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ON AN INTEGRAL EQUATION GOVERNING AN INTERNALLY INDENTED PENNY-SHAPED CRACK

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Introduction

In a recent study Selvadurai and Singh [1] examined the problem of the internal indentation of a penny-shaped crack, located in an isotropic elastic solid of infinite extent, by a smoothly embedded centrally placed rigid disc inclusion. This problem is of interest to the mathematical modelling of fracture of multiphase materials and resource bearing geological media. The analysis focusses on the evaluation of the stress intensity factor for the internally indented penny-shaped crack. In [1], the analysis of the integral equation governing the problem is achieved by employing a power series approximation technique. The non-dimensional parameter governing the expansion corresponds to the ratio of the radius of the penny-shaped inclusion (a) to that of the penny-shaped crack (b). In this paper the integral equation is further simplified and it is solved by employing a quadrature technique. Results for the stress intensity factor at the boundary of the penny-shaped crack derived from the quadrature scheme are compared with equivalent results derived from the power series expansion scheme.

Analysis

We consider the axisymmetric problem related to the internal indentation of the penny-shaped crack, of radius b, by a smoothly embedded penny-shaped inclusion of radius a and thickness 2h (Fig. 1). Since the problem exhibits a state of symmetry about the plane z=0 we can restrict our attention to a single halfspace region occupying z>0. The mixed boundary conditions associated with the problem are

$$u_z(r,0) = h \quad ; 0 \le r \le a \tag{1}$$

$$u_z(r,0) = 0 \quad ; b \le r < \infty \tag{2}$$

$$\sigma_{zz}(r,0) = 0 \qquad ; \; a < r < b \tag{3}$$

$$\sigma_{rz}(r,0) = 0 \qquad ; \; r > 0 \tag{4}$$

where u_z is the axial displacement and σ_{zz}, σ_{rz} etc., are the components of the Cauchy stress tensor referred to the cylindrical polar coordinate system (r,θ,z). For the analysis of the mixed boundary value problem in elasticity posed by (1) - (4), we employ the solution of the axisymmetric problem based on Love's strain potential approach. Using a Hankel transform development of the strain potential the mixed boundary conditions (1) - (4) can be reduced to a system of triple integral equations [1]. These in turn can be reduced to a Fredholm integral equation of the second kind for an unknown function $\psi(\eta)$: i.e.

$$\psi(\eta) = 1 + \int_0^1 \psi(\xi)\, K(\xi,\eta)\, d\xi \quad ; \quad 0 \le \eta \le 1 \tag{5}$$

It can be shown that the expression for the kernel function $K(\xi,\eta)$ given in [1] can be further reduced to the compact closed form

$$K(\xi,\eta) = -\frac{2\xi(1-c^2\xi^2)^{1/2}}{\pi^2\,(1-\xi^2)^{1/2}}\left[\frac{\Phi(\xi) - \Phi(\eta)}{\xi^2 - \eta^2}\right] \tag{6}$$

where

$$\Phi(\alpha) = \zeta \ln\left\{\frac{1+c\zeta}{1-c\zeta}\right\} \quad ; \quad \zeta = \left[\frac{1-\alpha^2}{1-c^2\alpha^2}\right]^{1/2} \quad ; \quad (\alpha=\xi,\eta) \tag{7}$$

and $c=a/b$. The expression for $\lim_{\xi\to\eta}[K(\xi,\eta)]$ can be found by invoking l'Hospital's rule; the resulting limit can be expressed as

$$\lim_{\xi\to\eta}[K(\xi,\eta)] = L(\eta) = \frac{(1-c^2)\,\eta\,\Omega(\eta)}{\pi^2\,(1-c^2\eta^2)\,(1-\eta^2)} \tag{8}$$

with

$$\Omega(\eta) = \frac{2\beta}{1-\beta^2} + \ln\left(\frac{1+\beta}{1-\beta}\right) \quad ; \quad \beta = c\left[\frac{1-\eta^2}{1-c^2\eta^2}\right]^{1/2} \tag{9}$$

THE STRESS INTENSITY FACTOR

The result of engineering interest concerns the flaw opening mode stress intensity factor at the boundary of the penny-shaped crack. Avoiding details of calculation it can be shown that this stress intensity factor can be obtained from the expression

$$K_1 = \frac{4\ hc\ G}{\pi^2\ (1-\nu)\ \sqrt{b}} \cdot \int_0^1 \frac{\xi\ \psi(\xi)\ d\xi}{(1-\xi^2)^{1/2}\ (1-c^2\xi^2)^{1/2}} \tag{10}$$

where G is the linear elastic shear modulus and ν is Poisson's ratio. The integral equation (5) can be solved, in a numerical fashion, by employing a Gaussian quadrature scheme. The resulting solutions for $\psi(\xi)$ at the Gaussian points can then be used in (10) to evaluate the stress intensity factor. This approach represents a departure from the technique adopted by Selvadurai and Singh [1] where in the basic integral equation is solved by adopting a power series expansion scheme. The expansion parameter corresponds to the radii ratio c. As a result of the restriction $c<1$, the solution developed by the power series expansion scheme is valid for only small values of $c \in (0,0.6)$. The series expansion for K_1 takes the form

$$K_1 = \frac{h\ G}{\pi(1-\nu)\ \sqrt{b}} \left[\frac{4c}{\pi} + \frac{16}{\pi^3} c^2 + c^3 \left\{\frac{64}{\pi^5} + \frac{4}{3\pi}\right\} + c^4 \left\{\frac{80}{9\pi^3} + \frac{256}{\pi^7}\right\}\right.$$

$$\left. + c^5 \left\{\frac{448}{9\pi^5} + \frac{1024}{\pi^9} + \frac{4}{5\pi}\right\} + 0\ (c^6)\right] \tag{11}$$

The results derived from (10), however, have no such restriction and the accuracy of the numerical evaluation can be improved by suitably increasing the number of Gauss points in the quadrature scheme. It is therefore, of interest to examine the relative accuracy of the estimates for the stress intensity factor obtained via the two schemes. The Table 1 shows such a comparison. The results for the quadrature scheme is obtained by employing 32 Gauss points. These results indicate that the approximate series solution gives reliable results for the stress intensity factor for $c \in (0,0.6)$. As $c \to 1$, the quadrature scheme provides more reliable results. It must of course be appreciated that as $c \to 1$, the condition of complete contact may not be physically realized at a smoothly indenting interface.

References

[1] A.P.S. SELVADURAI and B.M. SINGH, Int. J. Fracture, Vol.25, pp.69-77, 1984; Vol.26 pp.

Table 1
Stress Intensity Factors for a Penny-Shaped Crack
Indented by a Smooth Rigid Circular Inclusion
($\bar{K}_1 = K_1 \pi (1-\nu) \sqrt{b}/hG$)

	$\bar{K}_1$	
c	Quadrature Scheme	Series Solution Eq.(11)
0.1	0.132	0.133
0.2	0.278	0.281
0.3	0.445	0.450
0.4	0.641	0.647
0.5	0.881	0.882
0.6	1.189	1.170
0.7	1.616	1.527
0.8	2.293	1.973
0.9	3.709	2.536

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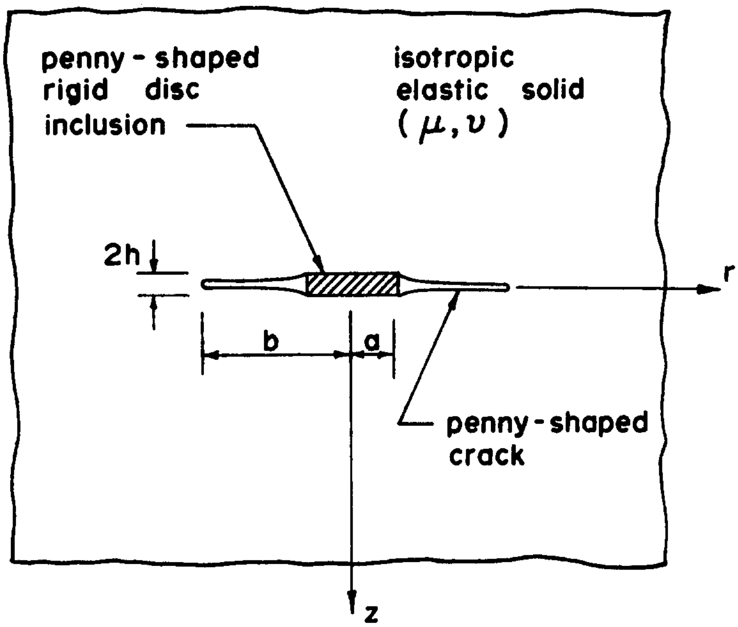

Figure 1. Internal indentation of a penny-shaped crack