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THE DISTRIBUTION OF STRESS IN A RUBBER-LIKE ELASTIC MATERIAL BOUNDED INTERNALLY BY A RIGID SPHERICAL INCLUSION SUBJECTED TO A CENTRAL FORCE

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Introduction

In this paper exact closed form solutions are obtained for the second-order effects in an infinite incompressible elastic medium containing a bonded rigid spherical inclusion which is subjected to a central force. The second-order elasticity theory adequately describes the mechanical behaviour of most rubber-like materials at moderately large strains. A displacement function technique is employed in the solution of the second-order problem

Analysis

The differential equations which arise in formulating the mathematical theory of highly elastic rubber-like materials are generally non-linear in character. Approximate methods of analysis are therefore of particular value in instances where exact solutions of these non-linear differential equations are not readily obtainable. The method of successive approximations is one such technique which has received considerable interest. Second-order elasticity theory considers the successive approximation procedure to include terms which are quadratic in the displacement gradients, and adequately describes the mechanical behaviour of most rubber-like materials at moderately large strains.

The general theory of second-order elasticity for axially symmetric deformations of isotropic incompressible elastic

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materials was developed by Selvadurai and Spencer [1]. This particular method of analysis of second-order effects is facilitated by the introduction of a displacement function, Ψ, which reduces the problem to the solution of a single equation of the form $E^4\Psi = f(S,\Theta)$, where E^2 is Stokes' differential operator, $f(S,\Theta)$ depends only on the first-order or classical elasticity solution and S,Θ are the spherical polar coordinates of the reference configuration.

In this paper we consider the problem of an incompressible infinite elastic medium containing a bonded rigid spherical inclusion which is subjected to a central resultant force. This problem is of importance in the analysis of stress concentrations in multiphase rubber-like materials and in bonded rubber mountings. It also serves as a useful mechanical analogue of Kelvin's problem for the concentrated force acting at the interior of an infinite elastic solid. The solutions for the second-order displacement and stress components are presented in exact closed form.

Basic equations

The general theory of second-order elasticity is given in Green and Adkins [2]. A detailed account of the displacement function formulation of the axially symmetric second-order problem is given by Selvadurai and Spencer [1]. For completeness, we shall briefly outline the relevant results.

The spherical polar coordinates of the reference configuration are denoted by (S,Φ,Θ) such that

$$R = S\sin\Theta, \qquad Z = S\cos\Theta, \tag{1}$$

where R,Z are the cylindrical polar coordinates. We restrict our attention to a state of stress which is symmetric about the Z-axis. By using the displacement function technique the solution of the first-order or the classical elasticity problem is reduced to the solution of the equations

$$E^4\Psi_1 = 0, \qquad \nabla^2 p_1 = 0, \tag{2}$$

where Ψ_1 is the first-order displacement function and p_1 is the first-order hydrostatic pressure, subject to the particular boundary conditions of the problem. The operators E^2 and ∇^2 are, respectively, Stokes' and Laplace's differential operators given by

$$E^2 = \frac{\partial^2}{\partial S^2} + \frac{1}{S^2}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{S^2}\frac{\partial}{\partial\theta}, \tag{3}$$

$$\nabla^2 = \frac{\partial^2}{\partial S^2} + \frac{2}{S}\frac{\partial}{\partial S} + \frac{1}{S^2}\frac{\partial^2}{\partial\theta^2} + \frac{\cot\theta}{S^2}\frac{\partial}{\partial\theta}, \tag{4}$$

and $$E^4 = E^2E^2 .$$

The first-order displacement components u_1 and w_1 in the S,θ directions are completely determined from the first-order displacement function Ψ_1 by means of the relations

$$u_1 = -\frac{1}{S^2\sin\theta}\frac{\partial\Psi_1}{\partial\theta}, \qquad w_1 = \frac{1}{S\sin\theta}\frac{\partial\Psi_1}{\partial S}. \tag{5}$$

The first-order constitutive equation is

$$\underset{\sim}{T}_1 = -p_1\underset{\sim}{I} + \mu\underset{\sim}{C}_1 , \tag{6}$$

where $\underset{\sim}{T}_1$ is the first-order component of the Cauchy stress tensor; μ is the linear elastic shear modulus; p_1 is the first-order hydrostatic pressure; $\underset{\sim}{I}$ is the unit matrix and

$$\underset{\sim}{C}_1 = \underset{\sim}{G}_1 + \underset{\sim}{G}_1^T , \tag{7}$$

where

$$\underset{\sim}{G}_1 = \begin{pmatrix} \frac{\partial u_1}{\partial S} & 0 & \frac{1}{S}\frac{\partial u_1}{\partial\theta} - \frac{w_1}{S} \\ 0 & \frac{u_1}{S} + \frac{w_1}{S}\cot\theta & 0 \\ \frac{\partial w_1}{\partial S} & 0 & \frac{u_1}{S} + \frac{1}{S}\frac{\partial w_1}{\partial\theta} \end{pmatrix} \tag{8}$$

The solution of the second-order problem for an isotropic incompressible elastic material reduces to the solution of the equations

$$E^4\Psi_2 = \frac{\sin\theta}{\mu}\left\{\frac{\partial}{\partial S}(SH_2) - \frac{\partial H_1}{\partial\theta}\right\}, \tag{9}$$

$$\nabla^2 p_2 = -\left\{\frac{\partial H_1}{\partial S} + \frac{2H_1}{S} + \frac{1}{S}\frac{\partial H_2}{\partial\theta} + H_2\frac{\cot\theta}{S}\right\}, \tag{10}$$

where Ψ_2 is the second-order displacement function, p_2 is the second-order hydrostatic pressure and

$$\begin{aligned} H_1 &= -\left\{\frac{\partial P'_{Ss}}{\partial S} + \frac{1}{S}\frac{\partial P'_{\theta s}}{\partial\theta} + \frac{1}{S}\left[2P'_{Ss} - P'_{\phi\phi} - P'_{\theta\theta} + P'_{\theta s}\cot\theta\right]\right\}, \\ H_2 &= -\left\{\frac{\partial P'_{S\theta}}{\partial S} + \frac{1}{S}\frac{\partial P'_{\theta\theta}}{\partial\theta} + \frac{1}{S}\left[\left(P'_{\theta\theta} - P'_{\phi\phi}\right)\cot\theta + 2P'_{S\theta} + P'_{\theta s}\right]\right\}. \end{aligned} \tag{11}$$

The terms $P'_{Ss}, \ldots,$ etc. are the elements of a non-symmetric matrix $\underset{\sim}{P}'_2$ given by

$$\underset{\sim}{P}'_2 = \begin{pmatrix} P'_{Ss} & 0 & P'_{S\theta} \\ 0 & P'_{\Phi\phi} & 0 \\ P'_{\Theta s} & 0 & P'_{\Theta\theta} \end{pmatrix} , \quad (12)$$

and

$$\underset{\sim}{P}'_2 = p_1 \underset{\sim}{G}_1 + \tfrac{1}{2}\mu\left[D\underset{\sim}{C}_1 - \underset{\sim}{G}_1^2 + \left(\underset{\sim}{G}^T\right)^2 + \frac{w_1}{S}(\underset{\sim}{Q}\underset{\sim}{C}_1 - \underset{\sim}{C}_1\underset{\sim}{Q})\right] - 2C_2\underset{\sim}{C}_1^2 . \quad (13)$$

In (13) the operator D is given by

$$D = u_1 \frac{\partial}{\partial S} + \frac{w_1}{S}\frac{\partial}{\partial\theta} , \quad (14)$$

$\underset{\sim}{Q}$ denotes the skew-symmetric matrix

$$\underset{\sim}{Q} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \quad (15)$$

and C_2 is a material constant. The second-order displacement components u_2 and w_2 can be represented in the form

$$u_2 = u'_2 + u''_2 , \qquad w_2 = w'_2 + w''_2 , \quad (16)$$

where

$$u'_2 = -\frac{1}{S^2 \sin\Theta}\frac{\partial\Psi_2}{\partial\Theta} , \qquad w'_2 = \frac{1}{S\sin\Theta}\frac{\partial\Psi_2}{\partial S} , \quad (17)$$

and

$$u''_2 = \tfrac{1}{2}Du_1 - \tfrac{1}{2}\frac{w_1^2}{S} , \qquad w''_2 = \tfrac{1}{2}Dw_1 + \tfrac{1}{2}\frac{u_1 w_1}{S} . \quad (18)$$

The second-order constitutive equation is

$$\underset{\sim}{T}_2 = -p_2\underset{\sim}{I} + \mu\underset{\sim}{C}'_2 + \underset{\sim}{P}'_2 + \frac{w_1}{S}\left\{\underset{\sim}{T}_1\underset{\sim}{Q} - \underset{\sim}{Q}\underset{\sim}{T}_1\right\} + \underset{\sim}{G}_1\underset{\sim}{T}_1 , \quad (19)$$

where $\underset{\sim}{T}_2$ is the second-order component of the Cauchy stress tensor and

$$\underset{\sim}{C}'_2 + \underset{\sim}{G}'_2 + \underset{\sim}{G}'^T_2 , \quad (20)$$

where

$$\underset{\sim}{G}'_2 = \begin{pmatrix} \partial u'_2 & 0 & \frac{1}{S}\frac{\partial u'}{\partial\Theta} - \frac{w_2}{S} \\ 0 & \frac{u'_2}{S} + \frac{w'_2}{S}\cot\Theta & 0 \\ \frac{\partial w'_2}{\partial S} & 0 & \frac{u'_2}{S} + \frac{1}{S}\frac{\partial w'_2}{\partial\Theta} \end{pmatrix} . \quad (21)$$

To the second-order in terms of the non-dimensional parameter ε

$$u = \varepsilon u_1 + \varepsilon^2 u_2; \qquad w = \varepsilon w_1 + \varepsilon^2 w_2; \qquad \underset{\sim}{T} = \varepsilon \underset{\sim}{T}_1 + \varepsilon^2 \underset{\sim}{T}_2. \qquad (22)$$

Spherical inclusion subjected to a resultant force

Consider an incompressible infinite elastic medium which is bounded internally by a rigid spherical inclusion of radius a. The inclusion is in welded contact with the elastic medium. The inclusion is subjected to a central force (F) which causes a rigid body translation ξ. The cylindrical polar coordinate system of the reference configuration is chosen such that the Z-axis coincides with the line of action of F (Fig.1).

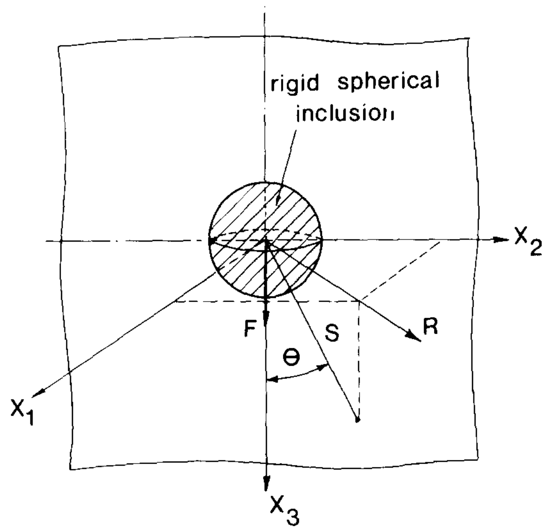

FIG.1
Geometry of the spherical inclusion and the coordinate system

It may be verified that the first-order displacement function

$$\varepsilon\Psi_1 = \frac{\xi}{4a}\left\{\frac{a^4}{S} - 3a^2 S\right\}\sin^2\Theta , \qquad (23)$$

satisfies displacement boundary conditions

$$u_1 = \xi\cos\Theta, \qquad w_1 = -\xi\sin\Theta, \qquad \text{on} \quad S = a, \qquad (24)$$

and gives stress components which tend to zero as $S \to \infty$. The small real dimensionless parameter is chosen to be equal to $(\xi/4a)$. The first-order displacement and stress components can be written as

$$u_1^* = \left\{-\frac{2}{S^{*3}} + \frac{6}{S^*}\right\}\cos\Theta , \qquad w_1^* = \left\{-\frac{1}{S^{*3}} - \frac{3}{S^*}\right\}\sin\Theta , \qquad (25)$$

$$T_{ss}^{(1)*} = \left\{\frac{12}{S^{*4}} - \frac{18}{S^{*2}}\right\}\cos\theta\,, \qquad T_{\phi\phi}^{(1)*} = T_{\theta\theta}^{(1)*} = -\frac{6}{S^{*4}}\cos\theta\,,$$
$$T_{s\theta}^{(1)*} = \frac{6}{S^{*4}}\sin\theta\,, \tag{26}$$

where ( )* denote the dimensionless variables which can be related to the physical variables according to the following:

$$u_1 = au_1^*\,, \qquad w_1 = aw_1^*\,, \qquad \underset{\sim}{T}_1 = \mu\underset{\sim}{T}_1^*\,. \tag{27}$$

If the expressions (25) and (26) for the first-order displacement and stress components are substituted in (9), the inhomogeneous differential equation for the second-order displacement function Ψ_2^* $(= \Psi_2/a^3)$ reduces to

$$E^{*4}\Psi_2^* = 216\left\{\frac{4C_2}{\mu} - 1\right\}\left\{\frac{2}{S^{*7}} - \frac{1}{S^{*5}}\right\}\sin^2\theta\cos\theta\,, \tag{28}$$

where

$$E^{*2} = \frac{\partial^2}{\partial S^{*2}} + \frac{1}{S^{*2}}\frac{\partial^2}{\partial\theta^2} - \frac{\cot\theta}{S^{*2}}\frac{\partial}{\partial\theta}\,. \tag{29}$$

A particular integral of (28) is

$$\Psi_{2P}^* = \chi\left\{\frac{3}{S^{*3}} + \frac{9}{S^*}\right\}\sin^2\theta\cos\theta\,, \tag{30}$$

where $\chi = (C_2 - C_1)/(C_2 + C_1)$, and C_1 is an elastic constant such that $2(C_1 + C_2) = \mu$. It can be verified that the second-order displacement components derived from (30) and (16) do not satisfy displacement boundary conditions $u_2^*(1,\theta) = w_2^*(1,\theta) = 0$, on the inclusion-elastic medium interface. These boundary conditions can be explicitly satisfied by employing solutions of the homogeneous equation $E^{*4}\Psi_2^* = 0$. The appropriate homogeneous solution is

$$\Psi_{2H}^* = \left\{\alpha_1 + \frac{\alpha_2}{S^{*2}}\right\}\sin^2\theta\cos\theta\,, \tag{31}$$

where α_1, α_2 are constants. The complete second-order displacement function is therefore

$$\Psi_2^* = \left[\frac{2C_2}{\mu}\left\{\frac{6}{S^{*3}} - \frac{18}{S^{*2}} + \frac{18}{S^*} - 6\right\} + \left\{-\frac{3}{S^{*3}} + \frac{15}{S^{*2}} - \frac{9}{S^*} - 3\right\}\right]\sin^2\theta\cos\theta\,. \tag{32}$$

The complete second-order displacement and stress components determined from (16), (19) and (32) are

$$u_2^* = \left[\frac{36C_2}{\mu}\left\{-\frac{1}{S^{*5}} + \frac{3}{S^{*4}} - \frac{3}{S^{*3}} + \frac{1}{S^{*2}}\right\} + \right.$$
$$+ \left.\left\{-\frac{9}{2}\frac{1}{S^{*7}} + \frac{36}{S^{*5}} - \frac{45}{S^{*4}} + \frac{9}{2}\frac{1}{S^{*3}} + \frac{9}{S^{*2}}\right\}\right] \cos^2\theta +$$
$$+ \frac{12C_2}{\mu}\left\{\frac{1}{S^{*5}} - \frac{3}{S^{*4}} + \frac{3}{S^{*3}} - \frac{1}{S^{*2}}\right\} +$$
$$+ \left\{-\frac{3}{2}\frac{1}{S^{*7}} - \frac{6}{S^{*5}} + \frac{15}{S^{*4}} - \frac{9}{2}\frac{1}{S^{*3}} - \frac{3}{S^{*2}}\right\}, \tag{33}$$
$$w^* = \left[\frac{36C_2}{\mu}\left\{-\frac{1}{S^{*5}} + \frac{2}{S^{*4}} - \frac{1}{S^{*3}}\right\} - \right.$$
$$\left. - \frac{3}{2}\frac{1}{S^{*7}} + \frac{18}{S^{*5}} - \frac{30}{S^{*4}} + \frac{27}{2}\frac{1}{S^{*3}}\right] \sin\theta\cos\theta ,$$

and

$$T_{ss}^{(2)*} = \left[\frac{72C_2}{\mu}\left\{-\frac{1}{S^{*8}} + \frac{7}{S^{*6}} - \frac{12}{S^{*5}} + \frac{9}{S^{*4}} - \frac{3}{S^{*3}}\right\} + \right.$$
$$\left. + \frac{84}{S^{*8}} - \frac{333}{S^{*6}} + \frac{360}{S^{*5}} - \frac{54}{S^{*3}}\right] \cos^2\theta +$$
$$+ \frac{12C_2}{\mu}\left\{-\frac{10}{S^{*6}} + \frac{24}{S^{*5}} - \frac{21}{S^{*4}} + \frac{6}{S^{*3}}\right\} +$$
$$+ \frac{9}{S^{*8}} + \frac{3}{S^{*6}} - \frac{120}{S^{*5}} + \frac{36}{S^{*4}} + \frac{18}{S^{*3}} ,$$
$$T_{\phi\phi}^{(2)*} = \left[\frac{72C_2}{\mu}\left\{\frac{1}{S^{*8}} - \frac{6}{S^{*6}} + \frac{5}{S^{*5}} - \frac{1}{S^{*4}}\right\} + \right.$$
$$+ \left.\left\{-\frac{21}{S^{*8}} + \frac{135}{S^{*6}} - \frac{150}{S^{*5}} + \frac{45}{S^{*4}}\right\}\right] \cos^2\theta +$$
$$+ \frac{12C_2}{\mu}\left\{\frac{6}{S^{*8}} + \frac{2}{S^{*6}} - \frac{6}{S^{*5}} + \frac{3}{S^{*4}}\right\} +$$
$$+ \left\{-\frac{12}{S^{*8}} - \frac{15}{S^{*6}} + \frac{30}{S^{*5}} - \frac{9}{S^{*4}}\right\} ,$$
$$T_{\theta\theta}^{(2)*} = \left[\frac{72C_2}{\mu}\left\{\frac{2}{S^{*8}} - \frac{7}{S^{*6}} + \frac{7}{S^{*5}} - \frac{2}{S^{*4}}\right\} + \right.$$
$$+ \left.\left\{-\frac{45}{S^{*8}} + \frac{117}{S^{*6}} - \frac{210}{S^{*5}} + \frac{63}{S^{*4}}\right\}\right] \cos^2\theta +$$
$$+ \frac{12C_2}{\mu}\left\{\frac{8}{S^{*6}} - \frac{14}{S^{*5}} + \frac{9}{S^{*4}}\right\} + \frac{12}{S^{*8}} + \frac{3}{S^{*6}} + \frac{90}{S^{*5}} - \frac{27}{S^{*4}} ,$$
$$T_{s\theta}^{(2)*} = \left[\frac{72C_2}{\mu}\left\{-\frac{1}{S^{*8}} + \frac{5}{S^{*6}} - \frac{6}{S^{*5}} + \frac{5}{S^{*4}} - \frac{1}{S^{*3}}\right\} + \right.$$
$$\left. + \frac{48}{S^{*8}} - \frac{126}{S^{*6}} + \frac{240}{S^{*5}} - \frac{144}{S^{*4}} - \frac{18}{S^{*3}}\right] \sin\theta\cos\theta , \tag{34}$$

respectively.

Force-displacement relationship for the spherical inclusion

Consider a closed spherical surface $\partial\Sigma$ in the deformed body. The component of the resultant force in the X_3 direction, over $\partial\Sigma$ is given by

$$F_z = \int_{\partial\Sigma} \left\{ T_{ss}\cos\theta - T_{s\theta}\sin\theta \right\} s_0 \sin\theta \, d\theta \,, \tag{35}$$

where s_0, θ are the coordinates of a particle on $\partial\Sigma$. Also on $\partial\Sigma$

$$s_0 = S + \varepsilon u_1(s_0, \Theta) \,, \qquad \theta = \Theta + \varepsilon \frac{w_1}{S}(s_0, \Theta) \,, \tag{36}$$

to order ε; and to this order S may be replaced by s_0. By considering the series expansions for T_{ss} and $T_{s\theta}$, (35) can be reduced to a form

$$F_z = \varepsilon F_z^{(1)} + \varepsilon^2 F_z^{(2)} \,, \tag{37}$$

where

$$F_z^{(1)} = \int_0^{\pi} \left\{ T_{ss}^{(1)} \cos\Theta - T_{s\theta}^{(1)} \sin\Theta \right\} s_0 \sin\Theta \, d\Theta \,, \tag{38}$$

and

$$\begin{aligned} F_z^{(2)} = \int_0^{\pi} \Bigg\{ & T_{ss}^{(2)} \cos\Theta - T_{s\theta}^{(2)} \sin\Theta - \frac{w_1}{S}\left[T_{ss}^{(1)} \sin\Theta + T_{s\theta}^{(1)} \cos\Theta \right] + \\ & - u_1 \left(\frac{\partial T_{ss}^{(1)}}{\partial S} \cos\Theta - \frac{\partial T_{s\theta}^{(1)}}{\partial S} \sin\Theta \right) + \\ & + \left[T_{ss}^{(1)} \cos\Theta - T_{s\theta}^{(1)} \sin\Theta \right] \left[\frac{1}{S} \frac{\partial w_1}{\partial \Theta} + \frac{w_1}{S} \cot\Theta \right] \Bigg\} s_0 \sin\Theta \, d\Theta \,. \end{aligned} \tag{39}$$

Evaluating the integrals (38) and (39) we obtain

$$\varepsilon F_z^{(1)} = -6\pi\mu a\xi \,, \qquad F_z^{(2)} = 0 \,. \tag{40}$$

The result $F_z^{(2)} = 0$ confirms the fact that from the particular spatial symmetry of the problem the force F must be an odd function of the inclusion displacement ξ. Therefore the result $F = -6\pi\mu a\xi$ is valid to order ε^3.

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