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AXISYMMETRIC PROBLEMS FOR AN EXTERNALLY CRACKED ELASTIC SOLID—II. EFFECT OF A PENNY-SHAPED INCLUSION

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Abstract—The present paper examines the problem of the axisymmetric translation of a rigid disc shaped inclusion which is embedded in bonded contact in an isotropic elastic solid. The plane containing the inclusion is weakened by an external circular crack. The mathematical analysis of the mixed boundary value problem concentrates on the evaluation of the load-displacement relationship for the penny-shaped inclusion and the stress intensity factor at the boundary of the externally cracked region.

1. INTRODUCTION

The stress analysis of elastic bodies which are reinforced with either rigid or elastic inclusions occupies a prominent position in the mathematical theory of three-dimensional elastostatics. Results derived from such analyses are of fundamental interest to studies related to composite materials. Detailed accounts of inclusion problems in classical elasticity are given by Mura [1], Willis [2] and Walpole [3]. Flat disc shaped inclusions are particular limiting cases of the general class of three-dimensional ellipsoidal or spheroidal inclusions. The articles by Collins [4], Kanwal and Sharma [5], Kassir and Sih [6], Keer [7] and Selvadurai [8–14] are primarily concerned with the study of disc inclusions which are embedded in bonded contact with isotropic and transversely isotropic elastic solids. Several investigators [1, 2, 15–21] have extended these studies to include a variety of other features such as flexural behaviour of the disc inclusion, annular configuration of the inclusion, influence of externally applied loads and the influence of free boundaries. Recently, Selvadurai [22–24] examined the class of problems pertaining to the translation and rotation of a rigid elliptical disc inclusions embedded at a bimaterial elastic interface. An exact formulation of the associated elasticity problems yield systems of simultaneous integral equations with complicated kernel functions which are not amenable to exact solution. For this reason, the analyses in [22–24] concentrate on the development of a set of bounds for the evaluation or the stiffness of the inclusion. These bounds are developed by invoking kinematic and traction constraints at the bi-material interface. The accuracy of these bounds have also been verified by appeal to a boundary element analysis of the elliptical inclusion embedded at a bi-material elastic interface [25]. The behaviour of inclusions embedded in bi-material elastic interfaces is of interest to the study of precipitation hardening effects in multiphase composites.

An examination of the literature on both inclusion and crack problems indicates that the class of problems which deal with the interaction between cracks and inclusions located in elastic media has received only limited attention. The work of Keer [16] which examines the partial adhesive contact between a disc inclusion and an elastic medium can be considered as a typical problem which examines this type of interactive behaviour. Recently, Selvadurai and Singh [26] examined the problem of the internal indentation of a penny-shaped crack by a rigid penny-shaped inclusion. This particular problem is of interest to the study of thermally induced fracturing and degradation of multiphase composites. In this paper, we examine the problem of the axial loading of a penny-shaped rigid inclusion which is located in the plane of an external circular crack (Fig. 1). The mathematical formulation of the mixed boundary value problem employs a Hankel transform development of the govening field equations. The integral equations associated with the mixed boundary conditions are reduced to a system of

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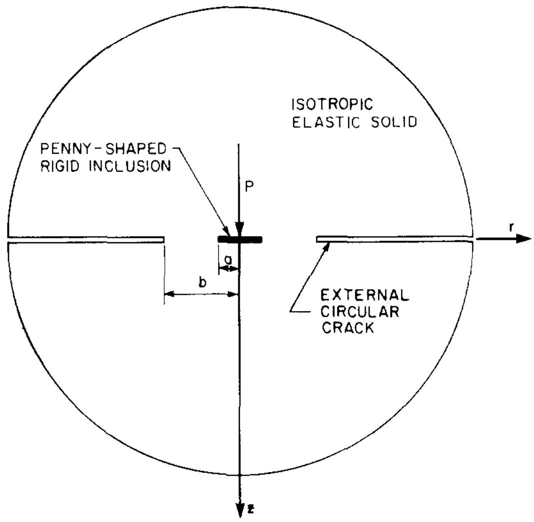

Fig. 1. Penny shaped rigid inclusion embedded in an externally cracked elastic solid.

Abel-type integral equations which are solved in an approximate fashion. The approximate method involves expansion of the governing functions in power series in terms of a small non-dimensional parameter. The small parameter corresponds to the ratio of the radius of the penny-shaped inclusion to the radius of the external circular crack. The numerical results obtained for the axial stiffness of the disc-inclusion are compared with exact analytical results derived for the stiffness of the inclusion embedded either in an uncracked elastic solid or a completely cracked interface. Also, an expression is derived for the stress intensity factor at the boundary of the externally cracked region.

2. BASIC EQUATIONS

For the analysis of the axisymmetric problem related to the interaction between the penny-shaped inclusion and the external circular crack, we employ the formulation based on the strain potential approach of Love [27]. Briefly, the solution of the displacement equations of equilibrium, for a medium free of body forces, can be represented in terms of a bi-harmonic function $\Phi(r, z)$, i.e.

$$\nabla^2\nabla^2\Phi(r, z) = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is the axisymmetric form of Laplace's operator referred to the cylindrical polar coordinate system. The components of the displacement vector $\mathbf{u}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ referred to the cylindrical polar coordinate system can be expressed in terms of the derivatives of Φ. We have

$$2Gu_r = -\frac{\partial^2\Phi}{\partial r\,\partial z} \tag{3}$$

$$2Gu_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \tag{4}$$

where G and ν are the linear elastic shear modulus and Poisson's ratio respectively.

Similarly, the components of the stress tensor are given by

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{\partial^2}{\partial r^2}\right)\Phi \tag{5}$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2 - \frac{1}{r}\frac{\partial}{\partial r}\right)\Phi \tag{6}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left([2-\nu]\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\Phi \tag{7}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left([1-\nu]\nabla^2 - \frac{\partial^2}{\partial z^2}\right)\Phi \tag{8}$$

3. THE INCLUSION-CRACK INTERACTION PROBLEM

We consider the problem of a penny-shaped rigid inclusion of radius a which is embedded in bonded contact with an isotropic elastic medium. The plane containing the disc inclusion is weakened by an external circular crack of radius $b(b>a)$. The inclusion is subjected to a central load P which acts in the +ve z-direction. The inclusion experiences a rigid body translation Δ along the +ve z-direction. It can be shown that this mode of deformation induces a state of asymmetry about the plane $z=0$. Therefore we may restrict the analysis to the examination of a single halfspace region occupying $z \geqslant 0$. The relevant mixed boundary conditions associated with the inclusion problem are as follows:

$$u_z(r,0) = \Delta; \qquad 0 \leqslant r \leqslant a \tag{9}$$

$$u_r(r,0) = 0; \qquad 0 \leqslant r \leqslant b \tag{10}$$

$$\sigma_{zz}(r,0) = 0; \qquad a < r < \infty \tag{11}$$

$$\sigma_{rz}(r,0) = 0; \qquad b < r < \infty \tag{12}$$

For the integral equation formulation of the mixed boundary value problem posed by (9)–(12), we seek solutions of (1) which can be obtained by Hankel transform development of the governing differential equation. Also, the displacement and stress fields derived from Φ should reduce to zero as $(r^2+z^2)^{1/2} \to \infty$. Following Sneddon [28], the relevant solution is given by

$$\Phi = \int_0^\infty \xi[A(\xi) + zB(\xi)]\mathrm{e}^{-\xi z}J_0(\xi r)\,\mathrm{d}\xi \tag{13}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions which are to be determined by satisfying the boundary conditions (9)–(12) on the plane $z=0$. Using the integral representation for Φ given by (13) in the expressions for $\mathbf{u}$ and $\boldsymbol{\sigma}$, it can be shown that the mixed boundary conditions (9)–(12) reduce to the following system of integral equations:

$$H_0[\xi\{\xi A(\xi) + 2(1-2\nu)B(\xi)\}; r] = -2G\Delta; \qquad 0 \leqslant r \leqslant a \tag{14}$$

$$H_1[\xi\{-\xi A(\xi) + B(\xi)\}; r] = 0; \qquad 0 \leqslant r \leqslant b \tag{15}$$

$$H_0[\xi^2\{\xi A(\xi) + (1-2\nu)B(\xi)\}; r] = 0; \qquad a < r < \infty \tag{16}$$

$$H_1[\xi^2\{\xi A(\xi) - 2\nu B(\xi)\}; r] = 0; \qquad b < r < \infty \tag{17}$$

where $H_n[f(\xi); r]$ is the Hankel transform of order n which is defined by

$$H_n[f(\xi); r] = \int_0^\infty \xi f(\xi) J_n(\xi r)\,\mathrm{d}\xi \tag{18}$$

We now make the assumption that as $b \to \infty$ we should recover from the solution developed the appropriate results for the problem of a penny-shaped rigid inclusion which is embedded in an uncracked elastic solid. We introduce functions $M(\xi)$ and $N(\xi)$ such that

$$2(1-\nu)\xi^3 A(\xi) = -(1-2\nu)M(\xi) + N(\xi) \tag{19}$$

$$2(1-\nu)\xi^3 B(\xi) = M(\xi) + N(\xi) \tag{20}$$

The integral equations (14)–(17) can now be written as

$$H_0\left[\xi^{-1}\left\{N(\xi) + \frac{(1-2\nu)}{(3-4\nu)}M(\xi)\right\}; r\right] = -\frac{4G\Delta(1-\nu)}{(3-4\nu)}; \qquad 0 \leqslant r \leqslant a \tag{21}$$

$$H_1[\xi^{-1}M(\xi); r] = 0; \qquad 0 \leqslant r \leqslant b \tag{22}$$

$$H_0[N(\xi), r] = 0; \qquad a < r < \infty \tag{23}$$

$$H_1[\{1-2\nu)N(\xi) - M(\xi)\}; r] = 0; \qquad b < r < \infty \tag{24}$$

In order to satisfy the boundary conditions (22) and (23) explicitly, we represent $M(\xi)$ and $N(\xi)$ in the following forms; we have

$$M(\xi) = \int_b^\infty \psi(t)\sin(\xi t)\,\mathrm{d}t = \frac{\psi(b)\cos(\xi b)}{\xi} + \frac{1}{\xi}\int_b^\infty \psi'(t)\cos(\xi t)\,\mathrm{d}t \tag{25}$$

where $\psi(\infty) = 0$; $\psi'(t) = \mathrm{d}\psi/\mathrm{d}t$ and

$$N(\xi) = \int_0^a \phi(t)\cos(\xi t)\,\mathrm{d}t = \frac{\phi(a)\sin(\xi a)}{\xi} - \int_0^a \phi'(t)\sin(\xi t)\,\mathrm{d}t \tag{26}$$

where $\phi'(t) = \mathrm{d}\phi/\mathrm{d}t = 0$. Substituting the values of $M(\xi)$ and $N(\xi)$ given by (25) and (26) into (22) and (23), we note that the latter equations are automatically satisfied. Using the results (25) and (26) the eqn (21) can be expressed in the form

$$\int_0^r \frac{\phi(t)\,\mathrm{d}t}{[r^2-t^2]^{1/2}} + \frac{(1-2\nu)}{(3-4\nu)}\int_b^\infty \frac{\phi(t)\,\mathrm{d}t}{[t^2-r^2]^{1/2}} = -\frac{4G\Delta(1-\nu)}{(3-4\nu)}; \qquad 0 \leqslant r \leqslant a \tag{27}$$

The above is an integral equation of the Abel type, the solution of which can be written in the form

$$\begin{aligned}\phi(t) = &-\frac{2(1-2\nu)}{(3-4\nu)\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{r\,\mathrm{d}r}{[t^2-r^2]^{1/2}}\int_b^\infty \frac{\psi(u)\,\mathrm{d}u}{[u^2-r^2]^{1/2}} \\ &-\frac{8G\Delta(1-\nu)}{(3-4\nu)\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{r\,\mathrm{d}r}{[t^2-r^2]^{1/2}}; \qquad 0 \leqslant r \leqslant a\end{aligned} \tag{28}$$

We observe that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \left\{1 : \frac{1}{[u^2-r^2]^{1/2}}\right\}\frac{r\,\mathrm{d}r}{[t^2-r^2]^{1/2}} = \left\{1 : \frac{u}{(u^2-t^2)}\right\} \tag{29}$$

With the aid of these results we can write (28) in the form

$$\phi(t) = -\frac{2(1-2\nu)}{(3-4\nu)\pi}\int_b^\infty \frac{u\psi(u)\,\mathrm{d}u}{(u^2-t^2)} - \frac{8G\Delta(1-\nu)}{(3-4\nu)\pi}; \qquad 0 \leqslant r \leqslant a \tag{30}$$

By multiplying (24) by $[r^2-y^2]^{-1/2}$ and integrating between the limits y to ∞ we obtain

$$\int_0^\infty \{(1-2\nu)N(\xi) - M(\xi)\}\sin(\xi y)\,\mathrm{d}\xi = 0; \qquad b < y < \infty \tag{31}$$

By substituting (25) and (26) in (31) we find that

$$\frac{(1-2\nu)}{2}\left[\phi(a)\ln\left|\frac{a+y}{a-y}\right| - \int_0^a \phi'(t)\ln\left|\frac{t+y}{t-y}\right|\mathrm{d}t\right] - \left[\frac{\pi}{2}\psi(b) + \frac{\pi}{2}\int_b^y \psi'(t)\,\mathrm{d}t\right] = 0 \tag{32}$$

Making use of the condition $\phi(0)=0$ we obtain

$$\int_0^a \phi'(t)\ln\left|\frac{t+y}{t-y}\right| dt = \phi(a)\ln\left|\frac{a+y}{a-y}\right| + 2y\int_0^a \frac{\phi(t)\,dt}{(t^2-y^2)} \tag{33}$$

Also,

$$\int_b^y \psi'(t)\,dt = \psi(y)-\psi(b) \tag{34}$$

The eqn (32) can now be expressed in the form

$$\frac{2(1-2\nu)u_1c}{\pi}\int_0^1 \frac{\phi(at_1)\,dt_1}{(u_1^2-c^2t_1^2)} = \psi(bu_1); \qquad 1<u_1<\infty \tag{35}$$

where

$$y=u_1b; \qquad t=at_1; \qquad c=\frac{a}{b} \tag{36}$$

The integral eqn (30) can also be written as

$$\phi(at_1) = -\frac{2(1-2\nu)}{(3-4\nu)\pi}\int_1^\infty \frac{u_1\psi(bu_1)\,du_1}{(u_1^2-c^2t_1^2)} - \frac{8G\Delta(1-\nu)}{(3-4\nu)\pi}; \qquad 0\leqslant t_1\leqslant 1 \tag{37}$$

In the ensuing, we develop a series solution of the coupled integral eqns (35) and (37) by assuming that $c<1$. We assume that $\phi(at_1)$ and $\psi(bt_1)$ admit power series representations of the form

$$\phi(at_1) = \sum_{i=0}^{N} c^i m_i(t_1) \tag{38}$$

$$\psi(bt_1) = \sum_{i=0}^{N} c^i n_i(t_1) \tag{39}$$

By substituting (38) and (39) into (37) and (35) and expanding the term $(u_1^2-c^2t_1^2)^{-1}$ in power series in c we obtain two equations of the form

$$\sum_{i=0}^{N} c^i m_i(t_1) = -\frac{8G\Delta(1-\nu)}{(3-4\nu)\pi} - \frac{2(1-2\nu)}{(3-4\nu)\pi}\int_1^\infty \sum_{i=0}^{N} c^i n_i(u_1)$$
$$\cdot\left\{\frac{1}{u_1}+\frac{c^2t_1^2}{u_1^3}+\frac{c^4t_1^4}{u_1^5}+\frac{c^6t_1^6}{u_1^7}+\frac{c^8t_1^8}{u_1^9}+0(c^{10})\right\}du_1; \qquad 0<t_1<1 \tag{40}$$

$$\sum_{i=0}^{N} c^i n_i(u_i) = \frac{2(1-2\nu)c}{\pi}\int_0^1 \sum_{i=0}^{N} c^i m_i(t_1)$$
$$\cdot\left\{\frac{1}{u_1}+\frac{c^2t_1^2}{u_1^3}+\frac{c^4t_1^4}{u_1^5}+\frac{c^6u_1^6}{u_1^7}+\frac{c^8t_1^8}{u_1^9}+0(c^{10})\right\}dt_1; \qquad 1<u_1<\infty \tag{41}$$

By comparing like terms in order c^i $(i=0,1,\ldots,N)$ in (40) and (41) we obtain integral expressions for $m_i(t_1)$ and $n_i(t_1)$. These are given in the Appendix. Explicit expressions for $m_i(t_1)$ and $n_i(t_1)$ are also given in the Appendix. This formally completes the analysis of the mixed boundary value problem defined by (9)–(12). The results for the stresses and displacements in the elastic medium can be determined by making use of the series expressions for $\phi(at_1)$ and $\psi(bt_1)$. The final expressions for these functions take the following forms:

$$\phi(at_1) = \frac{8\Delta G(1-\nu)}{(3-4\nu)\pi}\left[-1+\zeta c-\zeta^2c^2+\zeta c^3\left\{\frac{t_1^3}{3}+\frac{1}{9}+\zeta^2\right\}-\zeta^2c^4\left\{\frac{t_1^2}{3}+\frac{1}{3}+\zeta^2\right\}\right.$$
$$\left.+\zeta c^5\left\{\frac{t_1^4}{5}+t_1^2\left(\frac{1}{15}+\frac{\zeta^2}{3}\right)+\frac{1}{25}+\frac{\zeta^2}{9}+\zeta^2\left(\frac{4}{9}+\zeta^2\right)\right\}+O(c^6)\right] \tag{42}$$

$$\psi(bt_1) = \frac{16(1-\nu)(1-2\nu)\Delta G}{(3-4\nu)\pi^2}\left[-\frac{c}{t_1}+\frac{\zeta c^2}{t_1}-c^3\left\{\frac{1}{3t_1^3}+\zeta^2\right\}\right.$$
$$\left.+\zeta c^4\left\{\frac{1}{3t_1^3}+\frac{\zeta^2}{t_1}+\frac{2}{9t_1}\right\}-c^5\left\{\frac{1}{5t_1^5}+\frac{\zeta^2}{3t_1^3}+\frac{\zeta^2}{t_1}\left(\frac{4}{9}+\zeta^2\right)\right\}+O(c^6)\right] \tag{43}$$

where

$$\zeta = \frac{4(1-2\nu)^2}{(3-4\nu)\pi^2}. \tag{44}$$

4. LOAD DISPLACEMENT-RELATIONSHIP FOR THE PENNY-SHAPED INCLUSION

A result of engineering interest concerns the load-displacement relationship for the penny-shaped rigid inclusion located at the weakened plane. From (23), we observe that

$$\sigma_{zz}(r, 0) = \int_0^\infty \xi N(\xi) J_0(\xi r)\, d\xi \tag{45}$$

Substituting the value of $N(\xi)$ given by (26) into (45) we obtain

$$\sigma_{zz}(r, 0) = \frac{\phi(a)}{[a^2 - r^2]^{1/2}} - \int_0^a \frac{\phi'(t)\, dt}{[t^2 - r^2]^{1/2}}; \qquad 0 < r < a \tag{46}$$

Considering the equilibrium of the penny-shaped inclusion we have

$$P = -4\pi \int_0^a r\sigma_{zz}(r, 0)\, dr \tag{47}$$

Using (46), (47) can be reduced to the form

$$P = -4\pi a \int_0^1 \phi(at_1)\, dt_1 \tag{48}$$

Evaluating (48) we obtain

$$P = \frac{32G\Delta a(1-\nu)}{(3-4\nu)} \left[1 - c\zeta + c^2\zeta^2 - c^3\zeta\left\{\frac{2}{9} + \zeta^2\right\} + c^4\zeta^2\left\{\frac{4}{9} + \zeta^2\right\} - c^5\zeta\left\{\frac{23}{225} + \frac{2}{3}\zeta^2 + \zeta^4\right\} + O(c^6)\right] \tag{49}$$

5. STRESS INTENSITY FACTOR AT THE BOUNDARY OF THE EXTERNALLY CRACKED REGION

From the asymmetry of the deformation and the traction free nature of the cracked region ($b < r < \infty$), it is evident that $\sigma_{zz} = 0$ for $r \in (a, \infty)$. The non-zero stress intensity factor is of the K_2 type, which is induced by the shearing stress σ_{rz}. The expression (24) for the stress σ_{rz} can be reduced to the form

$$\sigma_{rz}(r, 0) = \frac{1}{2(1-\nu)} \int_0^\infty [-M(\xi) + (1-2\nu)N(\xi)]\xi J_1(\xi r)\, d\xi \tag{50}$$

By substituting (16) and (19) into (50) and making the appropriate reductions we obtain the following result for $\sigma_{rz}(r, 0)$: i.e.

$$\sigma_{rz}(r, 0) = \frac{1}{2(1-\nu)r}\left[-\psi(b)\left\{1 - \frac{b}{(b^2 - r^2)^{1/2}}\right\} + (1-2\nu)\phi(a)\left\{\frac{a}{(r^2 - a^2)^{1/2}}\right\} - \int_b^\infty \psi'(t)\left\{1 - \frac{t}{(t^2 - r^2)^{1/2}}\right\} dt - (1-2\nu)\int_0^a \frac{\phi'(t)t\, dt}{(r^2 - t^2)^{1/2}}\right]; \qquad 0 < r < b \tag{51}$$

The stress intensity factor K_2 at the boundary of the externally cracked region is defined by

$$K_2 = \lim_{r \to b^-} \{[2(b-r)]^{1/2}\sigma_{rz}(r, 0)\} \tag{52}$$

Evaluating (52) we obtain

$$K_2 = \frac{\psi(b)}{2(1-\nu)\sqrt{b}} \tag{53}$$

From (43) and (53) we obtain the following expression for the stress intensity factor:

$$K_2 = -\frac{(1-2\nu)}{4\pi^2(1-\nu)b^{3/2}}\left\{\frac{32G\Delta a(1-\nu)}{(3-4\nu)}\right\}\left[1 - \zeta c + c^2\left(\frac{1}{3}+\zeta^2\right)\right.$$
$$\left. - 4\zeta c^3\left(\frac{5}{9}+\frac{\zeta^3}{3}\right) + c^4\left(\frac{1}{5}+\frac{7}{9}\zeta^2+\zeta^4\right) + O(c^5)\right] \tag{54}$$

6. NUMERICAL EVALUATION OF THE INTEGRAL EQUATIONS

The integral eqns (35) and (37) governing the interaction between the external circular crack and the rigid penny-shaped inclusion can be solved in a variety of ways. The Section 3 presents a technique whereby power series approximations are developed, for $\phi(at_1)$ and $\psi(bt_1)$, in terms of the small non-dimensional parameter $c = a/b$. The accuracy of such approximate series solutions invariably depends upon the number of terms incorporated in the development of the series solution. For this reason it is instructive to consider a direct numerical solution of the relevant integral equations.† Prior to such a numerical analysis it is convenient to decouple the integral equations. By substituting (35) into (37) and making use of the integral relation

$$\int_1^\infty \frac{2cu_1^2\,du_1}{(u_1^2-c^2t_1^2)(u_1^2-c^2\xi^2)} = \frac{1}{(t_1^2-\xi^2)}\left[t_1\ln\left\{\frac{1+ct_1}{1-ct_1}\right\} - \xi\ln\left\{\frac{1+c\xi}{1-c\xi}\right\}\right] \tag{55}$$

and making use of the substitutions

$$\phi^*(at_1) = -\frac{(3-4\nu)\pi\phi(at_1)}{8G\Delta(1-\nu)}; \qquad c_1 = \frac{2(1-2\nu)^2}{(3-4\nu)\pi^2} \tag{56}$$

we obtain the following integral equation for $\phi^*(at_1)$:

$$\phi^*(at_1) + \int_0^1 K(t_1, \xi)\phi^*(a\xi)\,d\xi = 1; \qquad 0 \le t_1 \le 1 \tag{57}$$

where the kernal function $K(t_1, \xi)$ is given by

$$K(t_1, \xi) = \frac{c_1}{(t_1^2-\xi^2)}\left[t_1\ln\left\{\frac{1+ct_1}{1-ct_1}\right\} - \xi\ln\left\{\frac{1+c\xi}{1-c\xi}\right\}\right] \tag{58}$$

By using a Gaussian quadrature scheme, the integral eqn (57) can be represented as a matrix equation.

$$\sum_{j=1}^{N} A_{ij}\phi_j^* = 1, \qquad i = 1, 2, \ldots, N \tag{59}$$

where $\phi_j^* = \phi^*(az_j)$; $z_j = (1+g_j)/2$; g_i are the Gauss points, ω_i are the weights, $i = 1$ to N; N are the number of points. The coefficients of A_{ij} for $i \neq j$ are given by

$$A_{ij} = \frac{c_1\omega_j}{2(z_i^2-z_j^2)}\left[z_i\ln\left(\frac{1+cz_i}{1-cz_i}\right) - z_j\ln\left(\frac{1+cz_j}{1-cz_j}\right)\right] \tag{60}$$

when $i = j$, the coefficients of A_{ij} are given by

$$A_{ii} = 1 + \lim_{z_j\to z_i} A_{ij} = 1 + \frac{c_1\omega_i}{4z_i}\left[\ln\left(\frac{1+cz_i}{1-cz_i}\right) + \frac{2z_ic}{(1-c^2z_j^2)}\right] \tag{61}$$

† The authors are grateful for the referees comments which led to the inclusion of the ensuing numerical developments.

Using (48) and the solution of the matrix eqn (59) it can be shown that the load-displacement relationship for the inclusion is given by

$$P = \frac{32G\Delta a(1-\nu)}{(3-4\nu)} \sum_{j=1}^{N} \frac{1}{2} \omega_j \phi_j^* \tag{62}$$

Using (35) and (56) the function $\psi(bu_1)$ can be expressed in terms of $\phi^*(at_1)$ as follows:

$$\psi(bu_1) = -\frac{16G\Delta(1-\nu)(1-2\nu)}{(3-4\nu)\pi^2} \int_0^1 \frac{u_1 c\phi^*(at_1)\,dt_1}{[u_1^2 - c^2 t_1^2]} \tag{63}$$

Using (63) and the solution of (59), the result (53) for the stress intensity factor can be written in the form

$$K_2 = \frac{8G\Delta(1-2\nu)}{\pi^2(3-4\nu)\sqrt{b}} \sum_{j=1}^{N} \frac{c\omega_j \phi_j^*}{2(1-c^2 z_j^2)} \tag{64}$$

Other numerical schemes can be also developed for the analysis of the Fredholm-type integral equation given by (57). These are fully documented by Atkinson [29] and Baker [30].

7. LIMITING CASES AND NUMERICAL RESULTS

Before performing any numerical evaluations of the results (49), (54), (62) and (64), it is instructive to establish their accuracy in predicting the solutions to certain limiting cases.

In the limiting case when the radius of the externally cracked region becomes infinite (i.e. $c \to 0$), the result (49) reduces to

$$P = \frac{32G\Delta a(1-\nu)}{(3-4\nu)} \tag{65}$$

This expression is in agreement with the results obtained, independently, by Collins [4], Kanwal and Sharma [5] and Selvadurai [8] for the load displacement response of a penny-shaped rigid inclusion embedded in an isotropic elastic solid (Fig. 2(a)) by making use of integral equation methods, singularity methods and direct spheroidal harmonic function techniques respectively. For future reference we note that in the special instance when the externally cracked region extends to the boundary of the penny-shaped rigid inclusion (i.e. $c = 1$), the problem effectively reduces to the loading of a penny-shaped rigid inclusion which is embedded in bonded contact at the interface of two separate halfspace regions (Fig. 2b). By making use of the results by Gladwell [31] and Mossakovskii [32] it can be shown that the *exact* load-displacement relationship for the penny-shaped inclusion is given by

$$P = \frac{8G\Delta a \ln(3-4\nu)}{(1-2\nu)} \tag{66}$$

Also in the limiting case of material incompressibility both (65) and (66) reduce to the single result

$$P = 16G\Delta a \tag{67}$$

It may be noted that when the elastic medium is incompressible, the parameter $\zeta = 0$, and the result (49), also reduces to (67). From the above discussion it is evident that for incompressible elastic materials the extent of cracking at the plane containing the penny-shaped rigid inclusion, has no effect on the axial stiffness of the inclusion. Figure 3 illustrates the manner in which the non-dimensional stiffness $\bar{P}$ given by

$$\frac{P(3-4\nu)}{32G\Delta a(1-\nu)} = \bar{P}(c, \nu) \tag{68}$$

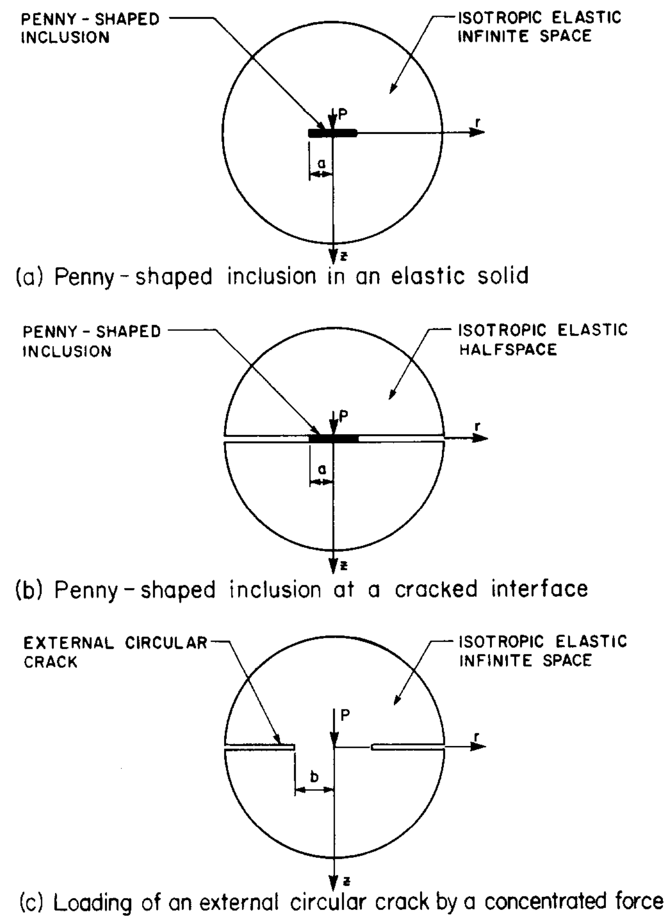

Fig. 2. Interaction between the external circular crack and the penny-shaped inclusion: limiting cases.

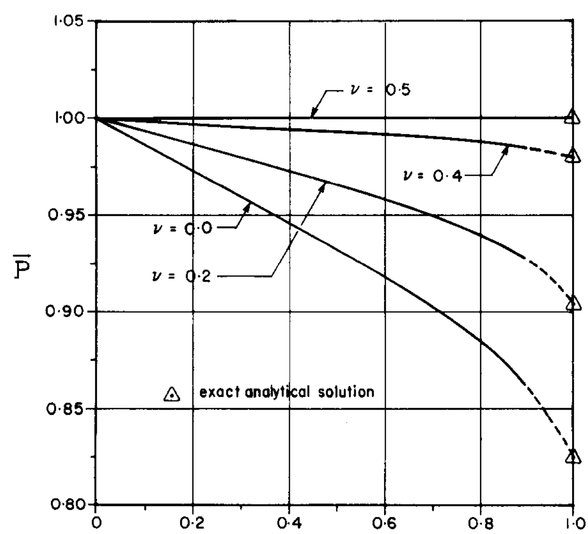

Fig. 3. Non-dimensional stiffness values for the embedded penny-shaped rigid inclusion: influence of the cracked region [$\bar{P} = P(3-4\nu)/32G\Delta a(1-\nu)$].

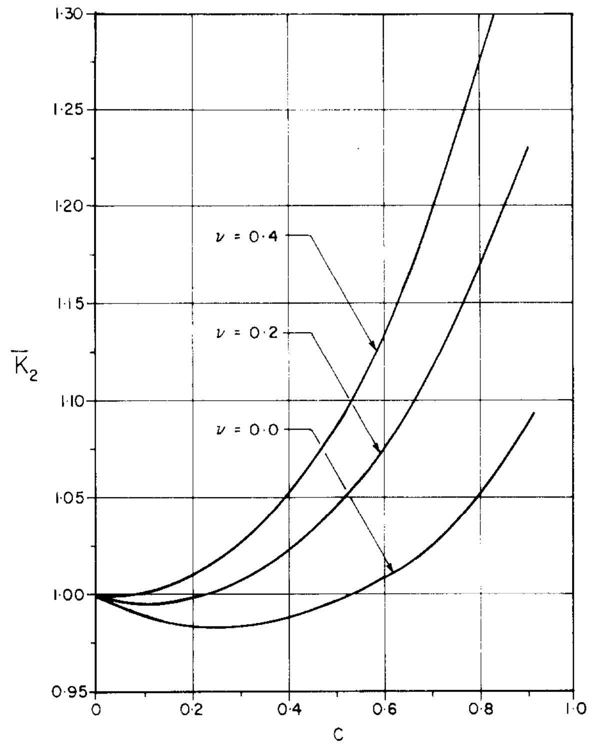

Fig. 4. Non-dimensional stress intensity factor at the boundary of the externally cracked region $[\bar{K}_2 = -(1-2\nu)/4(1-\nu)b^{3/2}\pi^2\{32G\Delta a(1-\nu)/(3-4\nu)\}]$.

is influenced by the extent of the cracked region and Poisson's ratio of the elastic material.

The accuracy of the expression (54) for the stress intensity factor can be established by invoking the following limiting procedure. In the special case when $a \to 0$, the geometry of the inclusion approaches that of a concentrated force with magnitude $P = 32G\Delta a(1-\nu)/(3-4\nu)$, which acts at the origin (Fig. 2c). It is also evident that as $a \to 0$, terms of order c^2 and higher can be neglected in (54). With these reductions (54) simplifies to the result

$$K_2 = -\frac{(1-2\nu)P}{4\pi^2(1-\nu)b^{3/2}} \tag{69}$$

The result (68) is in agreement with the expression derived by Kassir and Sih [33] for the stress intensity factor developed at the boundary of an external circular crack which is subjected to an axial force at the centre of the intact region. It is also evident that in the limit of material incompressibility, the shear stress on the plane $z = 0$ vanishes; consequently $K_2 \equiv 0$. Figure 4 illustrates the manner in which the normalized stress intensity factor $\bar{K}_2$ given by

$$-\frac{4\pi^2 K_2(1-\nu)b^{3/2}}{(1-2\nu)}\left\{\frac{(3-4\nu)}{32G\Delta a(1-\nu)}\right\} = \bar{K}_2(c, \nu) \tag{70}$$

is influenced by the extent of the cracked region and Poisson's ratio of the elastic material. In both cases the results for $\bar{P}$ and $\bar{K}_2$ are presented for $\nu \in (0, 0.5)$ and $c \in (0, 0.9)$. The solutions derived from the series expansion schemes give results which compare favourably with known exact limiting cases; the exact results for the stiffness of the inclusion embedded at a completely cracked interface are also presented for purposes of comparison.

The numerical expressions (62) and (64) respectively for the load-displacement response and the stress intensity factor are also evaluated to provide the necessary

Table 1. Non-dimensional load-displacement relationship

	$\bar{P}$							
	$\nu=0$		$\nu=0.2$		$\nu=0.3$		$\nu=0.5$	
c	Quadrature solution	Series solution	Quadrature solution	Series solution	Quadrature solution	Series solution	Quadrature solution	Series solution
0.01	0.99865	0.99865	0.99934	0.99934	0.99964	0.99964	1.00000	1.00000
0.10	0.98664	0.98664	0.99340	0.99340	0.99640	0.99640	1.00000	1.00000
0.20	0.97346	0.97346	0.98679	0.98679	0.99278	0.99278	1.00000	1.00000
0.30	0.96027	0.96027	0.98009	0.98009	0.98909	0.98909	1.00000	1.00000
0.40	0.94687	0.94686	0.97319	0.97319	0.98526	0.98526	1.00000	1.00000
0.50	0.93301	0.93300	0.96595	0.96597	0.98121	0.98123	1.00000	1.00000
0.60	0.91836	0.91839	0.95818	0.95826	0.97684	0.97690	1.00000	1.00000
0.70	0.90241	0.90266	0.94958	0.94988	0.97196	0.97217	1.00000	1.00000
0.80	0.88427	0.88535	0.93959	0.94059	0.96625	0.96691	1.00000	1.00000
0.90	0.86182	0.86590	0.92690	0.93010	0.95888	0.96094	1.00000	1.00000

Table 2. Non-dimensional stress intensity factor

	$\bar{K}_2$							
	$\nu=0$		$\nu=0.2$		$\nu=0.3$		$\nu=0.5$	
c	Quadrature solution	Series solution	Quadrature solution	Series solution	Quadrature solution	Series solution	Quadrature solution	Series solution
0.01	0.00999	0.00999	0.00999	0.00999	0.01000	0.01000	0.01000	0.01000
0.10	0.09899	0.09900	0.09967	0.09967	0.09997	0.09997	0.10034	0.10034
0.20	0.19735	0.19735	0.20005	0.20005	0.20127	0.20127	0.20274	0.20273
0.30	0.29722	0.29723	0.30336	0.30334	0.30614	0.30612	0.30952	0.30949
0.40	0.40112	0.40110	0.41228	0.41214	0.41740	0.41720	0.42365	0.42338
0.50	0.51242	0.51203	0.53056	0.52967	0.53896	0.53785	0.54931	0.54792
0.60	0.63622	0.63394	0.66399	0.66008	0.67700	0.67234	0.69315	0.68755
0.70	0.78152	0.77184	0.82298	0.80862	0.84265	0.82609	0.86730	0.84795
0.80	0.96748	0.93210	1.03017	0.98193	1.06037	1.00593	1.09861	1.03620
0.90	1.25182	1.22696	1.35558	1.18825	1.40659	1.22030	1.47218	1.26110

comparison with the results derived via power series approximation scheme. The analysis indicates that the numerical scheme converges rapidly and satisfactory results are obtained with 8 Gaussian points. The results obtained via the numerical scheme are virtually identical to those derived via the power series approximation scheme for values of $c \in (0, 0.8)$ and $\nu \in (0, 0.5)$. The comparisons are shown in Tables 1 and 2. The non-dimensional load-displacement parameter $\bar{P}$ and the stress intensity factor $\bar{K}_2$ are given by

$$\bar{P} = \frac{P(3-4\nu)}{32G\Delta a(1-\nu)}; \qquad \bar{K}_2 = \frac{\pi^2(3-4\nu)\sqrt{b}K_2}{8G\Delta(1-2\nu)} \tag{71}$$

8. CONCLUSIONS

In this paper, we present certain analytical and numerical results for the problem of the interaction between an axially loaded penny-shaped rigid inclusion and an external circular crack located in the same plane. (i) The mixed boundary value problem associated with this axisymmetric three-dimensional elastostatic problem can be reduced to the solution of two coupled Abel type integral equations. It would appear that these equations are not amenable to exact solution. The integral equations can, however, be solved by adopting power series representations of the functions involved. The series expansion parameter corresponds to the ratio of the radius of the inclusion to the radius of the external circular crack. (ii) Expressions are derived for the effective stiffness of the inclusion located at weakened plane and the stress intensity factor at the boundary of the external circular crack. Several limiting results of a closed form nature can be recovered from these solutions. Owing to the nature of the power series approximation scheme, the results presented are valid for $c < 1$. (iii) The accuracy of the power series expansion scheme is verified by a numerical scheme whereby the coupled integral equations are reduced to a single Fredholm integral equation of the second kind and the latter is solved by a Gaussian quadrature scheme. (iv) The results for the load-displacement response and the stress intensity factor derived from the two schemes show good agreement for $c \in (0, 0.8)$. (v) The techniques proposed in this paper can also be employed to examine asymmetric loadings of the inclusion and the category of interaction problems in which the penny-shaped rigid inclusion is embedded within a penny-shaped crack with a larger radius.

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APPENDIX

The integral expressions for the functions $m_i(t)$ and $n_i(t)$ $(i = 0, 1, \ldots, 6)$ take the following forms:

$$m_0(t_1) = -\frac{8\Delta G(1-\nu)}{(3-4\nu)\pi}$$

$$m_1(t_1) = -\frac{2(1-2\nu)}{(3-4\nu)\pi}\int_1^\infty \frac{n_1(u_1)\,\mathrm{d}u_1}{u_1}$$

$$m_2(t_1) = -\frac{2(1-2\nu)}{(3-4\nu)\pi}\left[\int_1^\infty \frac{n_2(u_1)\,\mathrm{d}u_1}{u_1} + t_1^2\int_1^\infty \frac{n_0(u_1)\,\mathrm{d}u_1}{u_1^3}\right]$$

$$m_3(t_1) = -\frac{2(1-2\nu)}{(3-4\nu)\pi}\left[t_1^2\int_1^\infty \frac{n_1(u_1)\,\mathrm{d}u_1}{u_1^3} + \int_1^\infty \frac{n_3(u_1)\,\mathrm{d}u_1}{u_1}\right]$$

$$m_4(t_1) = -\frac{2(1-2\nu)}{(3-4\nu)\pi}\left[\int_1^\infty \frac{n_4(u_1)\,\mathrm{d}u_1}{u_1} + t_1^4\int_1^\infty \frac{n_0(u_1)\,\mathrm{d}u_1}{u_1^5} + t_1^2\int_1^\infty \frac{n_2(u_1)\,\mathrm{d}u_1}{u_1^3}\right]$$

$$m_5(t_1) = -\frac{2(1-2\nu)}{(3-4\nu)\pi}\left[t_1^4\int_1^\infty \frac{n_1(u_1)\,\mathrm{d}u_1}{u_1^5} + t_1^2\int_1^\infty \frac{n_3(u_1)\,\mathrm{d}u_1}{u_1^3} + \int_1^\infty \frac{n_5(u_1)\,\mathrm{d}u_1}{u_1}\right]$$

and

$$n_0(t_1) = 0$$

$$n_1(t_1) = \frac{2(1-2\nu)}{\pi t_1}\int_0^1 m_0(u_1)\,\mathrm{d}u_1$$

$$n_2(t_1) = \frac{2(1-2\nu)}{\pi t_1}\int_0^1 m_1(u_1)\,\mathrm{d}u_1$$

$$n_3(t_1) = \frac{2(1-2\nu)}{\pi}\left[\frac{1}{t_1^3}\int_0^1 u_1^2 m_0(u_1)\,\mathrm{d}u_1 + \frac{1}{t_1}\int_0^1 m_1(u_1)\,\mathrm{d}u_1\right]$$

$$n_4(t_1) = \frac{2(1-2\nu)}{\pi}\left[\frac{1}{t_1}\int_0^1 m_3(u_1)\,\mathrm{d}u_1 + \frac{1}{t_1^3}\int_0^1 u_1^2 m_1(u_1)\,\mathrm{d}u_1\right]$$

$$n_5(t_1) = \frac{2(1-2\nu)}{\pi}\left[\frac{1}{t_1^5}\int_0^1 u_1^4 m_0(u_1)\,\mathrm{d}u_1 + \frac{1}{t_1^3}\int_0^1 u_1^2 m_2(u_1)\,\mathrm{d}u_1 + \frac{1}{t_1}\int_0^1 m_4(u_1)\,\mathrm{d}u_1\right]$$

Explicit expressions for $m_i(t_1)$ and $n_i(t_1)$ take the following forms:

$$m_0(t_1) = -\frac{8\Delta G(1-\nu)}{(3-4\nu)\pi}$$

$$m_1(t_1) = \frac{32(1-\nu)(1-2\nu)^2\Delta G}{(3-4\nu)^2\pi^3}$$

$$m_2(t_1) = -\frac{128(1-\nu)(1-2\nu)^4}{(3-4\nu)^3\pi^5}$$

$$m_3(t_1) = \frac{32(1-\nu)(1-2\nu)^2\Delta G}{(3-4\nu)^2\pi^3}\left[\frac{t_1^2}{3} + \frac{1}{9} + \frac{16(1-2\nu)^4}{(3-4\nu)^2\pi^4}\right]$$

$$m_4(t_1) = -\frac{128(1-\nu)(1-2\nu)^4\Delta G}{(3-4\nu)^3\pi^5}\left[\frac{t_1^2}{3} + \frac{1}{3} + \frac{16(1-2\nu)^4}{(3-4\nu)^2\pi^4}\right]$$

$$m_5(t_1) = \frac{32(1-\nu)(1-2\nu)^2\Delta G}{(3-4\nu)^2\pi^3}\left[\frac{t_1^4}{5} + t_1^2\left\{\frac{1}{15} + \frac{16(1-2\nu)^4}{3(3-4\nu)^2\pi^4}\right\} + \frac{1}{25} + \frac{16(1-2\nu)^4}{9(3-4\nu)^2\pi^4}\right.$$
$$\left. + \frac{16(1-2\nu)^4}{(3-4\nu)^2\pi^4}\left\{\frac{4}{9} + \frac{16(1-2\nu)^4}{(3-4\nu)^2\pi^4}\right\}\right]$$

and

$$n_0(t_1) = 0$$

$$n_1(t_1) = -\frac{16(1-\nu)(1-2\nu)\Delta G}{(3-4\nu)\pi^2 t_1}$$

$$n_2(t_1) = \frac{64(1-\nu)(1-2\nu)^3\Delta G}{(3-4\nu)^2\pi^4 t_1}$$

$$n_3(t_1) = -\frac{16(1-\nu)(1-2\nu)\Delta G}{3(3-4\nu)t_1^3\pi^2} - \frac{256(1-\nu)(1-2\nu)^5\Delta G}{(3-4\nu)^3\pi^6 t_1}$$

$$n_4(t_1) = \frac{64(1-\nu)(1-2\nu)^3\Delta G}{(3-4\nu)^2\pi^4}\left[\frac{1}{3t_1^3} + \frac{16(1-2\nu)^4}{(3-4\nu)^2\pi^4 t_1} + \frac{2}{9t_1}\right]$$

$$n_5(t_1) = -\frac{16(1-\nu)(1-2\nu)\Delta G}{(3-4\nu)\pi^2}\left[\frac{1}{5t_1^5} + \frac{16(1-2\nu)^4}{3(3-4\nu)^2\pi^2 t_1^3} + \frac{16(1-2\nu)^4}{(3-4\nu)^2\pi^4 t_1}\left\{\frac{4}{9} + \frac{16(1-2\nu)^4}{(3-4\nu)^2\pi^4}\right\}\right]$$