International Journal of Fracture 29 (1985) 191–204.

Body force loading of an annular adhesive contact region

A.P.S. SELVADURAI and B.M. SINGH

Department of Civil Engineering, Carleton University, Ottawa, Ontario, Canada K1S 5B6

(Received 27 February 1985)

Abstract

This paper examines the problem related to the axisymmetric loading of an annular adhesive contact region between two identical isotropic elastic halfspaces. The paper focusses on the evaluation of the stress intensity factors at the boundaries of the annular adhesive region. These stress intensity factors are evaluated in the form of power series in terms of a non-dimensional parameter which represents the ratio of the radii of the annular region.

1. Introduction

The stress analysis of defects located at bonded material interfaces is of interest to the study of adhesive mechanics and fracture mechanics. In situations where the bonded regions possess identical elastic properties, the stress analysis of the defect can be achieved by appeal to crack problems related to a homogeneous elastic solid. Extensive accounts of the stress analysis of penny-shaped, elliptical and annular cracks located in isotropic elastic solids are given by Sneddon and Lowengrub [1], Kassir and Sih [2], Cherepanov [3] and Mura [4]. Similarly, the stress analyses of cracks and other defects located at bi-material elastic interfaces are considered by Mossakovskii and Rybka [5], Willis [6], Arin and Erdogan [7], Kassir and Bregman [8], Lowengrub and Sneddon [9] and Keer et al. [10].

This paper examines the elastostatic problem of the body force loading of an annular adhesive contact region bonding two identical halfspace regions (Fig. 1). Owing to the symmetry of the problem about the plane containing the adhesive region, attention can be restricted to the study of a single halfspace region in which the adhesive contact region imposes displacement and symmetry constraints. The body force loading induces a state of axial symmetry in the bonded solids. The mixed boundary value problem related to the body force loading can be formulated by adopting a Hankel transform development of the governing equations. The mixed boundary conditions yield a system of triple integral equations which are solved in an approximate fashion by adopting a power series expansion technique. The small non-dimensional parameter involved in the series expansions corresponds to the ratio of the inner to the outer radii of the annular region. The results of particular interest to fracture mechanics correspond to the stress intensity factors that are developed at the inner and outer boundaries of the adhesive region. These stress intensity factors are evaluated in power series form. The numerical results presented in the paper illustrate the manner in which the location of the body forces and the geometry of the adhesive region influence the stress intensity factors at the boundaries of the annular region. The problem examined in the paper is of some relevance to the study of fracture toughness testing of adhesively bonded elastic solids.

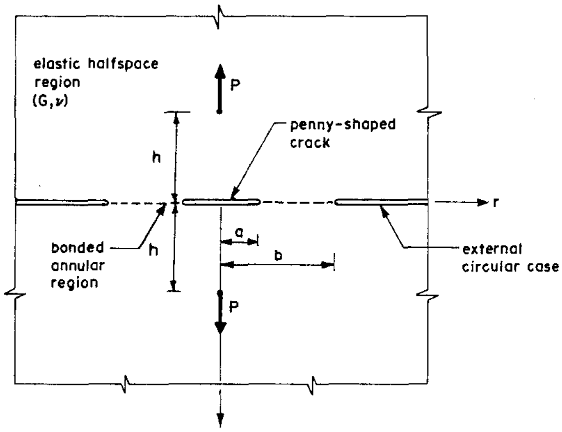

Figure 1. Body force loading of an annular adhesive contact region.

2. Fundamental equations

For the axially symmetric problem considered in this paper, the displacement components u_r, u_θ and u_z in the r, θ and z directions respectively, satisfy the constraint $u_r = u_r(r, z)$, $u_\theta = 0$ and $u_z = u_z(r, z)$. In the absence of body forces the displacement equations of equilibrium governing axially symmetric deformations take the form

$$\nabla^2 u_r + \frac{1}{(1-2\nu)}\frac{\partial e}{\partial r} - \frac{u_r}{r^2} = 0 \tag{1}$$

$$\nabla^2 u_z + \frac{1}{(1-2\nu)}\frac{\partial e}{\partial z} = 0 \tag{2}$$

where ∇^2 is the Laplacian for axially symmetric problems given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \tag{3}$$

In (1) and (2) ν is Poisson's ratio and e is the dilatation which is given by

$$e = \frac{\partial u_r}{\partial r} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z}. \tag{4}$$

Considering the mixed boundary value problem posed by the adhesive contact problem, it is evident that the state of stress exhibits symmetry about the plane $z = 0$. In this instance the solution of the displacement equations of equilibrium can be represented in terms of a single potential function $F(r, z)$ [2]. The resulting expressions for the displacement and stress components take the forms

$$u_r(r, z) = (1-2\nu)\frac{\partial F}{\partial r} + z\frac{\partial^2 F}{\partial r \partial z} \tag{5}$$

$$u_z(r, z) = -2(1-\nu)\frac{\partial F}{\partial z} + z\frac{\partial^2 F}{\partial z^2} \tag{6}$$

and

$$\sigma_{rr}(r, z) = 2G\left[(1-2\nu)\frac{\partial^2 F}{\partial r^2} - 2\nu\frac{\partial^2 F}{\partial z^2} + z\frac{\partial^3 F}{\partial r^2 \partial z}\right] \tag{7}$$

$$\sigma_{\theta\theta}(r, z) = 2G\left[\frac{1}{r}\frac{\partial F}{\partial r} + 2\nu\frac{\partial^2 F}{\partial r^2} + \frac{z}{r}\frac{\partial^2 F}{\partial r \partial z}\right] \tag{8}$$

$$\sigma_{zz}(r, z) = 2G\left[-\frac{\partial^2 F}{\partial z^2} + z\frac{\partial^3 F}{\partial z^3}\right] \tag{9}$$

$$\sigma_{rz}(r, z) = 2Gz\frac{\partial^3 F}{\partial r \partial z^2} \tag{10}$$

respectively, where G is the linear elastic shear modulus.

3. The adhesive contact problem

We consider the adhesive contact problem in which the interface between the two halfspace regions is bonded in the region $a \leqslant r \leqslant b$, and subjected to two axial loads of magnitude P which are directed away from the interface $z = 0$ (Fig. 1). We first consider the problem in which an intact elastic solid is subjected to a doublet of forces which are located at the points $(0, \pm h)$ and directed away from the plane $z = 0$ (Fig. 2). Owing to the state of symmetry, the shear stresses are zero on the plane $z = 0$. The tensile normal stress on the plane $z = 0$ due to the force doublet is given by [11]

$$\sigma_{zz}^0(r, 0) = \frac{Ph}{4\pi(1-\nu)}\left[\frac{(1-2\nu)r^2 + 2(2-\nu)h^2}{(r^2+h^2)^{5/2}}\right] = p(r). \tag{11}$$

The problem in which the adhesively bonded annular region is subjected to a doublet of forces can be obtained by considering a subsidiary problem in which the plane of symmetry is subjected to the mixed boundary conditions

$$\sigma_{rz}(r, 0) = 0; \qquad r \geqslant 0 \tag{12}$$

$$\sigma_{zz}(r, 0) = -p(r); \qquad 0 < r < a \tag{13}$$

$$u_z(r, 0) = 0; \qquad a \leqslant r \leqslant b \tag{14}$$

$$\sigma_{zz}(r, 0) = -p(r); \qquad b < r < \infty. \tag{15}$$

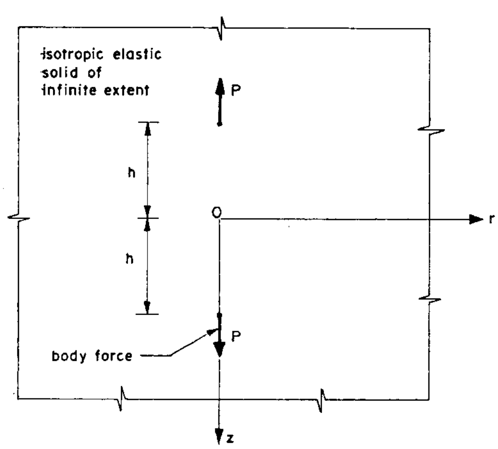

Figure 2. Action of a dipole of body forces in an isotropic elastic solid.

For the solution of the three part mixed boundary value problem posed by (12)–(15), we employ a Hankel transform based solution of $F(r, z)$ which satisfies the equation $\nabla^2 F = 0$, and gives bounded displacement and stress fields as $r, z \to \infty$, i.e.,

$$F(r, z) = \int_0^\infty \frac{A(\xi)}{\xi} \exp\{-\xi z\} J_0(\xi r) \mathrm{d}\xi \tag{16}$$

where $A(\xi)$ is an unknown function. By using (16) and (5)–(9), the mixed boundary conditions (13)–(15) yield the following system of triple integral equations for the unknown function $A^*(\xi)$ where $A^*(\xi) = 2G\xi A(\xi)$:

$$\int_0^\infty A^*(\xi) J_0(\xi r) \mathrm{d}\xi = p(r); \quad 0 < r < a \tag{17}$$

$$\int_0^\infty \xi^{-1} A^*(\xi) J_0(\xi r) \mathrm{d}\xi = 0; \quad a \leqslant r \leqslant b \tag{18}$$

$$\int_0^\infty A^*(\xi) J_0(\xi r) \mathrm{d}\xi = p(r); \quad b < r < \infty. \tag{19}$$

For the analysis of the system of triple integral equations, we adopt the general procedures described by Cooke [12]. Complete accounts of techniques that may be employed in the solution of systems of triple integral equations are given by Williams [13], Tranter [14], Sneddon [15] and Kanwal [16]. In the ensuing, we shall present a brief summary of the analytical procedure which focusses on the evaluation of an asymptotic series expansion solution in terms of a small parameter.

Following Cooke [12], we find that

$$\int_0^\infty \xi^{-1} A^*(\xi) J_0(\xi r) \mathrm{d}\xi = \begin{cases} f_1(r); & 0 < r < a \quad (20a) \\ f_2(r); & b < r < \infty \quad (20b) \end{cases}$$

where $f_1(r)$ and $f_2(r)$ are determined from the following integral equations:

$$f_1(r) = \frac{2}{\pi} \int_r^a \frac{G_1(u)\mathrm{d}u}{(u^2 - r^2)^{1/2}} + \frac{2}{\pi}(a^2 - r^2)^{1/2} \int_b^\infty \frac{t f_2(t)\mathrm{d}t}{(t^2 - a^2)^{1/2}(t^2 - r^2)}; \quad 0 < r < a \tag{21}$$

and

$$f_2(r) = \frac{2}{\pi} \int_b^r \frac{G_2(u)\mathrm{d}u}{(r^2 - u^2)^{1/2}} + \frac{2}{\pi}(r^2 - b^2)^{1/2} \int_0^a \frac{t f_1(t)\mathrm{d}t}{(r^2 - t^2)(b^2 - t^2)^{1/2}}; \quad b < r < \infty \tag{22}$$

where

$$G_1(u) = \int_0^u x(u^2 - x^2)^{-1/2} p(x)\mathrm{d}x \tag{23}$$

$$G_2(u) = \int_u^\infty x(x^2 - u^2)^{-1/2} p(x)\mathrm{d}x \tag{24}$$

and

$$p(x) = \frac{Ph}{4\pi(1-\nu)} \left[\frac{(1-2\nu)}{(x^2 + h^2)^{3/2}} + \frac{3h^2}{(x^2 + h^2)^{5/2}} \right]. \tag{25}$$

Performing the integrations, (23) and (24) give the following expressions for $G_1(u)$ and $G_2(u)$;

$$G_1(u) = \frac{Pu}{2\pi(1-\nu)} \left[\frac{(1-\nu)}{(u^2 + h^2)} + \frac{h^2}{(u^2 + h^2)^2} \right] \tag{26}$$

$$G_2(u) = \frac{Ph}{4\pi(1-\nu)}\left[\frac{(1-2\nu)}{(u^2+h^2)} + \frac{2h^2}{(u^2+h^2)^2}\right] \tag{27}$$

Introduce functions $F_1(u)$ and $F_2(u)$ such that

$$F_1(u) = \frac{\mathrm{d}}{\mathrm{d}u}\int_u^a \frac{rf_1(r)\mathrm{d}r}{(r^2-u^2)^{1/2}}; \quad u < a \tag{28}$$

$$F_2(u) = \frac{\mathrm{d}}{\mathrm{d}u}\int_b^u \frac{rf_2(r)\mathrm{d}r}{(u^2-r^2)^{1/2}}; \quad u > b. \tag{29}$$

The equations (28) and (29) are integral equations of the Abel type. Solutions of (28) and (29) take the form

$$f_1(r) = -\frac{2}{\pi}\int_r^a \frac{F_1(u)\mathrm{d}u}{(u^2-r^2)^{1/2}}; \quad r < a \tag{30}$$

$$f_2(r) = \frac{2}{\pi}\int_b^r \frac{F_2(u)\mathrm{d}u}{(r^2-u^2)^{1/2}}; \quad r > b. \tag{31}$$

Using the above representations, (22) can be rewritten in the form

$$\int_b^r \frac{[F_2(u)-G_2(u)]\mathrm{d}u}{(r^2-u^2)^{1/2}} = (r^2-b^2)^{1/2}\int_0^a \frac{tf_1(t)\mathrm{d}t}{(b^2-t^2)^{1/2}(r^2-t^2)}; \quad b < r < \infty \tag{32}$$

The equation (32) is also an integral equation of the Abel-type and its solution can be written in the following form:

$$F_2(u) - G_2(u) = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}u}\int_b^u \frac{rdr}{(u^2-r^2)^{1/2}}\int_0^a \frac{(r^2-b^2)^{1/2}tf_1(t)\mathrm{d}t}{(b^2-t^2)^{1/2}(r^2-t^2)}; \quad b < u < \infty \tag{33}$$

Now using the following integral

$$\int_b^u \frac{r(r^2-b^2)^{1/2}\mathrm{d}r}{(u^2-r^2)^{1/2}(r^2-t^2)} = \frac{\pi}{2}\left[1 - \frac{(b^2-t^2)^{1/2}}{(u^2-t^2)^{1/2}}\right]; \quad t < b \tag{34}$$

we can write the integral equation (33) in the following form

$$F_2(u) - G_2(u) = u\int_0^a \frac{tf_1(t)\mathrm{d}t}{(u^2-t^2)^{3/2}}; \quad 0 < u < a \tag{35}$$

With the aid of the result (30) we can write

$$\int_0^a \frac{tf_1(t)\mathrm{d}t}{(u^2-t^2)^{3/2}} = -\frac{2}{\pi}\int_0^a F_1(v)\mathrm{d}v\int_0^v \frac{t\mathrm{d}t}{(u^2-t^2)^{3/2}(v^2-t^2)^{1/2}}. \tag{36}$$

It can be shown that

$$\int_0^a \frac{tf_1(t)\mathrm{d}t}{(u^2-t^2)^{3/2}} = -\frac{2}{\pi u}\int_0^a \frac{vF_1(v)\mathrm{d}v}{(u^2-v^2)}. \tag{37}$$

Using (37) we can write (35) in the form

$$F_2(u) + \frac{2}{\pi}\int_0^a \frac{vF_1(v)\mathrm{d}v}{(u^2-v^2)} = G_2(u); \quad b < u < \infty. \tag{38}$$

In a similar manner using (21), (30) and (31) we observe that

$$F_1(u)+\frac{2u}{\pi}\int_b^{\infty}\frac{F_2(v)\mathrm{d}v}{(v^2-u^2)}=-G_1(u);\quad 0<u<a. \tag{39}$$

Introducing changes in variables according to

$$v=av_1;\quad u=bu_1;\quad c=\frac{a}{b} \tag{40}$$

we can write (38) in the following form

$$F_2(u_1b)+\frac{2c^2}{\pi}\int_0^1\frac{v_1F_1(av_1)\mathrm{d}v_1}{\left(u_1^2-v_1^2c^2\right)}=G_2(u_1b);\quad 1<u_1<\infty. \tag{41}$$

In a similar manner we can transform the integral equation (39) into the following form:

$$F_1(u_1a)+\frac{2c}{\pi}\int_1^{\infty}\frac{u_1F_2(bv_1)\mathrm{d}v_1}{\left(v_1^2-u_1^2c^2\right)}=-G_1(au_1);\quad 0<u_1<1. \tag{42}$$

In the ensuing, we develop a series solution of the pair of coupled integral equations (41) and (42) by assuming that $c<1$. We assume that $F_1(au_1)$ and $F_2(bu_1)$ admit series representations of the form

$$F_1(au_1)=\sum_{i=0}^{N}c^im_i(u_1) \tag{43}$$

$$F_2(bu_1)=\sum_{i=0}^{N}c^in_i(u_1). \tag{44}$$

By substituting (43) and (44) in (41) and (42) and expanding the terms $(u_1^2-v_1^2c^2)^{-1}$ and $(v_1^2-u_1^2c^2)^{-1}$ in power series in c, we obtain two equations of the form

$$\sum_{i=0}^{N}c^im_i(u_1)+\frac{2}{\pi}\int_1^{\infty}\left[u_1\left\{\frac{c}{v_1^2}+\frac{u_1^2}{v_1^4}c^3+\frac{u_1^4}{v_1^6}c^5+\dots\right\}\cdot\sum_{i=0}^{N}c^in_i(v_1)\right]\mathrm{d}v_1$$
$$=-G_1(au_1);\quad 0<u_1<1 \tag{45}$$

$$\sum_{i=0}^{N}c^in_i(u_1)+\frac{2}{\pi}\int_0^{1}\left[\frac{v_1}{u_1^2}\left\{c^2+\frac{v_1^2}{u_1^2}c^4+\frac{v_1^4}{u_1^4}c^6+\dots\right\}\cdot\sum_{i=0}^{N}c^im_i(v_1)\right]\mathrm{d}v_1$$
$$=G_2(bu_1);\quad 1<u_1<\infty. \tag{46}$$

By comparing like terms in c^i $(i=0,1,\dots,N)$ in (45) and (46), we obtain integral expressions for $m_i(u_1)$ and $n_i(u_1)$. These are given in Appendix 1. Explicit expressions for $m_i(u_1)$ and $n_i(u_1)$ $(i=0,1,\dots,6)$ are also given in Appendix 1. This formally completes the analysis of the system of triple integral equations defined by (17)–(19). The results for the stresses and displacements in the elastic halfspace regions can be determined by making use of the solutions for $f_1(r)$ and $f_2(r)$.

4. The stress intensity factors

In this section we shall evaluate the stress intensity factors at the boundaries of the annular adhesive contact region. By making use of (18) and (20) we have

$$\int_0^{\infty}\xi^{-1}A^*(\xi)J_0(\xi r)\mathrm{d}\xi=\begin{cases}f_1(r); & 0<r<a\\ 0; & a<r<b\\ f_2(r); & b<r<\infty.\end{cases} \tag{47}$$

From the Hankel inversion theorem we have

$$A^*(\xi)=\xi^2\left[\int_0^a rf_1(r)J_0(\xi r)\mathrm{d}r+\int_b^\infty rf_2(r)J_0(\xi r)\mathrm{d}r\right]. \tag{48}$$

From (17) and (19) we observe that

$$\sigma_{zz}(r,0)=-\int_0^\infty A^*(\xi)J_0(\xi r)\mathrm{d}\xi. \tag{49}$$

Using (30) and (31) we find that (48) can be expressed in the form

$$A^*(\xi)=\frac{2\xi^2}{\pi}\left[-\int_0^a rJ_0(\xi r)\mathrm{d}r\int_r^a\frac{F_1(u)\mathrm{d}u}{(u^2-r^2)^{1/2}}+\int_b^\infty rJ_0(\xi r)\mathrm{d}r\int_b^r\frac{F_2(u)\mathrm{d}u}{(r^2-u^2)^{1/2}}\right]. \tag{50}$$

Changing the order of integration in (50) we obtain

$$A^*(\xi)=\frac{2\xi^2}{\pi}\left[-\int_0^a F_1(u)\mathrm{d}u\int_0^u\frac{rJ_0(\xi r)\mathrm{d}r}{(u^2-r^2)^{1/2}}+\int_b^\infty F_2(u)\mathrm{d}u\int_u^\infty\frac{rJ_0(\xi r)\mathrm{d}r}{(r^2-u^2)^{1/2}}\right] \tag{51}$$

or

$$A^*(\xi)=\frac{2\xi}{\pi}\left[-\int_0^a F_1(u)\,\sin(\xi u)\mathrm{d}u+\int_b^\infty F_2(u)\,\cos(\xi u)\mathrm{d}u\right]. \tag{52}$$

Using (52) in (49) and noting that $F_1(0)=0$ and $F_2(\infty)=0$, it can be shown that

$$\sigma_{zz}(r,0)=\frac{2}{\pi}\left[-\frac{F_1(a)}{(r^2-a^2)^{1/2}}+\int_0^a\frac{F_1'(u)\mathrm{d}u}{(r^2-u^2)^{1/2}}\right.$$
$$\left.+\frac{F_2(b)}{(b^2-r^2)^{1/2}}-\int_b^\infty\frac{F_2'(u)\mathrm{d}u}{(b^2-r^2)^{1/2}}\right];$$
$$a<r<b \tag{53}$$

where the primes denote derivatives with respect to u. The stress intensity factors are defined by

$$K_a=\lim_{r\to a^+}[2(r-a)]^{1/2}\sigma_{zz}(r,0) \tag{54}$$

$$K_b=\lim_{r\to b^-}[2(b-r)]^{1/2}\sigma_{zz}(r,0). \tag{55}$$

Making use of (53), (54) and (55) we observe that

$$K_a=-\frac{2}{\pi}\frac{F_1(a)}{\sqrt{a}} \tag{56}$$

$$K_b=\frac{2}{\pi}\frac{F_2(b)}{\sqrt{b}}. \tag{57}$$

Explicit expressions for the stress intensity factors take the following non-dimensional forms:

$$\overline{K}_a=\frac{K_a}{P/\pi^2a^{3/2}};\quad \overline{K}_b=\frac{K_b}{P/\pi^2a^{3/2}} \tag{58}$$

where

$$\overline{K}_a=\frac{1}{(1-\nu)}\left\{\frac{(1-\nu)}{(1+\eta^2)}+\frac{\eta^2}{(1+\eta^2)^2}\right\}$$

$$+c\left[\frac{1}{\pi\eta(1-\nu)}\left\{(1-2\nu)\left[1-\frac{1}{c\eta}\tan^{-1}(c\eta)\right]\right\}\right.$$

$$\left.+2\left\{1-\frac{3}{2c\eta}\tan^{-1}(c\eta)+\frac{1}{2(1+c^2\eta^2)}\right\}\right]$$

$$+c^3\left[\frac{1}{\pi\eta(1-\nu)}\left\{(1-2\nu)\left[\frac{1}{3}-\frac{1}{c^2\eta^2}+\frac{1}{c^3\eta^3}\tan^{-1}(c\eta)\right]\right.\right.$$

$$\left.-\frac{c^2\eta^2}{(1+c^2\eta^2)}+5\left(\frac{1}{3}-\frac{1}{c^2\eta^2}\right)+\frac{5}{c^3\eta^3}\tan^{-1}(c\eta)\right\}$$

$$\left.+\frac{4}{3\pi^2(1-\nu)}\left\{(1-\nu)\left[1-\eta\tan^{-1}\left(\frac{1}{\eta}\right)\right]-\frac{\eta^2}{2(1+\eta^2)}+\frac{\eta}{2}\left[\frac{\pi}{2}-\tan^{-1}(\eta)\right]\right\}\right]$$

$$+c^4\left[\frac{4}{9\pi^3(1-\nu)\eta}\left\{(1-2\nu)\left[1-\frac{1}{c\eta}\tan^{-1}(c\eta)\right]\right.\right.$$

$$\left.\left.+2\left[1-\frac{3}{2c\eta}\tan^{-1}(c\eta)+\frac{1}{2(1+c^2\eta^2)}\right]\right\}\right]$$

$$+c^5\left[\frac{4}{5\pi^2(1-\nu)}\left\{(1-\nu)\left[1-\tan^{-1}\left(\frac{1}{\eta}\right)\right]-\frac{\eta^2}{2(1+\eta^2)}+\frac{\eta}{2}\left[\frac{\pi}{2}-\tan^{-1}(\eta)\right]\right.\right.$$

$$+(1-\nu)\left[\frac{1}{3}-\eta^2+\eta^3\tan^{-1}\left(\frac{1}{\eta}\right)\right]-\frac{\eta^2}{2(1+\eta^2)}$$

$$\left.\left.+\frac{3}{2}\eta^2\left[1-\eta\tan^{-1}\left(\frac{1}{\eta}\right)\right]\right\}\right]+0(c^6) \tag{59}$$

$$\overline{K}_b=\sqrt{c}\left[\frac{1}{2(1-\nu)}\left\{\frac{(1-2\nu)c^2\eta}{(1+c^2\eta^2)}+\frac{2c^4\eta^3}{(1+c^2\eta^2)^2}\right\}\right.$$

$$+c^2\left\{\frac{1}{\pi(1-\nu)}\left(2(1-\nu)\left[1-\tan^{-1}\left(\frac{1}{\eta}\right)\right]-\frac{\eta^2}{(1+\eta^2)}+\eta\left[\frac{\pi}{2}-\tan^{-1}(\eta)\right]\right)\right\}$$

$$+c^3\left\{\frac{2}{3\pi^2(1-\nu)\eta}\left((1-2\nu)\left[1-\frac{1}{c\eta}\tan^{-1}(c\eta)\right]\right.\right.$$

$$\left.\left.+2\left[1-\frac{3}{2c\eta}\tan^{-1}(c\eta)+\frac{1}{2(1+c^2\eta^2)}\right]\right)\right\}$$

$$+c^4\left\{\frac{2}{\pi(1-\nu)}\left((1-\nu)\left[\frac{1}{3}-\eta^2+\eta^3\tan^{-1}\left(\frac{1}{\eta}\right)\right]\right.\right.$$

$$\left.\left.-\frac{\eta^2}{2(1+\eta^2)}+\frac{3\eta^2}{2}\left[1-\eta\tan^{-1}\left(\frac{1}{\eta}\right)\right]\right)\right\}$$

$$+c^5\left\{\frac{2}{5\pi^2(1-\nu)\eta}\left((1-2\nu)\left[1-\frac{1}{c\eta}\tan^{-1}(c\eta)\right]\right.\right.$$

$$+2\left[1-\frac{3}{2c\eta}\tan^{-1}(c\eta)+\frac{1}{2(1+c^2\eta^2)}\right]$$

$$+(1-2\nu)\left[\frac{1}{3}-\frac{1}{c^2\eta^2}+\frac{1}{c^3\eta^3}\tan^{-1}(c\eta)\right]$$

$$\left.+5\left(\frac{1}{3}-\frac{1}{c^2\eta^2}\right)-\frac{c^2\eta^2}{(1+c^2\eta^2)}+\frac{5}{c^3\eta^3}\tan^{-1}(c\eta)\right)$$

$$+\frac{8}{9\pi^3(1-\nu)}\left((1-\nu)\left[1-\eta\tan^{-1}\left(\frac{1}{\eta}\right)\right]-\frac{\eta^2}{2(1+\eta^2)}\right.$$

$$\left.\left.+\frac{\eta}{2}\left[\frac{\pi}{2}-\tan^{-1}(\eta)\right]\right)\right\}+0(c^6)] \tag{60}$$

and

$$\eta=\frac{h}{a};\quad c=\frac{a}{b}.$$

5. Numerical results and conclusions

From the results given by (58)–(60) for the stress intensity factors K_a and K_b, it is possible to recover appropriate stress intensity factors for certain special cases. Firstly,

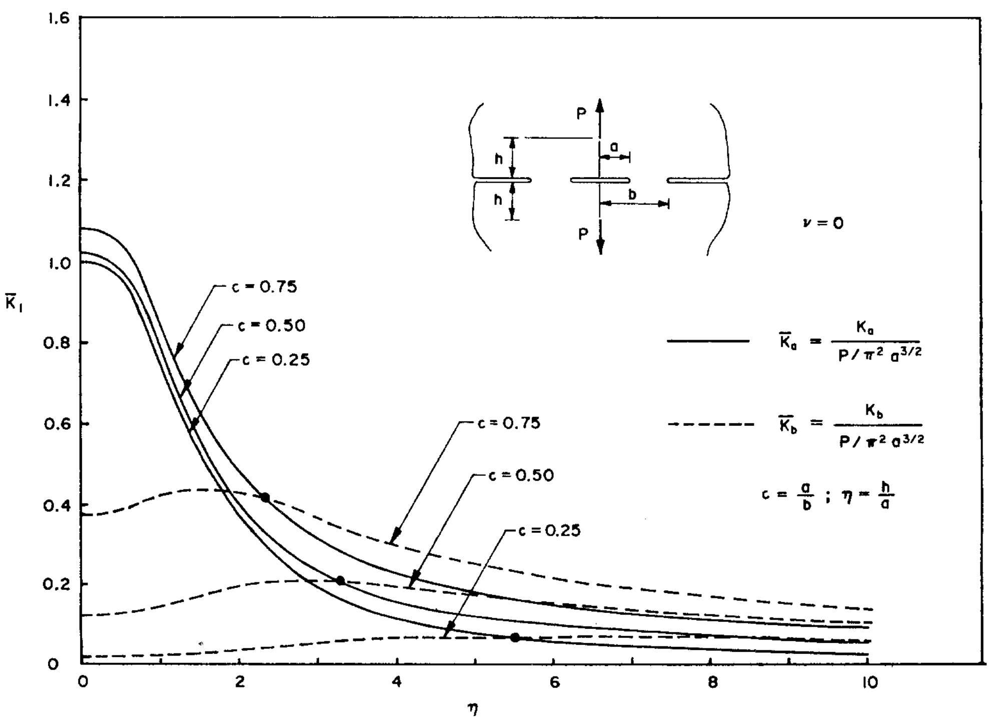

Figure 3. Stress intensity factors for the annular adhesive region loaded by a dipole of body forces.

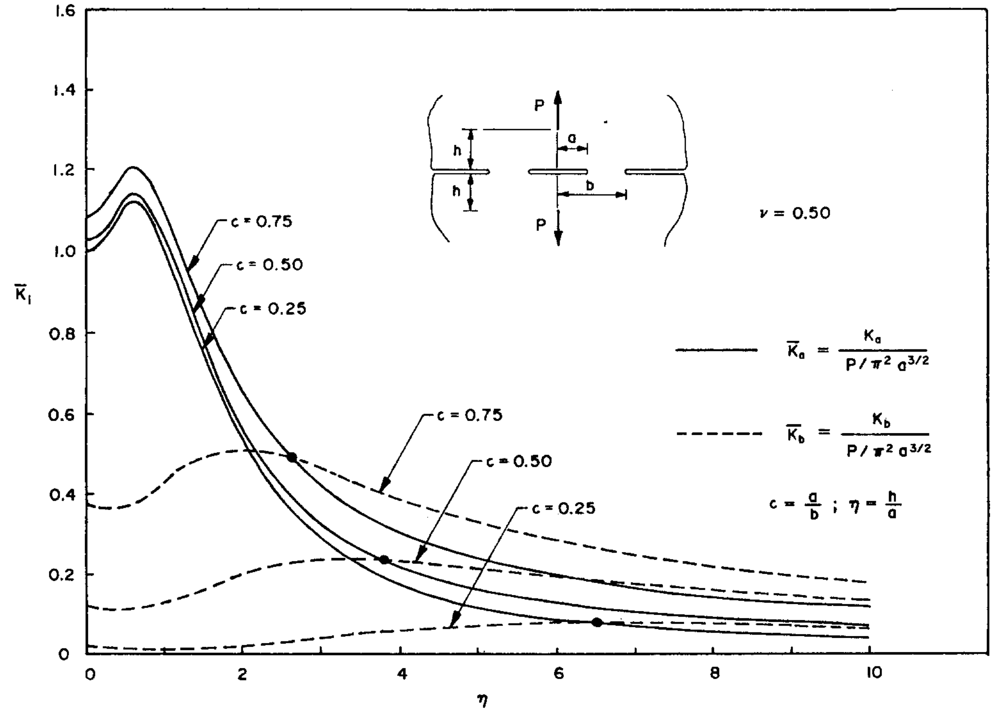

Figure 4. Stress intensity factors for the annular adhesive region loaded by a dipole of body forces.

when $b \to \infty$, the problem reduces to the case of the body force loading of a penny-shaped crack of radius a located in an infinite elastic solid. From (6) we note that $K_b \to 0$ as $c \to 0$ and (58) and (59) for the stress intensity factor K_a reduce to

$$K_a = \frac{P}{\pi^2 a^{3/2}(1-\nu)}\left[\frac{(1-\nu)}{(1+\eta^2)} + \frac{\eta^2}{(1+\eta^2)^2}\right] \tag{61}$$

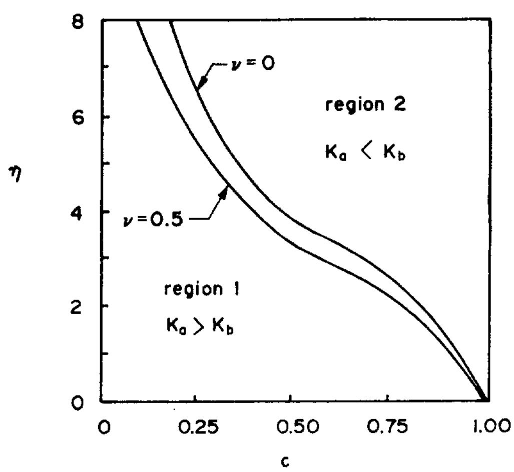

Figure 5. Influence of the geometry of the adhesive region and the location of the body forces on the stress intensity factors.

which is in agreement with the result obtained by Sneddon and Tweed [17]. In the limit as $a \to 0$, the results (58) and (60) for the stress intensity factor K_b give:

$$K_b = \frac{P}{2\pi b^{3/2}(1-\nu)}\left[\frac{(1-2\nu)\lambda}{(1+\lambda^2)} + \frac{2\lambda^3}{(1+\lambda^2)^2}\right] \tag{62}$$

where $\lambda = h/b$. The result (62) is in agreement with the expression given by Kassir and Sih [2] for the stress intensity factor at the boundary of an external circular crack which is loaded by a doublet of forces of magnitude P separated by a distance $2h$.

Also as $h \to 0$, the results (58)–(60) yield the stress intensity factors K_a and K_b for the annular adhesive region which is loaded by a dipole of forces located at the surfaces of the adhesively bonded region. Figures 3 and 4 illustrate the manner in which the normalized stress intensity factors $\bar{K}_a$ and $\bar{K}_b$ vary with the location ($\eta = h/a$) of the body forces and the geometric aspect ratio ($c = a/b$) of the bonded region. It is of interest to note that depending upon the values of η and c, the stress intensity factor $\bar{K}_a$ may be either greater than or less than $\bar{K}_b$. Particularly as η becomes large (for $c > 0$), the stress intensity factor at the outer boundary becomes larger than the stress intensity factor at the inner boundary. In these circumstances brittle fracture can be initiated at the outer boundary of the annular adhesive region. Figure 5 illustrates the manner in which η and c render either $K_a > K_b$ or $K_a < K_b$.

Appendix 1

The integral expressions for $m_i(u_1)$ and $n_i(u_1)$ $(i = 0, 1, \ldots, 6)$ take the following forms:

$$m_0(u_1) = -G_1(au_1),$$

$$m_1(u_1) = -\frac{2}{\pi}u_1\int_1^\infty \frac{n_0(v_1)\mathrm{d}v_1}{v_1^2},$$

$$m_2(u_1) = 0,$$

$$m_3(u_1) = -\frac{2}{\pi}u_1^3\int_1^\infty \frac{n_0(v_1)\mathrm{d}v_1}{v_1^4} - \frac{2}{\pi}u_1\int_1^\infty \frac{n_2(v_1)\mathrm{d}v_1}{v_1^2},$$

$$m_4(u_1) = -\frac{2}{\pi}u_1\int_1^\infty \frac{n_3(v_1)\mathrm{d}v_1}{v_1^2} - \frac{2}{\pi}u_1^3\int_1^\infty \frac{n_1(v_1)\mathrm{d}v_1}{v_1^4},$$

$$m_5(u_1) = -\frac{2}{\pi}u_1^5\int_1^\infty \frac{n_1(v_1)\mathrm{d}v_1}{v_1^6} - \frac{2}{\pi}u_1^3\int_1^\infty \frac{n_2(v_1)\mathrm{d}v_1}{v_1^4} - \frac{2}{\pi}u_1\int_1^\infty \frac{n_4(v_1)\mathrm{d}v_1}{v_1^2},$$

$$m_6(u_1) = -\frac{2}{\pi}u_1^5\int_1^\infty \frac{n_1(v_1)\mathrm{d}v_1}{v_1^6} - \frac{2}{\pi}u_1^3\int_1^\infty \frac{n_3(v_1)\mathrm{d}v_1}{v_1^4} - \frac{2}{\pi}u_1\int_1^\infty \frac{n_5(v_1)\mathrm{d}v_1}{v_1^2},$$

$$n_0(u_1) = G_2(bu_1),$$

$$n_1(u_1) = 0,$$

$$n_2(u_1) = -\frac{2}{\pi u_1^2}\int_0^1 v_1 m_0(v_1)\mathrm{d}v_1,$$

$$n_3(u_1) = -\frac{2}{\pi u_1^2}\int_0^1 v_1 m_1(v_1)\mathrm{d}v_1,$$

$$n_4(u_1) = -\frac{2}{\pi u_1^2}\int_0^1 v_1 m_2(v_1)\mathrm{d}v_1 - \frac{2}{\pi u_1^4}\int_0^1 v_1^3 m_0(v_1)\mathrm{d}v_1,$$

$$n_5(u_1) = -\frac{2}{\pi u_1^4}\int_0^1 v_1^3 m_1(v_1)\mathrm{d}v_1 - \frac{2}{\pi u_1^2}\int_0^1 v_1 m_3(v_1)\mathrm{d}v_1,$$

$$n_6(u_1) = -\frac{2}{\pi u_1^2}\int_0^1 v_1 m_4(v_1)\mathrm{d}v_1 - \frac{2}{\pi u_1^4}\int_0^1 v_1^3 m_2(v_1)\mathrm{d}v_1 - \frac{2}{\pi u_1^6}\int_0^1 v_1^5 m_0(v_1)\mathrm{d}v_1,$$

Explicit expressions for $m_i(u_1)$ and $n_i(u_1)$ take the following forms:

$$m_0(u_1) = -\frac{Pau_1}{2\pi(1-\nu)}\left[\frac{(1-\nu)}{(a^2u_1^2+h^2)} + \frac{h^2}{(a^2u_1^2+h^2)^2}\right],$$

$$m_1(u_1) = -\frac{Pu_1}{2\pi^2 h(1-\nu)}\left[(1-2\nu)\left\{1-\frac{b}{h}\tan^{-1}\left(\frac{h}{b}\right)\right\}\right.$$
$$\left. +2\left\{1-\frac{3b}{2h}\tan^{-1}\left(\frac{h}{b}\right)+\frac{b^2}{2(h^2+b^2)}\right\}\right],$$

$$m_2(u_1) = 0,$$

$$m_3(u_1) = -\frac{Pu_1^3}{2\pi^2(1-\nu)h}\left[(1-2\nu)\left\{\frac{1}{3}-\frac{b^2}{h^2}+\frac{b^3}{h^3}\tan^{-1}\left(\frac{h}{b}\right)\right\}\right.$$
$$\left. -\frac{h^2}{(h^2+b^2)}+5\left(\frac{1}{3}-\frac{b^2}{h^2}\right)+\frac{5b^3}{h^3}\tan^{-1}\left(\frac{h}{b}\right)\right]$$
$$-\frac{2Pu_1a}{3\pi^3(1-\nu)}\left[\frac{(1-\nu)}{a^2}\left\{1-\frac{h}{a}\tan^{-1}\left(\frac{a}{h}\right)\right\}\right.$$
$$\left. -\frac{h^2}{2a^2(a^2+h^2)}+\frac{h}{2a^3}\left\{\frac{\pi}{2}-\tan^{-1}\left(\frac{h}{a}\right)\right\}\right],$$

$$m_4(u_1) = -\frac{2Pu_1}{9\pi^4 h(1-\nu)}\left[(1-2\nu)\left\{1-\frac{b}{h}\tan^{-1}\left(\frac{h}{b}\right)\right\}\right.$$
$$\left. +2\left\{1-\frac{3b}{2h}\tan^{-1}\left(\frac{h}{b}\right)+\frac{b^2}{2(b^2+h^2)}\right\}\right],$$

$$m_5(u_1) = -\frac{2Pu_1^3a}{5\pi^3(1-\nu)}\left[\frac{(1-\nu)}{a^2}\left\{1-\frac{h}{a}\tan^{-1}\left(\frac{a}{h}\right)\right\}-\frac{h^2}{2a^2(a^2+h^2)}\right.$$
$$\left. +\frac{h}{2a^3}\left\{\frac{\pi}{2}-\tan^{-1}\left(\frac{h}{a}\right)\right\}\right]$$
$$-\frac{2Pau_1}{5\pi^3(1-\nu)}\left[(1-\nu)\left\{\frac{1}{3a^2}-\frac{h^2}{a^4}+\frac{h^3}{a^5}\tan^{-1}\left(\frac{a}{h}\right)\right\}\right.$$
$$\left. +h^2\left\{-\frac{1}{2a^2(a^2+h^2)}+\frac{3}{2a^4}\left[1-\frac{h}{a}\tan^{-1}\left(\frac{a}{h}\right)\right]\right\}\right],$$

$$n_0(u_1) = \frac{Ph}{4\pi(1-\nu)}\left[\frac{(1-2\nu)}{(b^2u_1^2+h^2)}+\frac{2h^2}{(h^2+b^2u_1^2)^2}\right],$$

$$n_1(u_1) = 0$$

$$n_2(u_1) = \frac{Pa}{\pi^2(1-\nu)u_1^2}\left[\frac{(1-\nu)}{a^2}\left\{1-\frac{h}{a}\tan^{-1}\left(\frac{a}{h}\right)\right\}\right.$$

$$- \frac{h^2}{2a^2(a^2+h^2)} + \frac{h}{2a^3}\left\{\frac{\pi}{2} - \tan^{-1}\left(\frac{h}{a}\right)\right\}\Bigg],$$

$$n_3(u_1) = \frac{P}{3\pi^3 u_1^2 h(1-\nu)}\left[(1-2\nu)\left\{1 - \frac{b}{h}\tan^{-1}\left(\frac{h}{a}\right)\right\}\right.$$

$$\left. + 2\left\{1 - \frac{3b}{2h}\tan^{-1}\left(\frac{h}{b}\right) + \frac{b^2}{2(b^2+h^2)}\right\}\right],$$

$$n_4(u_1) = \frac{Pa}{\pi^2(1-\nu)u_1^4}\left[(1-\nu)\left\{\frac{1}{3a^2} - \frac{h^2}{a^4} + \frac{h^3}{a^5}\tan^{-1}\left(\frac{a}{h}\right)\right\}\right.$$

$$\left. + h^2\left\{-\frac{1}{2a^2(a^2+h^2)} + \frac{3}{2a^4}\left[1 - \frac{h}{a}\tan^{-1}\left(\frac{a}{h}\right)\right]\right\}\right],$$

$$n_5(u_1) = \frac{P}{5\pi^3 h(1-\nu)u_1^4}\left[(1-2\nu)\left\{1 - \frac{b}{h}\tan^{-1}\left(\frac{h}{b}\right)\right\}\right.$$

$$\left. + 2\left\{1 - \frac{3b}{2h}\tan^{-1}\left(\frac{h}{b}\right) + \frac{b^2}{2(b^2+h^2)}\right\}\right]$$

$$+ \frac{P}{5(1-\nu)\pi^3 u_1^2 h}\left[(1-2\nu)\left\{\frac{1}{3} - \frac{b^2}{h^2} + \frac{b^3}{h^3}\tan^{-1}\left(\frac{h}{b}\right)\right\}\right.$$

$$\left. - \frac{h^2}{(h^2+b^2)} + 5\left(\frac{1}{3} - \frac{b^2}{h^2}\right) + \frac{5b^3}{h^3}\tan^{-1}\left(\frac{h}{b}\right)\right]$$

$$+ \frac{4Pa}{9u_1^2\pi^4(1-\nu)}\left[\frac{(1-\nu)}{a^2}\left\{1 - \frac{h}{a}\tan^{-1}\left(\frac{a}{h}\right)\right\} - \frac{h^2}{2a^2(a^2+h^2)}\right.$$

$$\left. + \frac{h}{2a^3}\left\{\frac{\pi}{2} - \tan^{-1}\left(\frac{h}{a}\right)\right\}\right].$$

References

[1] I.N. Sneddon and M. Lowengrub, *Crack Problems in the Classical Theory of Elasticity*, John Wiley, New York (1969).
[2] M.K. Kassir and G.C. Sih, *Three-Dimensional Crack Problems*; *Mechanics of Fracture*, Vol. 2 (G.C. Sih, Ed.) Noordhoff, Leyden (1975).
[3] G.P. Cherepanov, *Mechanics of Brittle Fracture* (Translation Edited by R. de Wit and W.C. Cooley) McGraw-Hill, New York (1979).
[4] T. Mura, *Micromechanics of Defects in Solids*, Sijthoff and Noordhoff, The Netherlands (1981).
[5] V.I. Mossakovskii and M.T. Rybka, Prikl. Math. Mekhanika 28 (1964) 1277–1286.
[6] J.R. Willis, *Quarterly Journal of Mechanics Applied Mathematics* 25 (1972) 367–385.
[7] K. Arin and F. Erdogan, *International Journal of Engineering Science* 9 (1971) 213–232.
[8] M.K. Kassir and A.M. Bregman, *Journal of Applied Mechanics* 39 (1972) 308–310.
[9] M. Lowengrub and I.N. Sneddon, *International Journal of Engineering Science* 12 (1974) 387–396.
[10] L.M. Keer, S.H. Chen and M. Comninou, *International Journal of Engineering Science* 16 (1978) 765–772.
[11] I.N. Sneddon, *Fourier Transforms*, McGraw-Hill, New York (1951).
[12] J.C. Cooke, *Proceedings Edinburgh Mathematical Society* 13 (1963) 317–323.
[13] W.E. Williams, *Proceedings Edinburgh Mathematical Society* Ser. 2, 13 (1963) 317–323.
[14] C.J. Tranter, *Proceedings Glasgow Mathematical Association* 4 (1960) 200–203.

[15] I.N. Sneddon, *Mixed Boundary Value Problems in Potential Theory*, North Holland, Amsterdam (1966).
[16] R.P. Kanwal, *Linear Integral Equations: Theory and Technique*, Academic Press, New York (1971).
[17] I.N. Sneddon and J. Tweed, *International Journal of Fracture* 3 (1967) 291–299.

Résumé

On examine le problème de la mise en charge axisymétrique de la région annulaire de contact avec adhésion entre deux demi espaces isotropes élastiques identiques. L'étude est centrée sur l'évaluation des facteurs d'intensité de contraintes aux frontières de la région annulaire adhésive. Ces facteurs d'intensité de contrainte sont exprimés sous forme de séries quadratiques en utilisant un paramètre sans dimension représentant le rapport des rayons de la zone annulaire.