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TWO MOVING CRACKS IN A LAYERED COMPOSITE

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Abstract—This paper deals with the problem of the uniform motion of two cracks located at the central plane of an elastic layer embedded in bonded contact with two elastic half-space regions. The material properties in the layer are assumed to be different from the material properties of the half-space regions. The uniform motion of the cracks is induced by antiplane and in-plane extension modes. The analysis of the problem employs successive application of Galilean and Fourier transformations. The results of primary interest to fracture mechanics, namely the dynamic stress-intensity factors, are illustrated in graphical form.

1. INTRODUCTION

THE CLASS of problems which deals with the propagation of cracks in layered elastic solids is of fundamental importance to the study of fracture processes in composite materials. Owing to the complexity of elastodynamic stress analysis usually associated with nonhomogeneous materials, this category of problem has received only scant attention. The splitting of an infinite strip by a semi-infinite crack propagating at a constant velocity was investigated by Sih and Chen[1]. The article by Sih and Chen[2] deals with the problem of uniform propagation of a crack in a strip of elastic material under plane extension. Detailed accounts of the classes of elastodynamic problems which deal with cracks moving either at a constant velocity or at an acceleration are given in the reference texts by Sih[3] and Sih and Chen[4]. In particular, [3] contains an approximate solution for the problem of a crack of finite length moving in an elastic layer subjected to antiplane shear stress. More recent results by Singh *et al.*[5] presents a closed-form solution for the same problem. Tait and Moodie[6] have also obtained a closed-form result for two moving cracks contained in a layer subjected to antiplane shear stresses. The work by Sih and Chen[7] represents an example of the problem of a moving crack contained in an elastic layered composite.

In this paper, we shall consider a more general type of problem related to two moving cracks in a layered composite elastic solid (Fig. 1). The composite consists of an elastic layer of finite thickness ($2h$)

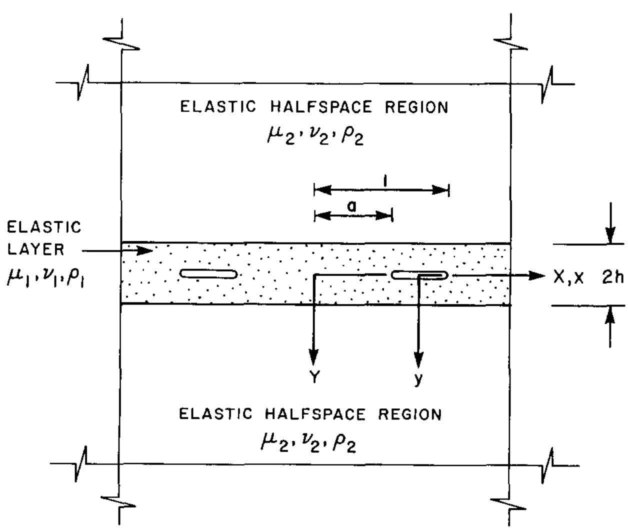

Fig. 1. Geometry of the cracks embedded in a layered composite.

which is embedded in bonded contact with two elastic half-space regions. The elastic properties of the layer are assumed to be different from the elastic properties of the adjacent half-space regions. The central plane of the layer contains two cracks of equal length which move at a uniform velocity. The motion of the cracks may be induced by either in-plane or antiplane shearing of the composite. The steady uniform motion of the cracks enables the application of a Galilean transformation to essentially reduce the elastodynamic problem to a pseudo-elastostatic one. With the aid of a Fourier transform development of the resulting equations, the boundary conditions associated with the two-crack problem yield a system of triple integral equations. These equations are solved in an approximate fashion. The results of particular interest to engineering applications concern the dynamic stress-intensity factors associated with the propagating crack. Numerical results presented in this paper illustrate the manner in which the properties of the composite region influence the dynamic stress-intensity factor.

2. THE ANTIPLANE PROBLEM

Consider the problem of two plane cracks of equal length which are moving at the central plane of thickness $2h$. The shear modulus, Poisson's ratio and mass density of the layer are denoted by μ_1, ν_1 and ρ_1 respectively. The equivalent properties for the surrounding elastic half-space regions are denoted by μ_2, ν_2 and ρ_2. The cracks spread with a constant width in the layer region. The uniform motion of the cracks is maintained by a system of antiplane shear stresses of magnitude $-\tau_0$ applied to the crack surfaces. Antiplane dynamic straining of the (layered) solid is characterized by a nonzero velocity component in the z direction:

$$[u_X]_j = [u_Y]_j = 0 \qquad (j = 1, 2); \tag{1a}$$

$$[u_Z]_j = w_j(X, Y, t) \qquad (j = 1, 2). \tag{1b}$$

The subscripts 1 and 2 refer to variables referred to the layer and half-space regions respectively. The rectangular Cartesian coordinate system (X, Y, Z) is fixed in the elastic region. By virtue of (1), the components of the Cauchy stress tensor reduce to the following:

$$[\sigma_X]_j = [\sigma_Y]_j = [\sigma_Z]_j = [\sigma_{XY}]_j = 0 \qquad (j = 1, 2); \tag{2}$$

$$[\sigma_{XZ}]_j = \mu_j \frac{\partial w_j}{\partial X}; \tag{3}$$

$$[\sigma_{YZ}]_j = \mu_j \frac{\partial w_j}{\partial Y}. \tag{4}$$

The nontrivial equations of motion reduce to

$$\nabla^2 w_j = \frac{1}{C_{2j}^2} \frac{\partial^2 w_j}{\partial t^2} \qquad (j = 1, 2), \tag{5}$$

where

$$C_{2j}^2 = \frac{\mu_j}{\rho_j} \qquad (j = 1, 2) \tag{6}$$

and

$$\nabla^2 = \frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}. \tag{7}$$

In (6), C_{2j} are the shear wave velocities, μ_j are the shear moduli and ρ_j the mass densities of the two materials. For cracks extending at uniform velocity it is convenient to introduce moving coordinates x_j, y_j and z_j ($j = 1,2$) which are related to X, Y and Z according to

$$x_j = X - vt, \qquad y_j = s_{2j}Y, \qquad z_j = Z;$$
$$s_{2j} = [1 - M_{2j}^2]^{1/2}, \qquad M_{2j} = \frac{v}{C_{2j}} \qquad (j = 1,2). \tag{8a}$$

The coordinate system (x_j, y_j, z_j) moves at a constant velocity v in the x direction and is spaced at the distance vt from the fixed system. Referred to the moving coordinate system, the eqn (5) reduces to

$$s_{2j}^2 \frac{\partial^2 w_j}{\partial x^2} + \frac{\partial^2 w_j}{\partial y^2} = 0 \qquad (j = 1,2). \tag{8b}$$

Applying the Fourier cosine transform to (8b) yields

$$w_1(x,y) = \frac{2}{\pi} \int_0^\infty [A_1(s) \exp(-ss_{21}y) + B_1(s) \exp(ss_{21}y)] \cos(sx)\, ds \tag{9}$$

and

$$w_2(x,y) = \frac{2}{\pi} \int_0^\infty [B_2(s) \exp(-ss_{22}y)] \cos(sx)\, ds. \tag{10}$$

As $M_{2j} \ll 1$ it implies that $(v/C_{2j}) < 1$, the crack velocity v is smaller than the shear-wave velocity and we observe that

$$(\sigma_{yz})_1 = \left(\frac{2\mu_1}{\pi}\right) \int_0^\infty s_{21}s[B_1(s) \exp(ss_{21}y) - A_1(s) \exp(-ss_{21}y)] \cos(sx)\, ds, \tag{11a}$$

$$(\sigma_{yz})_2 = -\left(\frac{2\mu_2}{\pi}\right) \int_0^\infty s_{22}s[B_2(s) \exp(-ss_{22}y)] \cos(sx)\, ds. \tag{11b}$$

The mixed boundary conditions for the plane $y = 0$ are given by

$$(\sigma_{yz})_1(x,0) = -\tau_0, \qquad a < |x| < 1; \tag{12a}$$

$$w_1(x,0) = 0, \qquad 0 < |x| < a, \qquad |x| > 1. \tag{12b}$$

Continuity of stresses across the interfaces yield the following:

$$(\sigma_{yz})_1(s, \pm h) = (\sigma_{yz})_2(x, \pm h). \tag{13}$$

Similarly, the continuity of displacements at the interfaces give

$$w_1(x, \pm h) = w_2(x, \pm h). \tag{14}$$

Making use of the boundary conditions (13) and (14) and the results (9)–(12), we obtain the following equations:

$$A_1(s) \exp(-shs_{21}) + B_1(s) \exp(shs_{21}) = B_2(s) \exp(-shs_{22}), \tag{15}$$

$$A_1(s)s_{21} \exp(-shs_{21}) - B_1(s)s_{21} \exp(shs_{21}) = \frac{\mu_2}{\mu_1} s_{22}B_2(s) \exp(-shs_{22}). \tag{16}$$

Solving (15) and (16) we find that

$$A_1(s) = \frac{(\mu_1 s_{21} + \mu_2 s_{22})B(s)}{[(\mu_1 s_{21} + \mu_2 s_{22}) + (\mu_1 s_{21} - \mu_2 s_{22}) \exp(-2shs_{21})]}, \tag{17a}$$

$$B_1(s) = \frac{(\mu_1 s_{21} - \mu_2 s_{22}) \exp(-2shs_{21}) B(s)}{[(\mu_1 s_{21} + \mu_2 s_{22}) + (\mu_1 s_{21} - \mu_2 s_{22}) \exp(-2shs_{22})]}, \tag{17b}$$

$$B_2(s) = \frac{2\mu_1 s_{21} \exp[-s(s_{21} - s_{22})h] B(s)}{[(\mu_1 s_{21} + \mu_2 s_{22}) + (\mu_1 s_{21} - \mu_2 s_{22}) \exp(-2shs_{21})]}. \tag{17c}$$

Using the above results, the mixed boundary conditions (12a) and (12b) yield the following system of triple-integral equations:

$$\int_0^\infty B(s) \cos(sx)\, ds = 0, \qquad 0 < x < a, \qquad x > 1; \tag{18}$$

$$\int_0^\infty s M_1(sh) B(s) \cos(sx)\, ds = \frac{\pi\tau_0\mu_1}{2s_{21}}, \qquad a < x < 1, \tag{19}$$

where

$$M_1(sh) = \frac{[(\mu_1 s_{21} + \mu_2 s_{22}) - (\mu_1 s_{21} - \mu_2 s_{22}) \exp(-2shs_{21})]}{[(\mu_1 s_{21} + \mu_2 s_{22}) + (\mu_1 s_{21} - \mu_2 s_{22}) \exp(-2shs_{21})]}. \tag{20}$$

Equation (19) can be rewritten in the form

$$\int_0^\infty sB(s) \cos(sx)\, ds + \int_0^\infty sB(s) M_2(sh) \cos(sx)\, ds = \frac{\pi\tau_0\mu_1}{2s_{21}}, \qquad a < x < 1, \tag{21}$$

where

$$M_2(sh) = \frac{2(\mu_2 s_{22} - \mu_1 s_{21}) \exp(-2shs_{21})}{[(\mu_1 s_{21} + \mu_2 s_{22}) + (\mu_2 s_{22} - \mu_1 s_{21}) \exp(-2shs_{21})]}. \tag{22}$$

The system of triple-integral equations can be solved by making use of the procedures outlined by Lowengrub and Srivastava[8]. In this method the triple-integral equations are reduced to a single Fredholm integral equation of the second kind by an application of a transformation of the type

$$B(\xi) = \left(\frac{\tau_0\mu_1}{ss_{21}}\right) \int_a^1 h(t^2) \sin(\xi t)\, dt. \tag{23}$$

The function $h(t^2)$ in (23) is derived from the solution of the Fredholm integral equation of the second kind

$$h(x^2) + \int_a^1 h(t^2) K(x^2, t)\, dt = F(x^2), \qquad a < x < 1, \tag{24}$$

and satisfying the condition

$$\int_a^1 h(t^2)\, dt = 0. \tag{25}$$

In (24),

$$K(x^2,t) = \frac{-4(x^2-a^2)^{1/2}}{\pi^2(1-x^2)^{1/2}} \int_a^1 \frac{(1-y^2)^{1/2}}{(y^2-a^2)^{1/2}} \frac{yK_1(y,t)\,\mathrm{d}y}{(y^2-x^2)}, \tag{26}$$

$$K_1(y,t) = \int_0^\infty M_2(\delta u)\cos(uy)\sin(ut)\,\mathrm{d}u, \tag{27}$$

and

$$F(x^2) = \frac{\tau_0\mu_1}{s_{21}} \frac{(x^2-a^2)^{1/2}}{(1-x^2)^{1/2}} + \frac{C}{[(x^2-a^2)(1-x^2)]^{1/2}}, \tag{28}$$

where C is an arbitrary constant which is to be determined from the consistency condition (25). For large values of h (i.e. $h \gg 1$), the Fredholm integral equation can be solved by employing a technique which involves expansion of $K_1(y,t)$ in powers of $(1/h)$. The details of the techniques are given by Lowengrub and Srivastava[8]. The result of primary importance concerns the estimation of the stress-intensity factors at the ends of the moving cracks. The stress-intensity factors at the tips $x = a$ and $x = 1$ of the moving cracks are given by

$$k_{3a} = \lim_{x\to a^-} (a-x)^{1/2}[(\sigma_{yz})_1(x,0)] \tag{29}$$

and

$$k_{31} = \lim_{x\to 1^+} (x-1)^{1/2}[(\sigma_{yz})_1(x,0)]. \tag{30}$$

From (11) we note that

$$(\sigma_{yz})_1(x,0) = -\left(\frac{2s_{21}}{\pi\mu_1}\right)\left[\int_a^1 \frac{th(t^2)\,\mathrm{d}t}{(t^2-x^2)} + \int_a^1 h(t^2)K_1(x,t)\,\mathrm{d}t\right]. \tag{31}$$

Performing the necessary reductions, it can be shown that

$$k_{3a} = \frac{\tau_0}{[2a(1-a^2)]^{1/2}}\left[\left(\frac{E}{F}-a^2\right)\left(1-\frac{I_0C_0}{2h^2}+\frac{I_0^2C_0^2}{4h^4}\right)+\frac{2I_1}{h^4}(3a^4+a^2C_1+C_2)+O(h^{-6})\right] \tag{32}$$

and

$$k_{31} = \frac{\tau_0}{[2(1-a^2)]^{1/2}}\left[\left(1-\frac{E}{F}\right)\left(1-\frac{I_0C_0}{2h^2}+\frac{I_0^2C_0^2}{4h^4}\right)-\frac{2I_1}{h^4}(3+C_1+C_2)+O(h^{-6})\right], \tag{33}$$

where

$$C_0 = (1+a^2)-2\frac{E}{F}, \tag{34a}$$

$$C_1 = \frac{(1-a^2)^2}{4C_0} - 1 - a^2, \tag{34b}$$

$$C_2 = a^2 + \frac{E}{F}C_1. \tag{34c}$$

In (32)–(34c), F and E are elliptic integrals of the first and second kinds respectively and

$$E\left[\frac{\pi}{2};(1-a^2)^{1/2}\right]=\int_a^1\frac{t^2\,dt}{\{(t^2-a^2)(1-t^2)\}^{1/2}},\qquad F\left[\frac{\pi}{2};(1-a^2)^{1/2}\right]=\int_a^1\frac{dt}{\{(t^2-a^2)(1-t^2)\}^{1/2}}. \tag{34d}$$

Also,

$$I_0=\int_0^\infty sM_2(s)\,ds, \tag{35a}$$

$$I_1=-\frac{1}{6}\int_0^\infty s^3M_2(s)\,ds; \tag{35b}$$

it also easily follows that

$$s_{22}=\sqrt{1-\frac{v^2}{c_{21}^2}\frac{\mu_1}{\mu_2}\frac{\rho_2}{\rho_1}}. \tag{35c}$$

3. THE IN-PLANE PROBLEM

In this section we examine the problem in which the cracks spread with a constant width in the homogeneous isotropic elastic layer embedded within the half-space regions. The cracks are maintained at a constant velocity v by uniform normal compressive tractions σ_0 which are applied to the crack surfaces. Again the stationary rectangular Cartesian coordinates are denoted by X, Y and Z. The displacement components in the layer and half-space regions referred to the (X, Y, Z) system are denoted by $(U_X)_j$, $(U_Y)_j$, $(U_Z)_j$, $j = 1, 2$, and the subscripts 1 and 2 refer to quantities specified in the layer and half-space regions respectively. For the analysis of the problem, we introduce displacement potentials $\phi_j(X, Y, t)$ and $\psi_j(X, Y, t)$ such that

$$(U_X)_j=\frac{\partial\phi_j}{\partial X}+\frac{\partial\phi_j}{\partial Y},\qquad (U_Y)_j=\frac{\partial\phi_j}{\partial Y}-\frac{\partial\psi_j}{\partial X},\qquad (U_Z)_j=0. \tag{36}$$

Making use of the stress–strain relationship for isotropic elastic materials, we obtain

$$\begin{aligned}(\sigma_X)_j&=\lambda_j\nabla^2\phi_j+2\mu_j\left(\frac{\partial^2\phi_j}{\partial X^2}+\frac{\partial^2\psi_j}{\partial X\,\partial Y}\right),\\ (\sigma_Y)_j&=\lambda_j\nabla^2\phi_j+2\mu_j\left(\frac{\partial^2\phi_j}{\partial Y^2}-\frac{\partial^2\psi_j}{\partial X\,\partial Y}\right),\\ (\sigma_Z)_j&=\frac{\lambda_j}{2}\left(\frac{\lambda_j+2\mu_j}{\lambda_j+\mu_j}\right)\nabla^2\phi_j,\\ (\sigma_{XY})_j&=\mu_j\left(2\frac{\partial^2\phi_j}{\partial X\,\partial Y}+\frac{\partial^2\psi_j}{\partial Y^2}-\frac{\partial^2\psi_j}{\partial X^2}\right),\\ (\sigma_{XZ})_j&=(\sigma_{YZ})_j=0,\end{aligned} \tag{37}$$

where λ_j and μ_j are Lamé's constants and ∇^2 is Laplace's operator referred to the (X, Y) coordinate system. Using (37), the nontrivial equations of motion can be expressed in the form

$$\nabla^2 \phi_j = \frac{1}{c_{1j}^2} \frac{\partial^2 \phi_j}{\partial t^2}, \tag{38a}$$

$$\nabla^2 \psi_j = \frac{1}{c_{2j}^2} \frac{\partial^2 \psi_j}{\partial t^2}, \tag{38b}$$

where

$$c_{1j} = \left\{\frac{\lambda_j + 2\mu_j}{\rho_j}\right\}^{1/2}, \qquad c_{2j} = \left(\frac{\mu_j}{\rho_j}\right)^{1/2} \tag{39}$$

are, respectively, the dilatational and shear-wave velocities in the layer and half-space regions. For steady-state motion of the cracks, we employ the Galilean transformation

$$x = X - vt; \qquad y = Y; \qquad z = Z. \tag{40}$$

In terms of the moving coordinate system (x, y, z), the equations of motion (38) can be expressed in the form

$$s_{1j}^2 \frac{\partial^2 \phi_j}{\partial x^2} + \frac{\partial^2 \phi_j}{\partial y^2} = 0, \tag{41}$$

$$s_{2j}^2 \frac{\partial^2 \psi_j}{\partial x^2} + \frac{\partial^2 \psi_j}{\partial y^2} = 0, \tag{42}$$

in which s_{1j} and s_{2j} are defined by the expressions

$$s_{1j}^2 = 1 - M_{1j}^2, \qquad s_{2j}^2 = 1 - M_{2j}^2. \tag{43}$$

The quantities M_{1j} and M_{2j} are given by

$$M_{1j} = \frac{v}{c_{ij}}, \qquad M_{2j} = \frac{v}{c_{2j}}. \tag{44}$$

The boundary conditions associated with the uniformly moving cracks are as follows:

$$(\sigma_y)_1(x, 0) = -\sigma_0, \qquad a < x < 1; \tag{45}$$

$$(u_y)_1(x, 0) = 0, \qquad 0 \leqslant x \leqslant a, \qquad 1 \leqslant x < \infty; \tag{46}$$

$$(\sigma_{xy})_1(x, 0) = 0, \qquad x \geqslant 0. \tag{47}$$

At the interfaces between the elastic layer and the half-space regions (i.e. $y = \pm h$) we require

$$(\sigma_y)_1(x, \pm h) = (\sigma_y)_2(x, \pm h), \tag{48}$$

$$(\sigma_{xy})_1(x, \pm h) = (\sigma_{xy})_2(x, \pm h), \tag{49}$$

and

$$(u_x)_1(x, \pm h) = (u_x)_2(x, \pm h), \tag{50}$$

$$(u_y)_1(x, \pm h) = (u_y)_2(x, \pm h). \tag{51}$$

The solutions of (41) and (42) applicable to the analysis of the moving-crack problem posed by

(45)–(51) are, for the layer region,

$$\phi_1(x,y) = \frac{2}{\pi}\int_0^\infty [A^{(1)}(s)\exp(-ss_{11}y)+A^{(2)}(s)\exp(ss_{11}y)]\cos(sx)\,\mathrm{d}s, \tag{52}$$

$$\psi_1(x,y) = \frac{2}{\pi}\int_0^\infty [B^{(1)}(s)\exp(-ss_{21}y)+B^{(2)}(s)\exp(ss_{21}y)]\sin(sx)\,\mathrm{d}s, \tag{53}$$

and for the half-space regions surrounding the layer

$$\phi_2(x,y) = \frac{2}{\pi}\int_0^\infty C^{(1)}(s)\exp(-ss_{12}y)\cos(sx)\,\mathrm{d}s, \tag{54}$$

$$\psi_2(x,y) = \frac{2}{\pi}\int_0^\infty C^{(2)}(s)\exp(-ss_{22}y)\cos(sx)\,\mathrm{d}s. \tag{55}$$

It can be shown that the boundary conditions (45)–(51) can be reduced to the following system of triple-integral equations for an unknown function $A(s)$:

$$\int_0^\infty A(s)\cos(sx)\,\mathrm{d}s = 0, \qquad 0 \leqslant x \leqslant a, \qquad x \geqslant 1; \tag{56}$$

$$\int_0^\infty sA(s)\cos(sx)\,\mathrm{d}s + \int_0^\infty s[F_1(sh)-1]A(s)\cos(sx)\,\mathrm{d}s = -\frac{\pi\sigma_0}{4\mu_1 f_0}, \qquad a < x < 1; \tag{57}$$

where

$$f_0 = \frac{(1+s_{21}^2)^2 - 4s_{11}s_{21}}{2s_{11}(1-s_{21}^2)} \tag{58}$$

and

$$F_1(sh) = \frac{(1+s_{21}^2)}{s_{11}f_0(1-s_{21}^2)}[\tfrac{1}{2}(1+s_{21}^2)c^{(1)}+c^{(2)}+c^{(3)}]-1. \tag{59}$$

The values of $A^{(i)}(s)$, $B^{(i)}(s)$ and $C^{(i)}(s)$ $(i = 1,2)$ can be expressed in terms of $A(s)$ and the values of $C^{(1)}$, $C^{(2)}$, etc. are given in the Appendix. The analysis of the triple system (56)–(57) follows the procedures outlined by Lowengrub and Srivastava[8] and in this paper we shall restrict our attention to the development of the stress-intensity factors at the locations $x = a$ and $x = 1$ of the moving cracks.

We find that

$$\sigma_{yy}(x,0) = \frac{4\mu f_0}{\pi}\left[\int_0^\infty sA(s)\cos(sx)\,\mathrm{d}s + \int_0^\infty s[F(sh)-1]A(s)\cos(sx)\,\mathrm{d}s\right]. \tag{60}$$

The appropriate stress-intensity factors are defined by the following:

$$k_{1a} = \lim_{x\to a^-}[a-x]^{1/2}[(\sigma_{yy})_1(x,0)], \tag{61}$$

$$k_{11} = \lim_{x\to 1^+}[x-1]^{1/2}[(\sigma_{yy})_1(x,0)]. \tag{62}$$

Explicit expressions for k_{1a} and k_{11} are given by

$$k_{1a} = \frac{\sigma_0}{[2a(1-a^2)]^{1/2}}\left[\left(\frac{E}{F}-a^2\right)\left(1-\frac{A_0C_0}{2h^2}+\frac{A_0^2C_0^2}{4h^4}\right)+\frac{2A_2}{h^4}(3a^4+C_2+C_1a^2)+0(h^{-6})\right], \tag{63}$$

$$k_{11} = \frac{\sigma_0}{[2(1-a^2)]^{1/2}}\left[\left(1-\frac{E}{F}\right)\left(1-\frac{A_0C_0}{2h^2}+\frac{A_0^2C_0^2}{4h^4}\right)-\frac{2A_2}{h^4}(3+C_1+C_2)+O(h^{-6})\right], \quad (64)$$

where

$$A_0 = \int_0^\infty sF_1(s)\,\mathrm{d}s, \qquad A_2 = -\frac{1}{6}\int_0^\infty s^3F_1(s)\,\mathrm{d}s, \quad (65)$$

and the expression for C_0, C_1 and C_2 are given in eqn (34).

4. CONCLUSIONS AND NUMERICAL RESULTS

This paper considers the analytical treatment of two moving cracks which are located in an elastic layer embedded between two elastic half-space regions. The analytical treatments culminate in the approximate solution of a set of triple-integral equations. The dynamic stress-intensity factors relevant to the antiplane flaw-shearing mode and the in-plane flaw-extension mode are developed for the tips of the moving crack located at $x = a$ and $x = 1$. Figures 2–5 illustrate the manner in which the stress-intensity factors k_{31} and k_{3a} (corresponding to the locations $x = 1$ and $x = a$ respectively) are influenced by (i) the shear modular ratio of the elastic half-space to the elastic layer, (ii) the normalized

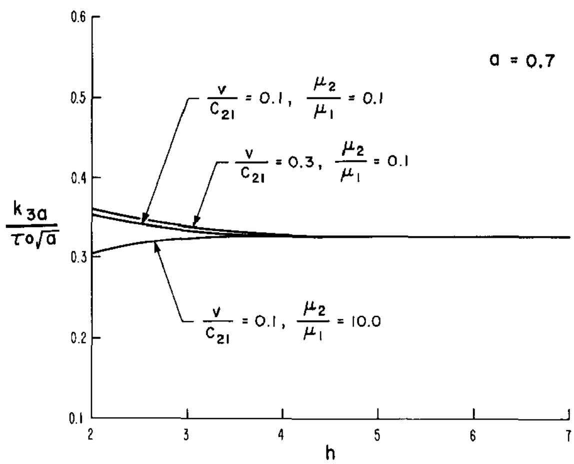

Fig. 2. Dynamic stress-intensity factor k_{3a} for the antiplane shearing mode.

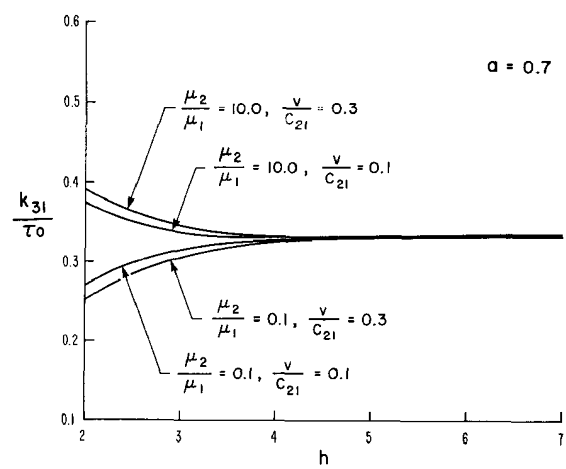

Fig. 3. Dynamic stress-intensity factor k_{31} for the antiplane shearing mode.

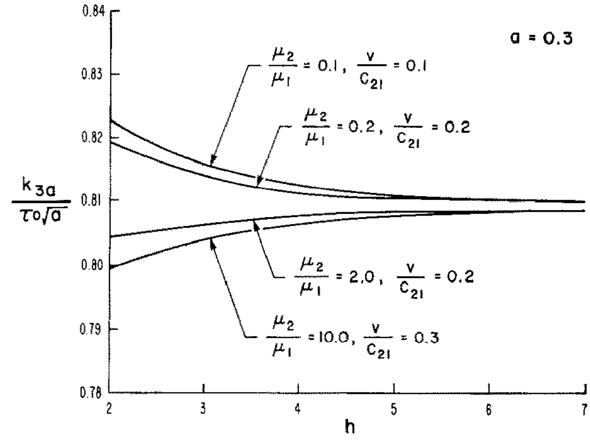

Fig. 4. Dynamic stress-intensity factor k_{3a} for the antiplane shearing mode.

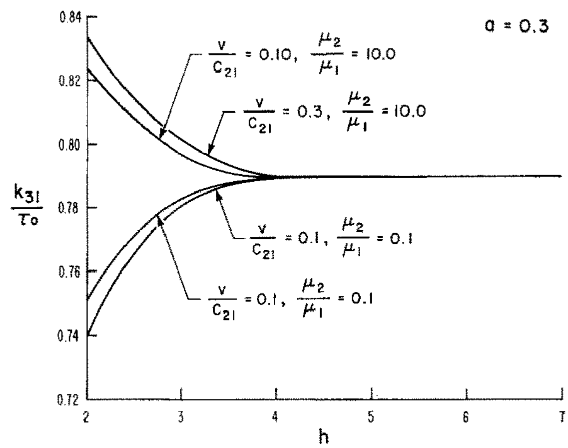

Fig. 5. Dynamic stress-intensity factor k_{31} for the antiplane shearing mode.

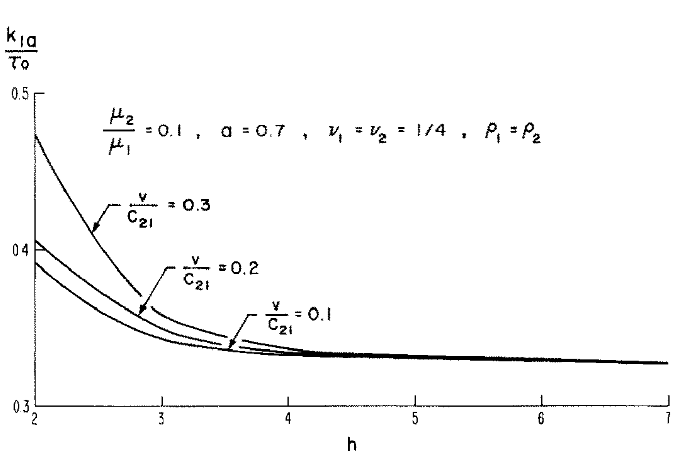

Fig. 6. Dynamic stress-intensity factor k_{1a} for the in-plane shearing mode.

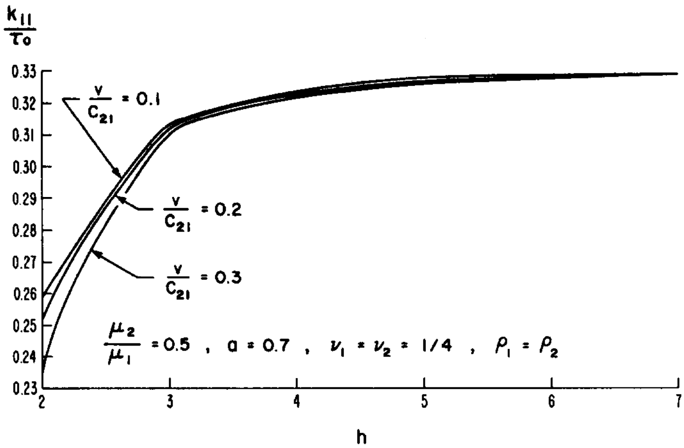

Fig. 7. Dynamic stress-intensity factor k_{11} for the in-plane shearing mode.

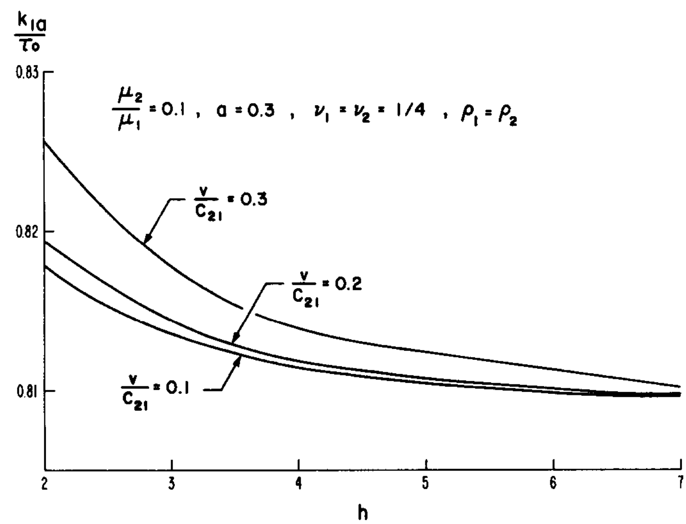

Fig. 8. Dynamic stress-intensity factor k_{1a} for the in-plane shearing mode.

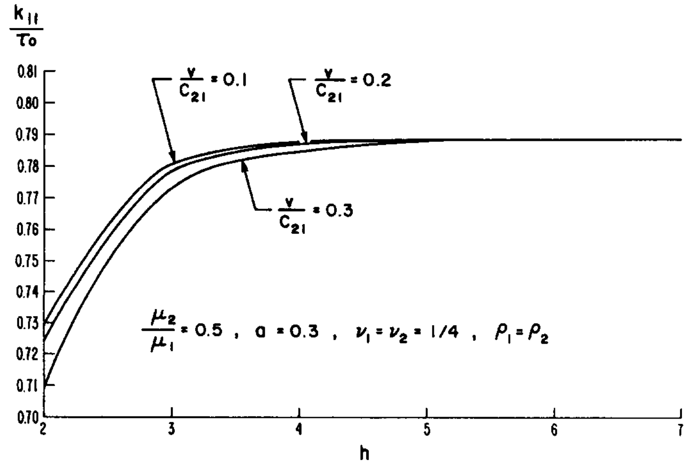

Fig. 9. Dynamic stress-intensity factor k_{11} for the in-plane shearing mode.

thickness of the embedded layer, (iii) the ratio of the velocity of propagation of the crack to the shear-wave velocity of the half-space region and (iv) the relative spacing between the two propagating cracks. Similar results for the stress-intensity factors k_{11} and k_{1a} are given in Figs. 6–9. The numerical results exhibit consistent trends and noticeable variations in the stress-intensity factors are observed only for small relative thicknesses of the embedded layer.

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APPENDIX

$$A^{(1)}(s) = \frac{(1+s_{21}^2)A(s)}{ss_{11}(1-s_{21}^2)(1-c^{(4)})},$$

$$A^{(2)}(s) = c^{(4)}A^{(1)}(s),$$

$$B^{(1)}(s) = \frac{\exp(ss_{21}h)}{b^{(1)}}[b^{(4)}A^{(1)}(s)\exp(-ss_{11}h)+b^{(2)}A^{(2)}(s)\exp(ss_{11}h)],$$

$$B^{(2)}(s) = \frac{\exp(-ss_{21}h)}{b^{(1)}}[b^{(5)}A^{(1)}(s)\exp(-ss_{11}h)-b^{(3)}A^{(2)}(s)\exp(ss_{11}h)],$$

$$\begin{aligned} C^{(1)}(s) = \frac{1}{(1-s_{12}s_{22})}&[(1-s_{11}s_{22})A^{(1)}(s)\exp(-s\{s_{11}-s_{12}\}h) \\ &+(1+s_{11}s_{22})A^{(2)}(s)\exp(s\{s_{11}+s_{22}\}h)+(s_{21}-s_{22})B^{(1)}(s)\exp(-s\{s_{21}-s_{12}\}h) \\ &-(s_{21}+s_{22})B^{(2)}(s)\exp(s\{s_{21}+s_{12}\}h)], \end{aligned}$$

$$\begin{aligned} C^{(2)}(s) = \frac{1}{(1-s_{12}s_{22})}&[(s_{11}-s_{12})A^{(1)}(s)\exp(-s\{s_{11}-s_{12}\}h) \\ &-(s_{11}+s_{12})A^{(2)}(s)\exp(s\{s_{11}+s_{12}\}h)+(1-s_{12}s_{21})B^{(1)}(s)\exp(-s\{s_{21}-s_{22}\}h) \\ &+(1+s_{21}s_{12})B^{(2)}(s)\exp(s\{s_{21}+s_{22}\}h)]. \end{aligned}$$

The quantities $b^{(1)}$, $b^{(2)}$, etc. are given by

$$\begin{aligned} b^{(1)} &= a^{(3)}a^{(8)}+a^{(4)}a^{(7)}, \\ b^{(2)} &= a^{(2)}a^{(8)}-a^{(4)}a^{(6)}, \\ b^{(3)} &= a^{(2)}a^{(7)}+a^{(3)}a^{(6)}, \\ b^{(4)} &= a^{(1)}a^{(8)}+a^{(4)}a^{(5)}, \\ b^{(5)} &= a^{(3)}a^{(5)}-a^{(1)}a^{(7)}, \end{aligned}$$

where

$$a^{(1)} = \frac{(1+s_{21}^2)}{2} - a^{(9)}\left[\frac{(1+s_{22}^2)}{2}(1-s_{11}s_{22})+s_{22}(s_{11}-s_{12})\right],$$

$$a^{(2)} = \frac{(1+s_{21}^2)}{2} - a^{(9)}\left[\frac{(1+s_{22}^2)}{2}(1+s_{11}s_{22})-s_{22}(s_{11}+s_{12})\right],$$

$$a^{(3)} = -s_{21} + a^{(9)}\left[\frac{(1+s_{22}^2)}{2}(s_{21}-s_{22})+s_{22}(1-s_{21}s_{12})\right],$$

$$a^{(4)} = -s_{21} + a^{(9)}\left[\frac{(1+s_{22}^2)}{2}(s_{21}+s_{22}) - s_{22}(1+s_{21}s_{12})\right],$$

$$a^{(5)} = s_{11} - a^{(9)}\left[\frac{(1+s_{22}^2)}{2}(s_{11}-s_{12}) + s_{12}(1-s_{11}s_{22})\right],$$

$$a^{(6)} = s_{11} - a^{(9)}\left[\frac{(1+s_{22}^2)}{2}(s_{11}-s_{12}) - s_{12}(1+s_{11}s_{22})\right],$$

$$a^{(7)} = -\frac{(1+s_{21}^2)}{2} + a^{(9)}\left[\frac{(1+s_{22}^2)}{2}(1-s_{21}s_{12}) + s_{12}(s_{21}-s_{22})\right],$$

$$a^{(8)} = -\frac{(1+s_{21}^2)}{2} + a^{(9)}\left[\frac{(1+s_{22}^2)}{2}(1+s_{21}s_{12}) - s_{12}(s_{21}+s_{22})\right],$$

$$a^{(9)} = \frac{\mu_2}{\mu_1(1-s_{12}s_{22})}.$$

Also

$$c^{(1)} = \frac{1+c^{(4)}}{1-c^{(4)}},$$

$$c^{(2)} = \frac{s_{11}s_{21}}{(1-c^{(4)})c^{(5)}}\{b^{(2)} + b^{(4)}\exp(-2ss_{11}h) + b^{(3)}\exp(-2ss_{21}h) - b^{(5)}\exp(-2s\{s_{11}+s_{21}\}h)\},$$

$$c^{(3)} = \frac{s_{21}(1+s_{21}^2)}{(1-c^{(4)})c^{(5)}b^{(1)}}[b^{(2)}b^{(5)} + b^{(3)}b^{(4)}]\exp(-s\{s_{11}+s_{21}\}h),$$

$$c^{(4)} = \frac{1}{c^{(5)}}[s_{11}b^{(1)}\exp(-s\{s_{11}+s_{21}\}h) + \tfrac{1}{2}(1+s_{21}^2)b^{(4)}\exp(-2ss_{11}h)$$
$$+ \tfrac{1}{2}(1+s_{21}^2)b^{(5)}\exp(-2s\{s_{11}+s_{21}\}h),$$

$$c^{(5)} = s_{11}b^{(1)}\exp(-s\{s_{11}+s_{21}\}h) - \tfrac{1}{2}(1+s_{21}^2)b^{(3)}\exp(-2ss_{21}h) - \tfrac{1}{2}(1+s_{21}^2)\,b^{(2)}.$$