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A Reissner-Sagoci problem for a non-homogeneous elastic solid

A.P.S. SELVADURAI [(1)],*, B.M. SINGH [(1)],** and J. VRBIK [(2)],***
[(1)] *Department of Civil Engineering, Carleton University, Ottawa, Ontario, Canada K1S 5B6*
[(2)] *Department of Civil Engineering, Brock University, St. Catherines, Ontario, Canada, L2S 3A1*

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Abstract The present paper examines the elastostatic problem related to the axisymmetric rotation of a rigid circular punch which is bonded to the surface of a non-homogeneous isotropic elastic halfspace. The non-homogeneity corresponds to an axial variation of the linear elastic shear modulus according to the exponential form $G(z) = G_1 + G_2\,e^{-\xi z}$. A Hankel transform development of the governing equations yields a set of dual integral equations which in turn can be reduced to a Fredholm integral equation of the second kind. A numerical evaluation of this integral equation yields results which can be used to estimate the torque-twist relationship for the circular punch.

1. Introduction

The elastostatic problem pertaining to the determination of stresses and displacements in a halfspace region in which a circular region of the plane boundary is forced to rotate about an axis which is normal to the undeformed plane surface is referred to as the Reissner-Sagoci problem. In their analysis, Reissner and Sagoci [1] utilized an oblate spheroidal coordinate formulation of the problem. The case of a circular rigid disc was obtained as a limiting case of the rigid spheroidal inclusion problem. In subsequent treatments, Sneddon [2,3] and Rostovtsev [4] used Hankel transform based dual integral equation schemes to examine the relevant mixed boundary value problem. The works of Uflyand [5] and Collins [6] investigated the associated problem of the torsional axisymmetric indentation of an elastic layer. Sneddon et al. [7] examined the problem of the torsional indentation of an infinitely long cylinder in which the cylindrical boundary is unrestrained. These studies were extended by Dhaliwal et al. [8] who examined the problem of the torsional indentation of a bimaterial cylindrical composite region. Selvadurai [9] has examined the Reissner-Sagoci problem related to a finitely deformed incompressible elastic halfspace. The theory of small deformations superposed on finite elastic deformations [10] is used to examine this particular problem. The torsional

* Professor and Chairman
** Research Fellow
*** Associate Professor

indentation corresponds to the incremental deformation. The results of this analysis serve as useful procedures for determining the mechanical properties of rubber-like elastic materials. Recently, Selvadurai [11] examined the Reissner-Sagoci problem for an internally loaded transversely isotropic elastic halfspace. In this case, the axisymmetric rotation of the bonded rigid circular punch is caused by a concentrated moment which acts at a finite distance from the punch along the axis of symmetry. A problem of related interest to the Reissner-Sagoci problem concerns the torsional loading of cylindrical rigid inclusions which are embedded in a halfspace region. The torsional loading of a rigid cylindrical inclusion embedded in an elastic halfspace was examined by Freeman and Keer [12]. Luco [13] extended this investigation to the case of a rigid cylindrical inclusion embedded in a two layer system. A similar result for the torsional loading of a prolate rigid inclusion embedded in a halfspace region was given by Selvadurai [14]. The work by Karasudhi et al. [15] also examined the static and dynamic problems of a cylindrical elastic inclusion embedded in elastic halfspace. Rajapakse and Selvadurai [16] and Selvadurai and Rajapakse [17] have examined problems related to rigid and elastic cylindrical inclusions, with uniform and non-uniform shapes, embedded in elastic media. A discretization procedure based on a fundamental solution for the torsional ring type internal loading of a halfspace region is used to solve these inclusion problems. It may be noted that in all problems dealing with the cylindrical inclusion problem [12–17] the Reissner-Sagoci result can be recovered as a limiting case.

The extension of the Reissner-Sagoci problem to include effects of material non-homogeneity is of interest to geomechanics and to non-destructive materials testing. The torsional indentation of a non-homogeneous halfspace region which exhibits either exponential or power law variations in the shear modulus was examined by Protsenko [18,19], Kassir [20], Kolybikhin [21], Singh [22] and Chuapresert and Kassir [23]. Similar problems related to the non-homogeneous elastic layer were investigated by Protsenko [24], Dhaliwal and Singh [25,26] and Hassan [27]. Gladwell and Coen [28] have examined the class of inverse problems which result from the torsional indentation of a non-homogeneous isotropic elastic halfspace. A complete and informative account of the torsional indentation problem was given by Gladwell [29]. In this paper we examine the torsional indentation problem related to a non-homogeneous elastic medium in which the shear modulus varies in an exponential fashion. In many of the treatments where the axial material inhomogeneity in the shear modulus is incorporated (see e.g. [18–23]), the shear modulus becomes unbounded as the axial coordinate $z \to \infty$. This represents a somewhat artificial constraint on the non-homogeneity. The class of problems in which the exponential non-homogeneity gives a finite linear elastic shear modulus as $z \to \infty$, is of interest to the study of material regions which experience surface hardening or surface softening effects. Therefore, we examine in this paper the Reissner-Sagoci problem related to a halfspace region in which the elastic shear modulus varies according to the relationship $G(z) = G_1 + G_2\, e^{-\xi z}$ (fig.

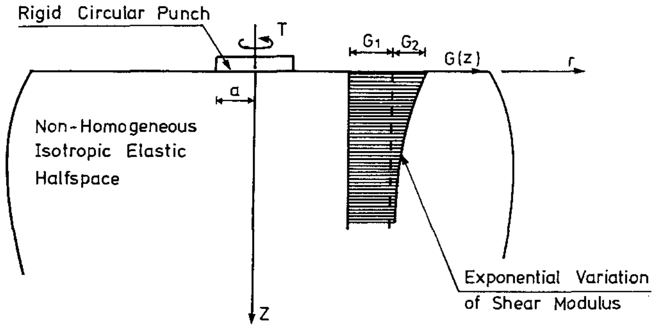

Fig. 1. Reissner-Sagoci problem for a non-homogeneous isotropic elastic halfspace.

1). Finally, it may be noted that the state of axisymmetric deformation induced by the torsional indentation is such that the medium experiences isochoric or volume preserving deformations. This does not however, imply that the medium is incompressible. Consequently the torsional indentation test provides a convenient method for determining the shear modulus and its non-homogeneity in isotropic elastic halfspace regions.

2. Fundamental equations

We consider the class of rotationally symmetric deformations which are referred to the cylindrical polar coordinate system (r, θ, z). The non-zero displacement component is the azimuthal displacement $u_\theta(r, z)$. For this category of rotationally symmetric problems the non-zero components of the Cauchy stress tensor are

$$\sigma_{\theta z}(r, z) = G(z)\left\{\frac{\partial u_\theta}{\partial z}\right\}, \tag{1}$$

$$\sigma_{r\theta}(r, z) = G(z)\left\{\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right\}, \tag{2}$$

where $G(z)$ is the linear elastic shear modulus which varies with depth z. In the absence of body forces, the non trivial equation of equilibrium is given by

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r} = 0. \tag{3}$$

Substituting the expressions for the stress components (1) and (2) into (3) we obtain the displacement equation of equilibrium

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \frac{1}{G(z)}\frac{\partial G}{\partial z}\frac{\partial u_\theta}{\partial z} = 0. \tag{4}$$

As discussed in the introduction, we shall restrict our attention to the specific form of elastic non-homogeneity which is characterized by

$$G(z) = G_1 + G_2\,\mathrm{e}^{-\zeta z}. \tag{5}$$

In order to ensure a bounded variation in $G(z)$ as $z \to \infty$, we require $\zeta > 0$. The values of G_1 and G_2 must be prescribed in such a way that the thermodynamic constraint which ensures positive definiteness of the elastic energy in the medium is satisfied, i.e.,

$$G(z) > 0; \qquad G_1 > 0; \qquad z \in (0, \infty).$$

It is also possible to interpret G_1 and G_2 in terms of the shear moduli that may be prescribed at $z = 0$ and $z \to \infty$. If we denote the surface shear modulus by G_0 and the shear modulus at $z \to \infty$ by G_∞ then

$$G_1 = G_\infty; \qquad G_2 = G_0 - G_\infty. \tag{7}$$

For the solution of the problem we also impose the following restriction on the values of G_1 and G_2:

$$\frac{G_2}{G_1} < 1 \text{ or} \frac{G_0}{G_\infty} < 2. \tag{8}$$

Therefore the particular form of elastic non-homogeneity discussed in the paper can be applied to media in which the shear modulus either increases or decreases with depth provided the criteria (6) and (8) are satisfied.

Using (5) the displacement equation of equilibrium (4) can be rewritten in the form

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{1}{r}\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} + \left[\frac{G_2 \zeta\,\mathrm{e}^{-\zeta z}}{\left(G_1 + G_2\,\mathrm{e}^{-\zeta z}\right)}\right]\frac{\partial u_\theta}{\partial z} = 0. \tag{9}$$

We now assume that u_θ admits a representation

$$u_\theta(r, z) = R(r)Z(z). \tag{10}$$

The equations (9) and (10) yield the following ordinary differential equations for $R(r)$ and $Z(z)$:

$$\frac{\mathrm{d}^2 R}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}R}{\mathrm{d}r} + \left(\xi^2 - \frac{1}{r^2}\right)R = 0, \tag{11}$$

$$\frac{\mathrm{d}^2 Z}{\mathrm{d}z^2} - \left\{\frac{G_2 \zeta\,\mathrm{e}^{-\zeta z}}{G_1 + G_2\,\mathrm{e}^{-\zeta z}}\right\}\frac{\mathrm{d}Z}{\mathrm{d}z} - \xi^2 Z = 0, \tag{12}$$

where ξ^2 denotes the constant of separation. By using the transformation

$$z_1 = \mathrm{e}^{-\zeta z}, \tag{13}$$

the differential equation (12) can be reduced to the form

$$z_1^2\left\{1 + \frac{G_2}{G_1}z_1\right\}\frac{\mathrm{d}^2 Z}{\mathrm{d}z_1^2} + z_1\left\{1 + \frac{2G_2}{G_1}z_1\right\}\frac{\mathrm{d}Z}{\mathrm{d}z_1} - \frac{\xi^2}{\zeta^2}\left\{1 + \frac{G_2}{G_1}z_1\right\}Z = 0. \tag{14}$$

Following Kamke [30], we see that the solution of (14) can be written in the form

$$Z(z_1) = z_1^{\xi/\zeta} F(\alpha, \beta; \gamma; -\rho z_1), \tag{15}$$

where F is the hypergeometric function [31],

$$\alpha = \frac{\xi}{\zeta} + \frac{1}{2}\left\{1 - \left[1 + \frac{4\xi^2}{\zeta^2}\right]^{1/2}\right\}, \tag{16a}$$

$$\beta = \frac{\xi}{\zeta} + \frac{1}{2}\left\{1 + \left[1 + \frac{4\xi^2}{\zeta^2}\right]^{1/2}\right\}, \tag{16b}$$

$$\gamma = 1 + \frac{2\xi}{\zeta}, \qquad \rho = \frac{G_2}{G_1}. \tag{16c}$$

For the solution (15) to be admissible,

$$|\rho z_1| < 1, \qquad \text{Re}\,\gamma > \text{Re}\,\beta > 0. \tag{17}$$

The solution of (11) can be obtained in a straightforward manner. The generalized integral solution for $u_\theta(r, z)$ derived from the above analysis can be written in the form

$$u_\theta(r, z) = \int_0^\infty A_1(\xi)\, \mathrm{e}^{-\xi z} F(\alpha, \beta; \gamma; -\rho\, \mathrm{e}^{-\zeta z}) J_1(\xi r)\, \mathrm{d}\xi. \tag{18}$$

3. The torsional indentation problem

We now focus on the problem where the non-homogeneous elastic halfspace region is subjected to a torsional indentation by a rigid circular punch of radius a which is bonded to its plane surface. The mixed boundary conditions associated with the indentation problem are

$$u_\theta(r, 0) = \Omega r, \qquad 0 \leqslant r \leqslant a, \tag{19}$$

$$\sigma_{\theta z}(r, 0) = 0, \qquad a < r < \infty, \tag{20}$$

where Ω is the rotation of the rigid punch. By making use of (1) and (18) the mixed boundary conditions (19) and (20) yield the following system of dual integral equations:

$$\int_0^\infty E_1(\xi) J_1(\xi r)\, \mathrm{d}\xi + \int_0^\infty E_1(\xi)[H(\xi) - 1]\, J_1(\xi r)\, \mathrm{d}\xi = \Omega r, \qquad 0 \leqslant r \leqslant a, \tag{21}$$

$$\int_0^\infty \xi E_1(\xi) J_1(\xi r)\, \mathrm{d}\xi = 0, \qquad a < r < \infty, \tag{22}$$

where

$$\xi E_1(\xi) = \left[\xi F(\alpha, \beta; \gamma; -\rho) - \frac{\zeta\rho\alpha\beta}{\gamma} F(\alpha+1, \beta+1; \gamma+1; -\rho)\right] A_1(\xi) \tag{23}$$

and

$$H(\xi) = \left[\frac{\xi F(\alpha, \beta; \gamma; -\rho)}{\xi F(\alpha, \beta; \gamma; -\rho) - \dfrac{\zeta\rho\alpha\beta}{\gamma} F(\alpha+1, \beta+1; \gamma+1; -\rho)} - 1\right]. \tag{24}$$

The solution of the dual system (21) and (22) can be attempted by using the techniques proposed by Sneddon [32] and others. We introduce the transformation

$$E_1(\xi) = \int_0^a \psi(t) \sin(\xi t)\, \mathrm{d}t \tag{25}$$

such that the traction boundary condition (22) is identically satisfied. The equation (21) now yields the following Fredholm integral equation of the second-kind for the unknown function $\psi(t)$:

$$\psi(t) + \int_0^a \psi(s) K(s, t)\, \mathrm{d}s = \frac{4\Omega t}{\pi}, \qquad 0 < t < a, \tag{26}$$

where

$$K(s, t) = \frac{2}{\pi} \int_0^\infty H(\xi) \sin(\xi s) \sin(\xi t)\, \mathrm{d}\xi. \tag{27}$$

The solution of (26) essentially completes the analysis of the torsional indentation problem related to a non-homogeneous elastic medium in which the shear modulus varies according to the relationship (5). Owing to the complicated form of the kernel function (27) it seems unlikely that a closed form solution can be developed for $\psi(t)$. The Fredholm integral equation (26) can however be solved by employing a variety of numerical schemes [33,34].

4. The torque-rotation relationship

A result of some interest to engineering applications concerns the torque-rotation relationship for the bonded rigid circular indentor. Considering the equilibrium of the disc we have

$$T = -2\pi \int_0^a r^2 \sigma_{\theta z}(r, 0)\, \mathrm{d}r. \tag{28}$$

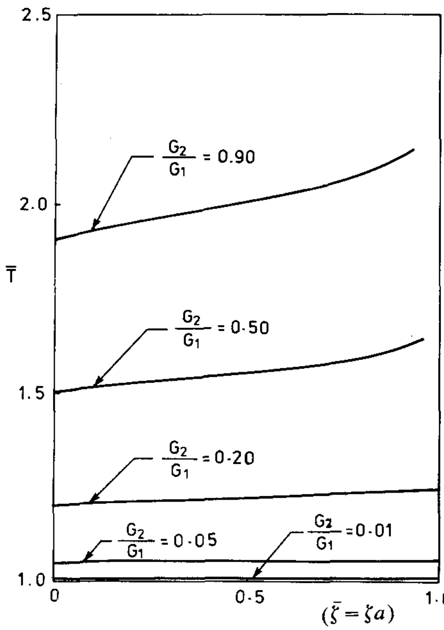

Fig. 2. Influence of the exponential non-homogeneity on the torsional stiffness of the rigid circular punch.

By using (1), (18) and (25), the above equation can be expressed in the form

$$T = 16(G_1 + G_2)\Omega a^3 \int_0^1 t^* \tilde{\psi}(t^*)\, \mathrm{d}t^*, \tag{29}$$

where

$$\tilde{\psi}(t) = \frac{\pi}{4}\psi(t), \qquad t^* = \frac{t}{a}. \tag{30}$$

In terms of shear moduli values at the surface at the halfspace region and at infinity, the expression (29) can be re-written in the form

$$\frac{3T}{16\Omega a^3 G_\infty} = 3\{1 + \rho\} \int_0^1 t^* \tilde{\psi}(t^*)\, \mathrm{d}t^*. \tag{31}$$

The approximate solution of the integral equation (26) can be obtained by reducing the problem to the solution of a set of simultaneous equations. The integrals encountered in this analysis can be evaluated by employing Gaussian quadrature. The details of the analysis will not be pursued here. The results of the analysis are presented in Fig. 2. The results indicate that the normalized torsional stiffness $\bar{T}(= 3T/16\Omega a^3 G_\infty)$ reduces the values given by Reissner and Sagoci [1] as $G_2 \to 0$. These results also indicate the influence of the exponential variation in the shear modulus on the torsional stiffness of the bonded indentor. It is evident that the effects of the exponential variation become noticeable only at relatively large values of G_2/G_1.

5. Conclusions

This paper focusses on the torsional indentation problem related to a rigid circular punch resting on a non-homogeneous isotropic elastic halfspace. The exponential form of the non-homogeneity considered in the paper represents a realistic variation which gives bounded values for the shear modulus at any point within the halfspace region. The mixed boundary value problem associated with the torsional indentation problem can be reduced to the solution of a Fredholm integral equation of the second-kind, with a somewhat complicated kernel function. Numerical results presented in the paper establish the influence of the modular ratio G_2/G_1 (or $\{(G_0/G_\infty) - 1\}$) and the parameter $\bar{\xi}$ (which represents the intensity of the exponential variation) on the torsional stiffness of the rigid circular punch bonded to a non-homogeneous elastic halfspace.

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