Engineering Fracture Mechanics Vol. 24, No. 1, pp. 39–44, 1986
Printed in Great Britain.

0013-7944/86 $3.00 + .00
Pergamon Press Ltd.

IMPACT RESPONSE OF A PRESSURIZED PENNY-SHAPED CRACK IN AN ELASTIC–PLASTIC MATERIAL

B. M. SINGH and H. T. DANYLUK
Department of Mechanical Engineering, University of Saskatchewan,
Saskatoon, Canada

J. VRBIK
Department of Mathematics, Brock University, St. Catharines,
Ontario, Canada

and

A. P. S. SELVADURAI
Department of Civil Engineering, Carleton University, Ottawa,
Ontario, Canada

Abstract—The transient response of a pressurized penny-shaped crack subjected to a time-dependent load is considered. The crack is embedded in an elastic–plastic solid and the plastic zone has the shape of an annulus of small thickness surrounding the crack and extending radially. The Dugdale hypothesis is applied to find the length of the plastic zone as a function of the applied time-varying load. The results are applicable for so-called quasibrittle solids.

1. INTRODUCTION

BY ACCEPTED terminology, "quasibrittle" fracture is fracture involving a highly localized plastic zone which precedes the crack tip. This type of behavior is exhibited by some low-carbon steels and aluminum alloys. A penny-shaped crack in an infinite elastic–plastic medium under static load has been considered by Wnuk[1] and Olesiak and Wnuk[2]. The transient response of a penny-shaped crack under a uniform tensile stress in an elastic medium was discussed by Embley and Sih[3]. References to the plastic deformation around cracks and the governing fracture criteria are found in the book by Parton and Morozov[4] and in the paper by Vitvitskii, Panasyuk and Yarema[5].

The purpose of this note is to determine the length of the plastic zone around the pressurized penny-shaped crack in an infinite elastic–plastic solid subjected to a time-varying load. The plastic zone surrounding the crack is considered to be very thin in comparison with the length of the crack. It is well known that for "quasibrittle" solids the Dugdale hypothesis[6] can be applied. In the plastic zone, the yield condition is satisfied for finite normal-stress values. By using Dugdale's hypothesis, the length of the plastic zone is obtained as a function of time and the results are shown graphically.

2. BASIC EQUATIONS AND FORMULATION OF THE PROBLEM

We consider that the impact load is applied symmetrically about the z axis about which the penny-shaped crack is centered. For axially symmetric deformation, material elements remain unchanged in the θ direction. In terms of the wave potentials $\phi(r, z, t)$ and $\psi(r, z, t)$, the displacement and the stress fields are

$$u_r(r,z,t) = \frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial z},$$

$$u_z(r,z,t) = \frac{\partial\phi}{\partial z} + \frac{\partial\psi}{\partial r} + \frac{\psi}{r},$$

$$\sigma_r(r,z,t) = 2\mu\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial r} + \frac{\partial\psi}{\partial z}\right) + \lambda\nabla^2\phi,$$

$$\sigma_\theta(r,z,t) = \frac{2\mu}{r}\left(\frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial z}\right) + \lambda\nabla^2\phi,$$

$$\sigma_z(r,z,t) = 2\mu\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial r} + \frac{\partial\psi}{\partial r} + \frac{\psi}{r}\right) + \lambda\nabla^2\phi,$$

$$\sigma_{rz}(r,z,t) = \mu\left[\frac{\partial}{\partial z}\left(2\frac{\partial\phi}{\partial r} - \frac{\partial\psi}{\partial z}\right) + \frac{\partial}{\partial r}\left(\frac{\partial\psi}{\partial r} + \frac{\psi}{r}\right)\right],$$

$$\sigma_{r\theta} = \sigma_{\theta z} = 0. \tag{1}$$

It can be shown that the equations of motion are satisfied if ϕ and ψ are governed by the wave equations

$$\begin{aligned}\nabla^2\phi &= \frac{1}{c_2^2}\frac{\partial^2\phi}{\partial t^2},\\ \nabla^2\psi - \frac{\psi}{r^2} &= \frac{1}{c_2^2}\frac{\partial^2\psi}{\partial t^2},\end{aligned} \tag{2}$$

where

$$\begin{aligned}c_1 &= \left(\frac{\lambda+2\mu}{\rho}\right)^{1/2},\\ c_2 &= \left(\frac{\mu}{\rho}\right)^{1/2},\end{aligned} \tag{3}$$

ρ being the mass density of the material and λ and μ the Lamé constants.

The material is perfectly elastic–plastic. The nonzero components of the stress tensor $\sigma_r, \sigma_\theta, \sigma_z, \sigma_{rz}$ satisfy the Huber–Mises–Hencky condition in the plastic zone. This condition is approximately satisfied if we assume that σ_z is equal to a constant Y in this region. This constant is called the "effective yield stress".

Suppose that the material is initially at rest. At time $t = 0$, the normal stress of magnitude $-\sigma_0$ is suddenly applied to both crack surfaces and maintained at this same value thereafter. Then

$$\sigma_z(r,0,t) = -\sigma_0 H(t), \qquad 0 < r < l, t > 0; \tag{4a}$$

$$\sigma_z(r,0,t) = Y, \qquad l < r < a, t > 0; \tag{4b}$$

$$u_z(r,0,t) = 0, \qquad r > a, t > 0; \tag{5}$$

$$\sigma_{rz}(r,0,t) = 0, \qquad 0 < r < \infty, t > 0. \tag{6}$$

The initial conditions at time $t = 0$ are all zeros. The plastic zone has the shape of a thin annulus, $l \leqslant r < a$, surrounding the crack. The problem discussed here addresses the case of large-scale yielding.

3. METHOD OF SOLUTION

The standard Laplace transform of $f(t)$ is

$$\overline{f(p)} = \int_0^\infty f(t)\,\mathrm{e}^{-pt}\,\mathrm{d}t, \tag{7}$$

whose inversion is

$$f(t) = \frac{1}{2\pi i}\int_{\mathrm{Br}} f(p)\,\mathrm{e}^{pt}\,\mathrm{d}p, \tag{8}$$

where Br denotes the Bromwich path of integration. Now in the p plane, eqns (2) and the boundary conditions (4a)–(6) become

$$\nabla^2\bar{\phi} = \left(\frac{p}{c_1}\right)^2 \bar{\phi}, \qquad \nabla^2\bar{\psi} - \frac{\bar{\psi}}{r^2} = \left(\frac{p}{c_2}\right)^2 \bar{\psi}, \tag{9}$$

$$\bar{\sigma}_z(r, 0, p) = \frac{1}{p} F(r), \qquad 0 < r < a, \tag{10}$$

$$\bar{u}_z(r, 0, p) = 0, \qquad r > a, \tag{11}$$

and

$$\bar{\sigma}_{rz}(r, 0, p) = 0, \qquad 0 < r < \infty, \tag{12}$$

where

$$F(r) = \begin{cases} -\sigma_0, & 0 < r < l, \\ Y, & l < r < a. \end{cases} \tag{13}$$

Equation (2) may be solved with the help of the Hankel transform to render

$$\bar{\phi}(r, z, p) = \int_0^\infty A_1(s, p) J_0(rs) \exp(-\gamma_1 z) \, \mathrm{d}s, \tag{14}$$

$$\bar{\psi}(r, z, p) = \int_0^\infty A_2(s, p) J_1(rs) \exp(-\gamma_2 z) \, \mathrm{d}s, \tag{15}$$

where

$$\gamma_j^2 = \left(s^2 + \frac{p^2}{c_j^2}\right), \qquad j = 1, 2, \tag{16}$$

and $J_\nu(rs)$ denotes the Bessel function of order $\nu \geqslant 0$. $A_1(s, p)$ and $A_2(s, p)$ are unknown functions to be determined from the boundary conditions. We denote the displacement and stress components in the p plane by $\bar{u}_r$, $\bar{u}_z$, $\bar{\sigma}_r$, $\bar{\sigma}_\theta$, $\bar{\sigma}_z$ and $\bar{\sigma}_{rz}$. Hence, making use of (1), (7), (13), (14) and (15), we find that

$$\bar{u}_r(r, z, p) = -\int_0^\infty [sA_1(s, p)\,\mathrm{e}^{-\gamma_1 z} - \gamma_2 A_2(s, p)\,\mathrm{e}^{-\gamma_2 z}] J_1(rs) \, \mathrm{d}s, \tag{17}$$

$$\bar{u}_z(r, z, p) = -\int_0^\infty [sA_2(s, p)\,\mathrm{e}^{-\gamma_1 z} - \gamma_1 A_1(s, p)\,\mathrm{e}^{-\gamma_2 z}] J_0(rs) \, \mathrm{d}s, \tag{18}$$

$$\bar{\sigma}_{rz}(r, z, p) = \mu \int_0^\infty \{2\gamma_1 s A_1(s, p)\,\mathrm{e}^{-\gamma_1 z} J_1(rs) + (p^2/c_2^2 - 2\gamma_2^2)\,\mathrm{e}^{-\gamma_2 z} A_2(s, p) J_1(rs)\} \, \mathrm{d}s, \tag{19}$$

$$\bar{\sigma}_z(r, z, p) = \int_0^\infty \{(2\mu\gamma_1^2 + p^2/c_1^2)\,\mathrm{e}^{-\gamma_1 z} A_1(s, p) - 2s\mu\gamma_2\,\mathrm{e}^{-\gamma_2 z} A_2(s, p)\} J_0(rs) \, \mathrm{d}s. \tag{20}$$

Making use of condition (12), we find with the help of eqn (19) that

$$A_1(s, p) = \frac{c_2^2[s^2 + p^2/2c_2^2] A(s, p)}{\mu(1 - k^2)\gamma_1 p^3} \tag{21}$$

and

$$A_2(s,p) = \frac{sc_2^2 A(s,p)}{\mu p^3(1-k^2)}, \tag{22}$$

where

$$k = \frac{c_2}{c_1} = \left[\frac{1-2\nu}{2(1-\nu)}\right]^{1/2}, \tag{23}$$

and ν denotes Poisson's ratio. Making use of eqns (21) and (22), we find that

$$\bar{\sigma}_z(r,0,p) = \frac{1}{p}\int_0^\infty sR(s,p)A(s,p)J_0(rs)\,\mathrm{d}s, \tag{24}$$

$$\bar{u}_z(r,0,p) = \frac{1}{2\mu p(1-k^2)}\int_0^\infty A(s,p)J_0(rs)\,\mathrm{d}s, \tag{25}$$

where

$$R(s,p) = \frac{2c_2^2[(s^2+p_2^2/2c_2^2)^2 - s^2\gamma_1\gamma_2]}{\gamma_1 p^2 s(1-k^2)}. \tag{26}$$

The conditions (10) and (11) lead to a pair of dual integral equations:

$$\int_0^\infty sA(s,p)J_0(rs)\,\mathrm{d}s + \int_0^\infty s[R(s,p)-1]A(s,p)J_0(rs)\,\mathrm{d}s = -F(r), \qquad 0<r<a; \tag{27}$$

$$\int_0^\infty A(s,p)J_0(rs)\,\mathrm{d}s = 0, \qquad r>a. \tag{28}$$

Solving the dual integral eqns (27) and (28) by the method of Copson[7] and using (13), we find that

$$A(s,p) = \left(\frac{2}{\pi}\right)a^2\int_0^1 \phi(n_1,P)\sin(an_1 s)\,\mathrm{d}n, \tag{29}$$

$$\phi(n_1,P) + \int_0^1 \phi(u_1,P)K(u_1,n_1,P)\,\mathrm{d}u_1 = F_2(n_1), \qquad 0<n_1<1, \tag{30}$$

$$K(u_1,n_1,P) = \left(\frac{2}{\pi}\right)\int_0^\infty [R(s_1,P)-1]\sin(s_1 n_1)\sin(s_1 u_1)\,\mathrm{d}s_1, \tag{31}$$

$$R(s_1,P) = \frac{[(2s_1^2+P^2)^2 - 4s_1^2\gamma_1'\gamma_2']}{2\gamma_1' s_1 P^2(1-k^2)}, \tag{32}$$

$$\gamma_1' = (s_1^2+k^2P^2)^{1/2}, \tag{33}$$

$$\gamma_2' = (s_1^2+P^2)^{1/2}, \tag{34}$$

$$P = \frac{ap}{c_2}, \tag{35}$$

$$F_2(n_1) = \begin{cases} \lambda Y n_1, & 0<n_1<m, \\ Y[\lambda n_1 - (1+\lambda)\sqrt{n_1^2-m^2}], & m<n_1<1, \end{cases} \tag{36}$$

where

$$\phi(0,P)=0,\qquad \lambda=\frac{\sigma}{Y},\qquad \text{and}\qquad m=\frac{l}{a}. \tag{37}$$

$1-m$ is the length of the plastic zone.

From eqn (29), we find that

$$A(s,p)=\frac{2a}{\pi s}\left[\phi(1,P)\cos(as)+\int_0^1\phi'(n,P)\cos(ans)\,\mathrm{d}n\right], \tag{38}$$

where the prime denotes the derivative with respect to n. We can write the expression for $\bar{\sigma}_z(r,0,p)$ in the following form:

$$\bar{\sigma}_z(r_1,0,p)=\frac{2}{\pi p}\left[\frac{\phi(1,P)}{\sqrt{r_1^2-1}}+\int_0^1\frac{\phi'(n,P)\,\mathrm{d}n}{\sqrt{r_1^2-n^2}}\right]+\frac{2a^2}{\pi p}\int_0^1\phi(n,P)\,\mathrm{d}n$$
$$\times\int_0^\infty s(R(s,P)-1)\sin(asn_1)J_1(r_1sa)\,\mathrm{d}s,\qquad 1<r_1, \tag{39}$$

where $r_1=r/a$. Now $\sigma_z(r,0,p)$ should be finite in the plastic zone. Hence, we find that

$$\phi(1,P)=\int_0^\infty\phi(T)\,\mathrm{e}^{-PT}\,\mathrm{d}T=0. \tag{40}$$

Applying the inverse Laplace transform, we get

$$\phi(T)=\frac{1}{2\pi i}\int_{\mathrm{Br}}\frac{\mathrm{e}^{PT}\phi(1,P)\,\mathrm{d}P}{P}=0, \tag{41}$$

$$T=\frac{c_2t}{a}. \tag{42}$$

The values of $\phi(1,P)$ are evaluated after solving numerically the integral eqn (30) at the discrete points P for sufficiently many values of λ and m. Then, with the help of the relation (41), the numerical Laplace inversion technique developed by Miller and Guy[8] is used to obtain the size of plastic zone, $1-m$, for different values of T. From Figs. 1 and 2, we see that the length of the plastic zone increases as

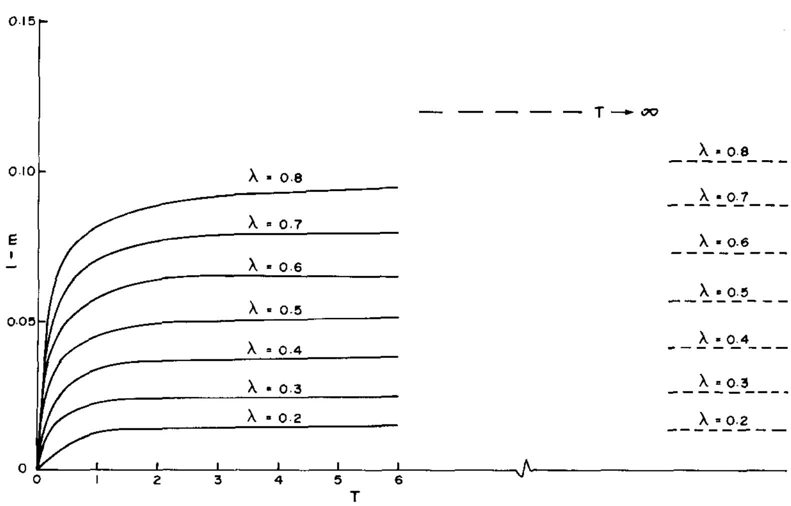

Fig. 1. Variation of length of plastic zone with T for different values of λ with $\nu=\frac{1}{4}$.

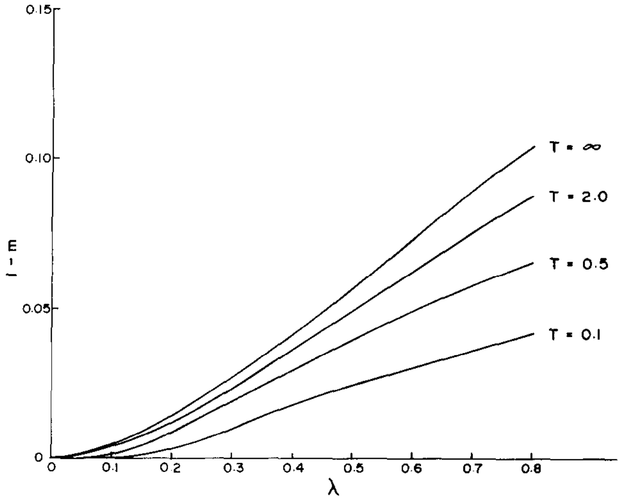

Fig. 2. Variation of length of plastic zone with λ for different values of T with $\nu = \frac{1}{4}$.

time increases. As $p \to 0$, we obtain the static solution of the problem. With the help of the book by Parton and Morozov[4] or Wnuk[1], we find the solution for the case $T \to \infty$ in the following form:

$$1 - m = \text{length of the plastic zone} = 1 - \frac{\sqrt{1+2\lambda}}{1+\lambda}. \tag{43}$$

The solution for $T \to \infty$ is shown by a dotted line in Fig. 1.

The difference observed between the transient solution and static solution obtained using (43) for large values of T is shown in Figs. 1 and 2.

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