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AXISYMMETRIC PROBLEMS FOR AN EXTERNALLY CRACKED ELASTIC SOLID. I. EFFECT OF A PENNY-SHAPED CRACK

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Abstract—The present paper examines the problem of a penny-shaped flaw which is located in the plane of an external crack in an isotropic elastic solid. The penny-shaped flaw is subjected to uniform internal pressure. The paper develops power series representations for the stress intensity factors at the boundary of the penny-shaped flaw and at the perimeter of the externally cracked region. These series representations are in terms of a non-dimensional parameter which is the ratio of the radius of the penny-shaped flaw to the radius of the externally cracked region.

1. INTRODUCTION

The stress analysis of a penny-shaped crack located in an isotropic elastic solid is a classical problem in linear elastostatics. It is also a problem of fundamental interest to the study of initiation and propagation of fracture in brittle solids. The classical studies of the penny-shaped flaw problem are given by Sneddon [1, 2] and Sack [3] and detailed accounts of further developments in the stress analysis of penny-shaped defects located in elastic media are given by Sneddon and Lowengrub [4], Kassir and Sih [5] and Cherepanov [6]. These latter references contain complete accounts of problems in which the penny-shaped crack is subjected to arbitrary surface tractions. The class of problems in which the surfaces of the penny-shaped flaw is subjected to displacement dependent traction boundary conditions has been investigated by Selvadurai [7, 8] in connection with the analysis of flaw-bridging in unidirectional fibre reinforced materials. Recently Selvadurai and Singh [9] have examined the problem where the surfaces of the penny-shaped crack are indented by a flat penny-shaped rigid inclusion. This particular problem is of interest to the study of indentation testing of brittle ceramic materials (Fett [10]).

In this paper we examine the problem of a penny-shaped crack located in an isotropic elastic solid which is weakened by an external crack situated in the plane of the penny-shaped crack (Fig. 1). The crack is subjected to uniform internal pressure of intensity p_0. The analysis of the axisymmetric mixed boundary value problem is achieved by employing a Hankel transform development of the governing field equations. The mixed boundary conditions yield a system of triple integral equations which are solved in an approximate fashion. The analysis of the problem concentrates on the evaluation of the stress intensity factors at the boundary of the penny-shaped crack and at the boundary of the externally cracked region. These stress intensity factors are evaluated in power series form in terms of a non-dimensional parameter which involves the ratio of the radius of the penny-shaped crack to the radius of the externally cracked region.

It may be noted that due to the existence of the infinite crack and in view of the uniform loading, the principle of superposition does not hold for the problem examined. Consequently, the solution to the problem where the elastic solid is subjected to a homogeneous state cannot be modelled by this study. Such solutions can be developed for situations where axial loads of finite magnitude (e.g. localized loads) are applied at the outer boundary.

2. FUNDAMENTAL EQUATIONS

For the analysis of the elastostatic problem discussed here it is convenient to adopt a formulation which is based on Love's strain potential approach applicable to axisymmetric problems [11, 12]. In the absence of body forces the solution of the displacement equations

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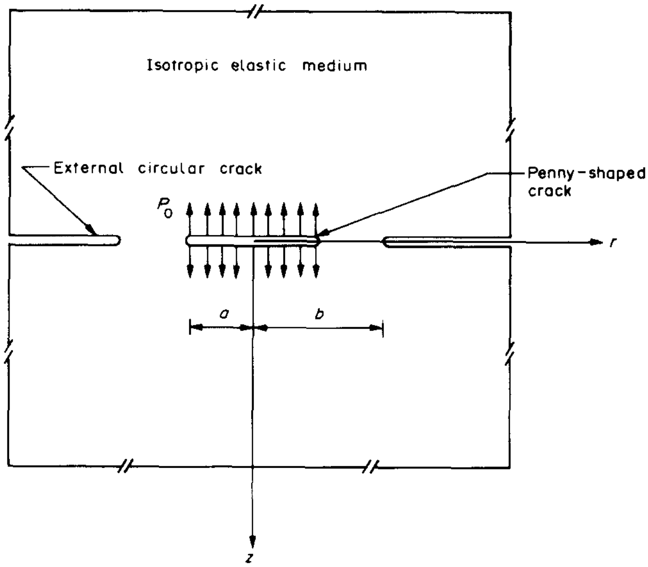

Fig. 1

of equilibrium can be represented in terms of a biharmonic function $\Phi(r, z)$; i.e.

$$\nabla^2\nabla^2\Phi(r, z) = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is the axisymmetric equivalent of Laplace's operator referred to the cylindrical polar coordinate system. The components of the displacement vector **u** and the Cauchy stress tensor $\boldsymbol{\sigma}$ referred to the cylindrical polar coordinate system can be expressed in terms of the derivatives of $\Phi(r, z)$. The expressions for the displacements take the forms

$$2Gu_r = -\frac{\partial^2\Phi}{\partial r\,\partial z} \tag{3}$$

$$2Gu_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \tag{4}$$

where G and ν are the linear elastic shear modulus and Poisson's ratio of the material respectively. Similarly the components of the stress tensor are given by

$$\sigma_{rr} = \frac{\partial}{\partial r}\left\{\nu\nabla^2\Phi - \frac{\partial^2\Phi}{\partial r^2}\right\} \tag{5}$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left\{\nu\nabla^2\Phi - \frac{1}{r}\frac{\partial\Phi}{\partial r}\right\} \tag{6}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left\{(2-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right\} \tag{7}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left\{(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right\}. \tag{8}$$

3. THE CRACK PROBLEM

We refer to the isotropic elastic region $[r\in(0,\infty); z\in(-\infty,\infty)]$ which is bounded internally by a penny-shaped flaw of radius "a" and bounded externally by an external circular crack of radius "b". The penny-shaped crack and the external crack are located in the plane $z=0$. The penny-shaped crack is subjected to the uniform internal pressure p_0. Since the problem exhibits a state of symmetry about the plane $z=0$, we can restrict our attention to a single halfspace occupying the region $z \geqslant 0$ and denote by $z = 0^+$ the plane of symmetry associated with that region. The mixed boundary conditions relevant to the crack problem are as follows:

$$\sigma_{zz}(r,0^+) = -f(r) = -p_0; \qquad 0<r<a \tag{9}$$

$$\sigma_{zz}(r,0^+) = 0; \qquad r>b \tag{10}$$

$$u_z(r,0^+) = 0; \qquad a \leqslant r \leqslant b \tag{11}$$

$$\sigma_{rz}(r,0^+) = 0; \qquad r \geqslant 0. \tag{12}$$

In order to examine the mixed boundary value problem defined by (9)–(12) it is convenient to employ a solution of Love's strain potential which is based on a Hankel transform development of the governing differential equation (1). The integral representation for $\Phi(r,z)$ can be chosen such that the stresses and displacements derived from $\Phi(r,z)$ reduce to zero as $(r^2+z^2)^{1/2} \to \infty$. The relevant solution is (see e.g. Sneddon [13])

$$\Phi(r,z) = \int_0^\infty [A_1(\xi) + zA_2(\xi)]\,\mathrm{e}^{-\xi z} J_0(\xi r)\,\mathrm{d}\xi \tag{13}$$

where $A_1(\xi)$ and $A_2(\xi)$ are arbitrary functions which need to be determined by satisfying the mixed boundary conditions (9)–(12). The displacements and stresses in the elastic medium can be determined by making use of the Love strain potential (13) and the expressions (3)–(8). The mixed boundary conditions (9)–(12) yield the following set of triple integral equations for a single unknown function $A(\xi)$ [the function $A_1(\xi)$ and $A_2(\xi)$ can be expressed in terms of $A(\xi)$]; we have

$$H_0[\xi^{-1}A(\xi); r] = f(r); \qquad 0<r<a \tag{14}$$

$$H_0[\xi^{-2}A(\xi); r] = 0; \qquad a \leqslant r \leqslant b \tag{15}$$

$$H_0[\xi^{-1}A(\xi); r] = 0; \qquad b<r<\infty. \tag{16}$$

where

$$H_0[F(\xi);r] = \int_0^\infty \xi F(\xi) J_0(\xi r)\,d\xi. \tag{17}$$

The system of triple integral equations defined by (14)–(16) can be solved by employing the procedures described by Cooke [14]. Complete accounts of the techniques that may be employed in the solution of systems of triple integral equations are given by Williams [15], Tranter [16], Sneddon [17] and Kanwal [18]. In the ensuing we shall present a brief summary of the analytical procedure which focusses on the evaluation of an asymptotic series solution in terms of a small parameter.

We assume that

$$H_0[\xi^{-2}A(\xi);r] = \begin{cases} f_1(r); & 0 < r < a \quad (18) \\ f_2(r); & b < r < \infty. \quad (19) \end{cases}$$

Employing the results given by Cooke [14] it can be shown that

$$f_1(r) = p_1(r) + \frac{2}{\pi}\int_b^\infty \frac{t f_2(t)(a^2 - r^2)^{1/2}\,dt}{(t^2 - a^2)^{1/2}[t^2 - r^2]}; \qquad 0 < r < a \tag{20}$$

and

$$f_2(r) = \frac{2}{\pi}\int_0^a \frac{t f_2(t)(r^2 - b^2)^{1/2}\,dt}{(b^2 - t^2)^{1/2}[r^2 - t^2]}; \qquad b < r < \infty \tag{21}$$

where

$$p_1(r) = \frac{2}{\pi}\int_r^a \left\{\int_0^s \frac{t f(t)\,dt}{(s^2 - t^2)^{1/2}}\right\} \frac{ds}{(s^2 - r^2)^{1/2}}. \tag{22}$$

Introduce functions $F_1(r)$ and $F_2(r)$ such that

$$F_1(r) = \frac{d}{dr}\int_r^a \frac{t f_1(t)\,dt}{(t^2 - r^2)^{1/2}}; \qquad 0 < r < a \tag{23}$$

and

$$F_2(r) = \frac{d}{dr}\int_b^r \frac{t f_2(t)\,dt}{(r^2 - t^2)^{1/2}}; \qquad b < r < \infty. \tag{24}$$

The solutions of the above Abel integral equations take the forms

$$f_1(t) = -\frac{2}{\pi}\int_t^a \frac{F_1(s)\,ds}{(s^2 - t^2)^{1/2}}; \qquad 0 < t < a \tag{25}$$

$$f_2(t) = \frac{2}{\pi}\int_b^t \frac{F_2(s)\,ds}{(t^2 - s^2)^{1/2}}; \qquad b < t < \infty. \tag{26}$$

By making use of (20) and (25) we have

$$\int_r^a \frac{F_1(s)\,\mathrm{d}s}{(s^2-r^2)^{1/2}} = -\frac{\pi}{2}p_1(r) - \int_b^\infty \frac{t(a^2-r^2)^{1/2}f_2(t)\,\mathrm{d}t}{(t^2-a^2)^{1/2}(t^2-r^2)}; \qquad 0<r<a. \tag{27}$$

Again we observe that (27) is an integral equation of the Abel type the solution of which can be written in the form

$$F_1(s) = \frac{\mathrm{d}}{\mathrm{d}s}\int_s^a \frac{rp_1(r)\,\mathrm{d}r}{(r^2-s^2)^{1/2}} + \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}s}\int_s^a \frac{r(a^2-r^2)^{1/2}}{(r^2-s^2)^{1/2}} \times\left\{\int_b^\infty \frac{tf_2(t)\,\mathrm{d}t}{(t^2-a^2)^{1/2}(t^2-r^2)}\right\}\mathrm{d}r; \qquad 0<s<a. \tag{28}$$

The second integral of (28) can be reduced to the form (see Appendix A)

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_b^\infty \frac{tf_2(t)\,\mathrm{d}t}{(t^2-a^2)^{1/2}}\int_s^a \frac{r(a^2-r^2)^{1/2}\,\mathrm{d}r}{(r^2-s^2)^{1/2}(t^2-r^2)} = s\int_b^\infty \frac{F_2(u)\,\mathrm{d}u}{(u^2-s^2)}. \tag{29}$$

Consequently (28) gives

$$F_1(s) + \frac{2s}{\pi}\int_b^\infty \frac{F_2(u)\,\mathrm{d}u}{(u^2-s^2)} = \frac{\mathrm{d}}{\mathrm{d}s}\int_s^a \frac{rp_1(r)\,\mathrm{d}r}{(r^2-s^2)^{1/2}}; \qquad 0<s<a. \tag{30}$$

In a similar fashion (21) can be reduced to the form

$$F_2(s) + \frac{2}{\pi}\int_0^a \frac{uF_1(u)\,\mathrm{d}u}{(s^2-u^2)} = 0; \qquad b<s<\infty. \tag{31}$$

The eqns (30) and (31) are a pair of simultaneous Fredholm integral equations which can be solved in an approximate fashion by using an asymptotic expansion procedure. In the particular instance when $f(r) = p_0$, the eqn (30) can be written

$$F_1(as_1) + \frac{2cs_1}{\pi}\int_1^\infty \frac{F_2(bu_1)\,\mathrm{d}u_1}{(u_1^2-s_1^2c^2)} = -p_0as_1; \qquad 0<s_1<1 \tag{32}$$

where $c = a/b$; $s_1 = s/a$ and $u_1 = u/b$. Similarly by introducing the substitutions $u_1 = u/a$ and $s_1 = s/b$, the eqn (31) can be written as

$$F_2(bs_1) + \frac{2c^2}{\pi}\int_0^1 \frac{u_1F_1(au_1)\,\mathrm{d}u_1}{(s_1^2-c^2u_1^2)} = 0; \qquad 1<s_1<\infty. \tag{33}$$

Assuming that $c<1$, the denominators of the integrands of (32) and (33) can be expressed in the forms

$$(u_1^2-s_1^2c^2)^{-1} = \frac{1}{u_1^2} + \frac{s_1^2c^2}{u_1^4} + \frac{s_1^4c^4}{u_1^6} + \frac{s_1^6c^6}{u_1^8} + \frac{s_1^8c^8}{u_1^{10}} + O(c^{10}) \tag{34}$$

$$(s_1^2-u_1^2c^2)^{-1} = \frac{1}{s_1^2} + \frac{u_1^2c^2}{s_1^4} + \frac{u_1^4c^4}{s_1^6} + \frac{u_1^6c^6}{s_1^6} + \frac{u_1^8c^8}{s_1^8} + O(c^{10}) \tag{35}$$

where $0(c^n)$ is the Landau symbol. We further assume that the functions F_1 and F_2 can be expressed in power series of the form

$$F_1(as_1) = \sum_{i=0}^{8} c^i m_i(s_1) \tag{36}$$

$$F_2(bs_1) = \sum_{i=0}^{8} c^i n_i(s_1). \tag{37}$$

By substituting (34)–(37) in eqns (32) and (33) and comparing like terms in c^i it is possible to determine the functions $m_i(s_i)$ and $n_i(s_1)$. We have

$$F_1(as_1) = ap_0\left[-s_1 - \frac{4s_1}{9\pi^2}c^3 - \frac{4c^5}{5\pi^2}\left(\frac{s_1}{5}+\frac{s_1^3}{3}\right) - \frac{16c^6 s_1}{81\pi^4} + \frac{4s_1c^7}{\pi^2}\left(\frac{1}{49}+\frac{s_1^2}{35}+\frac{s_1^4}{21}\right) - \frac{16c^8}{\pi^4}\left(\frac{s_1}{75}+\frac{s_1^3}{135}\right) + 0(c^9)\right]; \quad 0 < s_1 < 1 \tag{38}$$

$$F_2(bs_1) = ap_0\left[\frac{2c^2}{3\pi s_1^2} + \frac{2c^4}{5\pi s_1^4} + \frac{8c^5}{27\pi^3 s_1^2} + \frac{2c^6}{7\pi s_1^6} + \frac{16c^7}{\pi^3 s_1^2}\left(\frac{1}{75}+\frac{1}{90s_1^2}\right) + \frac{2c^8}{9\pi s_1^2}\left(\frac{16}{27\pi^4}+\frac{1}{s_1^6}\right) + 0(c^9)\right]; \quad 1 < s_1 < \infty. \tag{39}$$

This formally completes the analysis of the problem and the function $A(\xi)$ can be expressed in the form

$$A(\xi) = \xi^2\left[-\int_0^a F_1(s)\,ds\int_0^s \frac{rJ_0(\xi r)\,dr}{(s^2-r^2)^{1/2}} + \int_b^\infty F_2(s)\,ds\int_s^\infty \frac{rJ_0(\xi r)\,d\xi}{(r^2-s^2)^{1/2}}\right]. \tag{40}$$

4. THE STRESS INTENSITY FACTOR

In the ensuing we shall examine the stress intensity factors associated with the boundary of the penny-shaped crack and the circular boundary of the externally cracked region. Considering (40) and the result

$$\sigma_{zz}(r,0) = -H_0[\xi^{-1}A(\xi); r] \tag{41}$$

it can be shown that

$$\sigma_{zz}(r,0) = \left[\frac{-F'_1(a)}{(r^2-a^2)^{1/2}} + \int_0^a \frac{F_1(s)\,ds}{(r^2-s^2)^{1/2}} + \frac{F'_2(b)}{(b^2-r^2)^{1/2}} + \int_b^\infty \frac{F'_2(s)\,ds}{(s^2-r^2)^{1/2}}\right] \tag{42}$$

where $F'_1(s)$ and $F'_2(s)$ denote derivatives of the respective functions.

The stress intensity factors at the boundaries $r = a$ and $r = b$ are defined by

$$K_a = \lim_{r\to a^+} [2(r-a)]^{1/2}\sigma_{zz}(r,0) \tag{43}$$

and

$$K_b = \lim_{r\to b^-} [2(b-r)]^{1/2}\sigma_{zz}(r,0) \tag{44}$$

respectively. From (42), (43) and (44) it follows that

$$K_a = -\frac{F_1(a)}{\sqrt{a}} \tag{45}$$

$$K_b = \frac{F_2(b)}{\sqrt{b}} \tag{46}$$

Using eqns (38) and (39) in (45) and (46) we obtain the following expressions for the stress intensity factors:

$$K_a = p_0\sqrt{a}\left[1 + \frac{4c^3}{9\pi^2} + \frac{32c^5}{75\pi^2} + \frac{16c^6}{81\pi^4} + \frac{284c^7}{735\pi^2} + \frac{224c^8}{675\pi^4} + 0(c^9)\right] \tag{47}$$

$$K_b = p_0\sqrt{a}\sqrt{c}\left[\frac{2c^2}{3\pi} + \frac{2c^4}{5\pi} + \frac{8c^5}{27\pi^3} + \frac{2c^6}{7\pi} + \frac{88c^7}{225\pi^3} + \frac{2c^8}{9\pi}\left(1 + \frac{16}{27\pi^4}\right) + 0(c^9)\right] \tag{48}$$

where $c = a/b$.

The expression for the displacement u_z can be written in the form

$$u_z(r,0) = \begin{cases} \dfrac{-4}{\pi}\dfrac{(1-\nu^2)}{E}\displaystyle\int_r^a \frac{F_1(s)\,ds}{(s^2-r^2)^{1/2}}; & 0 \leqslant r \leqslant a \quad (49a) \\[2ex] \dfrac{4}{\pi}\dfrac{(1-\nu^2)}{E}\displaystyle\int_b^r \frac{F_2(s)\,ds}{(r^2-s^2)^{1/2}}; & b \leqslant r \leqslant \infty \quad (49b) \end{cases}$$

The work done in opening the penny-shaped crack is given by

$$W = 2\pi p_0 \int_0^a r u_z(r,0)\,dr \tag{50}$$

From (49a) and (50) we obtain

$$W = \frac{8p_0^2a^3(1-\nu^2)}{3E}\left[1 + \frac{4c^3}{9\pi^2} + \frac{8c^5}{25\pi^2} + \frac{16c^6}{81\pi^4} + \frac{284c^7}{1225\pi^2} + \frac{64c^8}{225\pi^4} + 0(c^9)\right]. \tag{51}$$

5. CONCLUSIONS

The computed stress intensity factors (47) and (48) indicate that when the penny-shaped crack is subjected to uniform internal pressure the stress intensity factor at the boundary of the penny-shaped crack ($r = a$) is greater than the stress intensity factor at the boundary of the weakened zone ($r = b$). Consequently brittle-elastic type of fracture will be initiated at the boundary of the penny-shaped crack. Such a conclusion, however, cannot be generalized to include other forms of axisymmetric loading. Also it is evident that as $b \to \infty$ (or $c \to 0$) the result (47) yields the classical result for the stress intensity factor at the boundary of an internally loaded penny-shaped crack in an infinite elastic solid [1]. Similar conclusions apply for the expression (51) derived for the work done in opening the penny-shaped crack. Fig. 2 illustrates the manner in which K_a and K_b are influenced by the aspect ratio a/b. It is of interest to note that $K_a \approx 1$ for $c\epsilon(0, 0.5)$ indicating that the stress intensity factor at the boundary of the penny-shaped crack remains at its classical value for

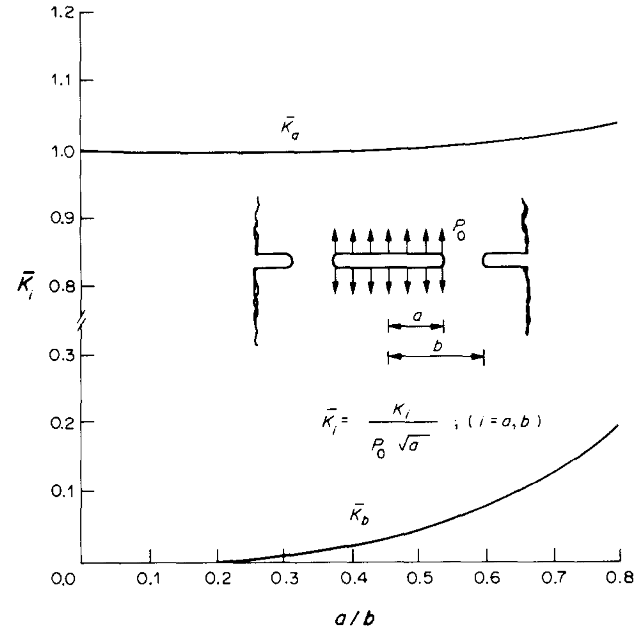

Fig. 2

sufficiently large values of a/b. Also the effects of the internal loading of the penny-shaped crack does not result in the development of appreciable stress intensity factors at the external crack region. Owing to the nature of the asymptotic analysis the accuracy of the solution presented is expected to be satisfactory for small values of $c\varepsilon(0, 0.7)$.

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APPENDIX A

The integral

$$I = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}s}\int_s^a \frac{r(a^2 - r^2)}{(r^2 - s^2)}\mathrm{d}r \int_b^\infty \frac{tf_2(t)\,\mathrm{d}t}{(t^2 - a^2)^{1/2}(t^2 - r^2)} \tag{A1}$$

$$= \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}s}\int_b^\infty \frac{tf_2(t)\,\mathrm{d}t}{(t^2 - a^2)^{1/2}}\int_s^a \frac{r(a^2 - r^2)^{1/2}\mathrm{d}r}{(r^2 - s^2)^{1/2}(t^2 - r^2)}. \tag{A2}$$

The second integral in (A2) is given by (see e.g. Gradshteyn and Ryzhik [19])

$$\int_s^a \frac{r(a^2 - r^2)\,\mathrm{d}r}{(r^2 - s^2)^{1/2}(t^2 - r^2)} = \frac{\pi}{2}\left[1 - \left\{\frac{t^2 - a^2}{t^2 - s^2}\right\}^{1/2}\right]; \qquad s < a. \tag{A3}$$

Hence

$$I = -s\int_b^\infty \frac{tf_2(t)\,\mathrm{d}t}{(t^2 - s^2)^{3/2}}. \tag{A4}$$

Substituting the value of $f_2(t)$ given by (26) in (A4) we have

$$I = -\frac{2s}{\pi}\int_b^\infty \frac{t\,\mathrm{d}t}{(t^2 - s^2)^{3/2}}\int_b^t \frac{F_2(u)\,\mathrm{d}u}{(t^2 - u^2)^{1/2}}. \tag{A5}$$

Changing the order of integrations we have

$$I = -\frac{2s}{\pi}\int_b^\infty F_2(u)\,\mathrm{d}u\int_u^\infty \frac{t\,\mathrm{d}t}{(t^2 - u^2)^{1/2}(t^2 - s^2)^{3/2}}. \tag{A6}$$