tfrac{1}{3}\int_0^1 s(t_1^2 + s^2)\,m_0(s)\,ds\right]$$

$$m_4(t_1) = -4\zeta^2\left[\int_0^1 \frac{s}{3}(t_1^2 + s^2)\,m_1(s)\,ds + \int_0^1 s\,m_3(s)\,ds\right]$$

$$m_5(t_1) = -4\zeta\left[\tfrac{1}{5}\int_0^1 s(t_1^4 + s^4 + s^2 t_1^2)\,ds + \int_0^1 s\,m_4(s)\,ds + \tfrac{1}{3}\int_0^1 s(t_1^2 + s^2)\,m_2(s)\,ds\right].$$

Explicit results for $m_i(t_1)$ $(i = 3, 4, 5)$ take the following forms:

$$m_3(t_1) = -4\zeta\,\{2\zeta^2 + \tfrac{1}{6}(\tfrac{2}{3} + t_1^2)\}$$

$$m_4(t_1) = 4\zeta^2\,\{\tfrac{1}{3}(t_1^2 + \tfrac{1}{2}) + 4(\tfrac{1}{12} + 2\zeta^2)\}$$

$$m_5(t_1) = -4\zeta\left[\frac{1}{10}\left(t_1^4 + \frac{t_1^2}{2} + \frac{1}{3}\right) + 4\zeta^2\left(\frac{1}{3} + 2\zeta^2\right) + \frac{2\zeta^2}{3}\left(t_1^2 + \frac{1}{2}\right)\right].$$

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Abstract

The paper examines the problem related to the axisymmetric interaction between an external circular crack and a centrally placed penny-shaped rigid inclusion located in the plane of the crack. The interface between the inclusion and the elastic medium exhibits a Mindlin-type imperfect bi-lateral contact. Analytical results presented in the paper illustrate the manner in which the lateral translational stiffness of the inclusion and the stress intensity factor at the boundary of the external circular crack are influenced by the inclusion/crack radii ratio.

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