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ROTATIONAL STIFFNESS OF A RIGID ELLIPTICAL DISC INCLUSION EMBEDDED AT A BI-MATERIAL ELASTIC INTERFACE

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1. INTRODUCTION

Problems dealing with inclusions embedded in elastic media is of some interest to the mathematical modelling of composite and multi-phase elastic solids. A very comprehensive account of current developments in this area of micromechanics of solids is given by Mura [1]. Specific applications of these theories to composite materials are also documented by Christensen [2], Willis [3] and Walpole [4].

This paper examines the asymmetric rotation of a rigid elliptical disc inclusion which is embedded at a bi-material elastic interface (Figure 1). A disc inclusion is a particular simplification of the general class of three-dimensional inhomogeneities considered in the classical studies by Eshelby [5], Lur'e [6] and others. Problems related to disc inclusions embedded in isotropic and transversely isotropic homogeneous elastic media have been studied by Collins [7], Keer [8], Kassir and Sih [9], Kanwal and Sharma [10] and others. References to further articles are also given by Selvadurai [11] and Selvadurai and Singh [12]. The majority of the articles dealing with disc inclusion problems concentrate on inclusions embedded in homogeneous elastic solids. The analysis of disc inclusion problems pertaining to bi-material elastic regions have received limited attention. In particular, the problem related to a disc shaped rigid inclusion located at a bi-material elastic interface can serve as a useful model for the study of precipitation hardening which occurs at the local scale of the elastic inclusion - elastic matrix interface. This paper focusses on the problem of the asymmetric rotation of a rigid elliptical disc inclusion embedded in bonded contact at a bi-material elastic interface. An exact formulation of this inclusion problem yields a set of three simultaneous integral equations. These integral equations cannot be solved analytically owing to the occurrence of complicated kernel functions. For this reason it is desirable to explore alternative analytical techniques which will yield results of engineering interest. In this paper we discuss the development of a set of bounds which can be used to estimate the asymmetric rotational stiffness of a rigid elliptical disc inclusion which is embedded in bonded contact at an isotropic bi-material elastic interface. These bounds are developed by imposing kinematic traction constraints at the bi-material interface. The upper bound solution imposes an inextensibility constraint at the interface and the lower bound imposes a frictionless bi-lateral contact at the interface.

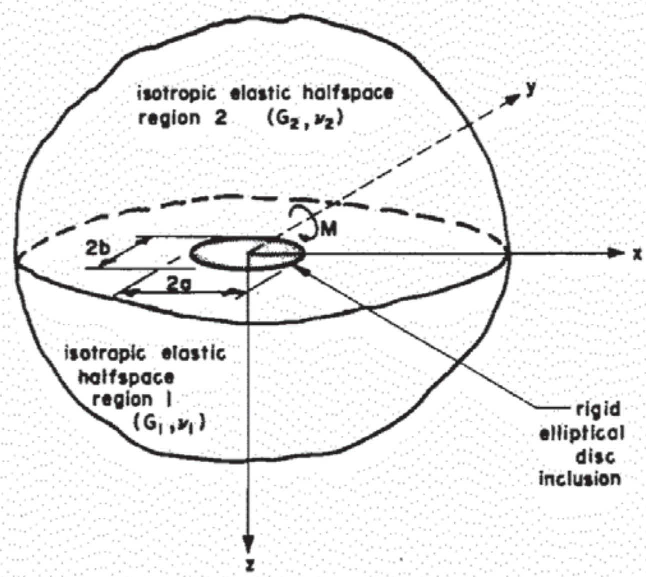

Figure 1 - Geometry of the rigid elliptical disc inclusion embedded at a bi-material elastic interface

2. BASIC EQUATIONS

For a three-dimensional problem in elasticity, the displacement components $u_i^{(\alpha)}$ (i = x,y,z; α = medium 1 or medium 2) in a medium free of body forces are governed by the Navier equations

$$G_\alpha \nabla^2 u_i^{(\alpha)} + (\lambda_\alpha + G_\alpha) u_{k,ki}^{(\alpha)} = 0 \ , \qquad (1)$$

where G_α and λ_α are the Lamé constants; $\lambda_\alpha = 2G_\alpha\nu_\alpha/(1-2\nu_\alpha)$; G_α are the shear moduli; ν_α are Poisson's ratios and ∇^2 is Laplace's operator referred to the rectangular Cartesian coordinate system. Here and in the sequel the Greek indices and superscripts will refer to quantities and variables pertaining to the two half-space regions.

The displacement equations of equilibrium (1) can be solved by using a variety of stress function techniques (see e.g., Truesdell [13] and Gurtin [14]). For example, in the generalized Papkovich-Neuber representation, the solution for $u_i^{(\alpha)}$ is

$$u_i^{(\alpha)} = (3-4\nu_\alpha)\phi_i^{(\alpha)} - x_j\phi_{j,i}^{(\alpha)} - \phi_{0,i}^{(\alpha)} , \tag{2}$$

where $x_1 = x$; $x_2 = y$; $x = z$ and $\phi_i^{(\alpha)}$ $(i = 1,2,3)$ and $\phi_0^{(i)}$ are harmonic functions, i.e.,

$$\nabla^2\phi_i^{(\alpha)} = 0 ; \qquad \nabla^2\phi_0^{(\alpha)} = 0 . \tag{3}$$

Once the displacement components are known, the stress components in the isotropic elastic media $(\alpha = 1,2)$ can be obtained from the stress-displacement relationships

$$\sigma_{ij}^{(\alpha)} = \lambda_\alpha\delta_{ij}\, u_{k,k}^{(\alpha)} + G_\alpha[u_{i,j}^{(\alpha)} + u_{j,i}^{(\alpha)}] , \tag{4}$$

where δ_{ij} is Kronecker's delta function

3. THE UPPER BOUND

Consider the problem of a rigid elliptical disc inclusion which is embedded in bonded contact at a bi-material elastic solid. For ease of reference we shall adopt the following nomenclature. Referring to the plane $z = 0$ (Figure 2(a)) the region occupied by the elliptical inclusion (i.e., $(x^2/a^2)+(y^2/b^2) \leq 1$; where a and b are the major and minor semi-axes of the ellipse) is denoted by S_I. The plane surface exterior to the inclusion is denoted by S_E; also $S = S_I \cup S_E$. To develop the upper bound estimate for the rotational stiffness of the elliptical inclusion we assume that the bi-material interface region S_E behaves as an inextensible surface. Since the rigid disc inclusion is embedded in bonded contact with the bi-material interface the inextensibility condition is also satisfied in the region S_I. The interface conditions associated with the upper bound problem can be stated as follows:

$$u_x^{(1)}(x,y,o) = u_x^{(2)}(x,y,o) = 0 \; ; \quad (x,y) \in S \; , \tag{5}$$

$$u_y^{(1)}(x,y,o) = u_y^{(2)}(x,y,o) = 0 \; ; \quad (x,y) \in S \; , \tag{6}$$

$$u_z^{(1)}(x,y,o) = u_z^{(2)}(x,y,o) \; ; \quad (x,y) \in S \; , \tag{7}$$

$$u_z^{(1)}(x,y,o) = u_z^{(2)}(x,y,o) = \Omega x \; ; \quad (x,y) \in S_I \; , \tag{8}$$

$$\sigma_{zz}^{(1)}(x,y,o) = \sigma_{zz}^{(2)}(x,y,o) \; ; \quad (x,y) \in S_E \; . \tag{9}$$

In order to satisfy the constraints (5) and (6) we select the following simplified forms of the Papkovich-Neuber potentials:

$$\phi_x^{(\alpha)} = \phi_y^{(\alpha)} = 0 \; ; \quad \phi_z^{(\alpha)} = \Phi^{(\alpha)} \; ; \quad \phi_0^{(\alpha)} = 0 \; . \tag{10}$$

The displacement components derived from (10) are

$$u_x^{(\alpha)} = -z \frac{\partial \Phi^{(\alpha)}}{\partial x} \; ; \quad u_y^{(\alpha)} = -z \frac{\partial \Phi^{(\alpha)}}{\partial y} \; , \tag{11}$$

$$u_z^{(\alpha)} = (3-4\nu_\alpha)\Phi^{(\alpha)} - z \frac{\partial \Phi^{(\alpha)}}{\partial z} \; , \tag{12}$$

and the relevant stress components $\sigma_{zz}^{(\alpha)}$ are given by

$$\sigma_{zz}^{(\alpha)} = 2G_\alpha \left[2(1-\nu_\alpha) \frac{\partial \Phi^{(\alpha)}}{\partial z} - z \frac{\partial^2 \Phi^{(\alpha)}}{\partial z^2} \right] . \tag{13}$$

Considering the boundary conditions (7) and the result (12) it is evident that

$$\Phi^{(1)} = \frac{(3-4\nu_2)}{(3-4\nu_1)} \Phi^{(2)} = \Phi^* \; . \tag{14}$$

The mixed boundary conditions (8) and (9) on S yield the following:

$$(3-4\nu_1)\Phi^* = \Omega x \; ; \quad (x,y) \in S_I \; , \tag{15a}$$

$$\frac{\partial \Phi^*}{\partial z} = 0 \; ; \quad (x,y) \in S_E \; . \tag{15b}$$

For the analysis of the mixed boundary value problem defined by (15) it is convenient to express the potential Φ^* in relation to a system of ellipsoidal coordinates (ξ,η,ζ) which are defined in such a way that ξ, η, and ζ

are the roots of the equation

$$\frac{x^2}{(a^2+s)} + \frac{y^2}{(b^2+s)} + \frac{z^2}{s} = 1 \ , \qquad (16)$$

where the coordinates have values subject to the restrictions

$$-a^2 < \zeta < -b^2 < \eta < 0 < \xi \ . \qquad (17)$$

In the ellipsoidal coordinate system S_I corresponds to $\xi = 0$ and S_E corresponds to $\eta = 0$. When $\xi = 0$, the remaining coordinates η, ζ are obtained from the roots of

$$\frac{x^2}{(a^2+s)} + \frac{y^2}{(b^2+s)} = 1 \ , \qquad (18)$$

and the products of these roots is found to be

$$\eta\zeta = a^2b^2\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right) . \qquad (19)$$

Using the techniques outlined by Segedin [15], Kassir and Sih [16] and Stallybrass [17] it can be shown that the appropriate solution for Φ^* is

$$\Phi^* = Cx \int_\xi^\infty \frac{ds}{(a^2+s)[s(a^2+s)(b^2+s)]^{1/2}} = \frac{2Cx[u-E(u)]}{a^3e_0^2} \ , \qquad (20)$$

where C is an arbitrary constant and $e_0^2 = (a^2-b^2)/a^2$. The variable u is related to the ellipsoidal coordinate ξ by

$$\xi^2 = a^2(\text{sn}^{-2}u-1) \ , \qquad (21)$$

$$E(u) = \int_0^u \text{dn}^2 t \ dt \ . \qquad (22)$$

The quantities sn u, dn u, etc., represent the Jacobian elliptic functions (see e.g., Greenhill [18]) which have real and imaginary roots 4K and 2iK' respectively corresponding to the moduli e_0 and $e_0' = b/a$. It may also be noted that

$$K(e_0) = \int_0^1 \frac{dt}{[(1-t^2)(1-e_0^2t^2)]^{1/2}} \ ; \quad E(e_0) = \int_0^1 \left[\frac{1-e_0^2t^2}{1-t^2}\right]^{1/2} dt \ . \qquad (23)$$

Considering the boundary condition (15a) and (20) it can be shown that

$$C = - \frac{\Omega a^3 e_0^2}{2[K(e_0)-E(e_0)](3-4\nu_1)} . \tag{24}$$

The normal stress $\sigma_{zz}^{(1)}$ acting on the plane z = 0 is given by

$$\sigma_{zz}^{(1)}(x,y,o) = - \frac{4x\Omega a^2 e_0^2 [\zeta\eta(a^2+\xi)(b^2+\xi)]^{1/2} G_1(1-\nu_1)}{b(a^2-\xi)[K(e_0)-E(e_0)](\xi-\eta)(\xi-\zeta)(3-4\nu_1)} . \tag{25}$$

The normal stress in the inclusion region is given by

$$[\sigma_{zz}^{(1)}]_{\xi=0} = - \frac{4G_1(1-\nu_1)e_0^2 x\Omega}{(3-4\nu_1)[K(e_0)-E(e_0)]b\left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]^{1/2}} . \tag{26}$$

Similarly, by using (14) and (20) it is possible to develop an equivalent expression for the normal stresses $\sigma_{zz}^{(2)}$ acting at the bonded interface. The upper bound estimate for the rotational stiffness of the embedded inclusion is obtained by evaluating the resultant of moments of $\sigma_{zz}^{(\alpha)}$ about the y-axis, i.e.,

$$M = \iint_{S_I} [\sigma_{zz}^{(1)}(x,y,o) - \sigma_{zz}^{(2)}(x,y,o)]x\,dx\,dy . \tag{27}$$

With the knowledge that the rotation Ω occurs in the direction of application of M it can be shown that

$$M = \frac{8\pi a^3 G_1 e_0^2 \Omega}{3[K(e_0)-E(e_0)]} \left\{ \frac{(1-\nu_1)(3-4\nu_2)+\Gamma(1-\nu_2)(3-4\nu_1)}{(3-4\nu_1)(3-4\nu_2)} \right\}, \tag{28}$$

where $\Gamma = G_2/G_1$.

4. THE LOWER BOUND

To develop a lower bound for the rotational stiffness of the embedded rigid elliptical disc inclusion we impose a frictionless bi-lateral contact at the bi-material interface. According to this assumption the interface is capable of transmitting only normal tractions. The smoothly embedded rigid elliptical inclusion is subjected to a couple M about the y-axis. Also, it is assumed that during the application of M, the two half-space regions remain in contact with each other over the entire interface region. To physically realize this condition the bi-material interface can be subjected to a sufficiently large uniform compression σ_0 (Figure 2(b)). As long as no separation takes place at the smoothly interacting bi-material interface the action of σ_0 does not influence the rotational stiffness of the embedded rigid elliptical inclusion. For the lower bound estimate the interface conditions are as follows:

$$\sigma_{xz}^{(1)}(x,y,o) = \sigma_{xz}^{(2)}(x,y,o) = 0 \; ; \qquad (x,y) \in S \; , \tag{29}$$

$$\sigma_{yz}^{(1)}(x,y,o) = \sigma_{yz}^{(2)}(x,y,o) = 0 \; ; \qquad (x,y) \in S \; , \tag{30}$$

$$u_{z}^{(1)}(x,y,o) = u_{z}^{(2)}(x,y,o) \; ; \qquad (x,y) \in S \; , \tag{31}$$

$$u_{z}^{(1)}(x,y,o) = u_{z}^{(2)}(x,y,o) = \Omega x \; ; \qquad (x,y) \in S_I \; , \tag{32}$$

$$\sigma_{zz}^{(1)}(x,y,o) = \sigma_{zz}^{(2)}(x,y,o) \; ; \qquad (x,y) \in S_E \; . \tag{33}$$

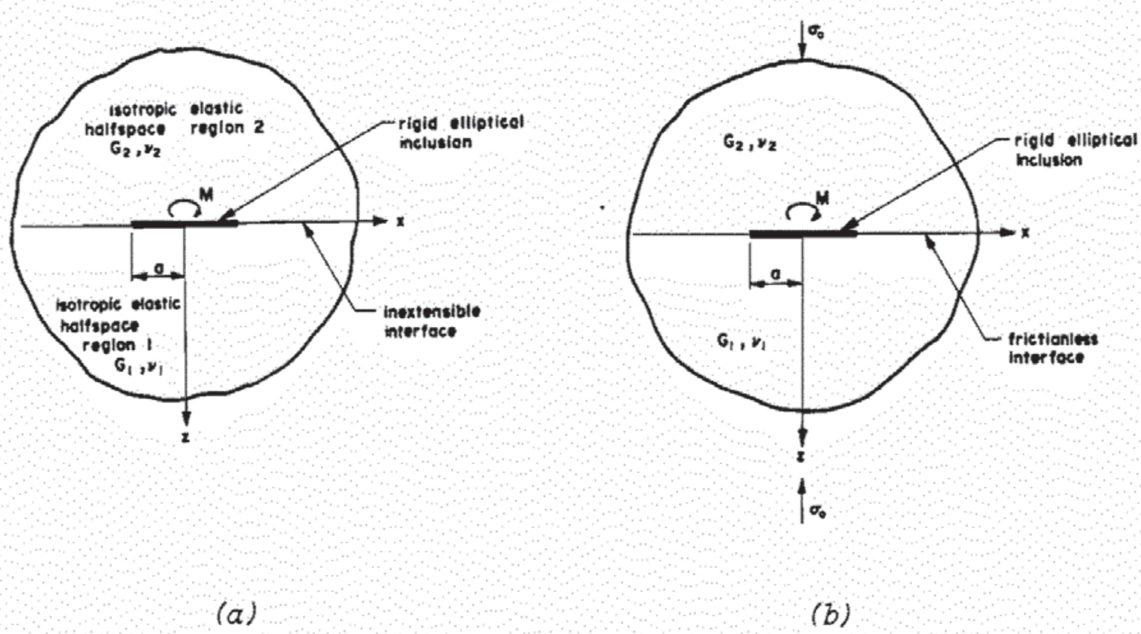

Figure 2 - Embedded Inclusion at Interfaces with Kinematic/Traction Constraints

(a) bonded inextensible interface
(b) frictionless interface

In order to satisfy the interface conditions (29) and (30) we select the following simplified forms of the Papkovich-Neuber potentials:

$$\phi_x^{(\alpha)} = \phi_y^{(\alpha)} = 0 \; ; \qquad \phi_z^{(\alpha)} = \Phi^{(\alpha)} \; . \tag{34}$$

The function $\phi_0^{(\alpha)}$ is chosen such that $\sigma_{xz}^{(\alpha)}$ and $\sigma_{yz}^{(\alpha)}$ are zero at the interface, i.e.,

$$\frac{\partial\phi_0^{(\alpha)}}{\partial z} = (1-2\nu_\alpha)\Phi^{(\alpha)} . \tag{35}$$

The relevant displacement and stress components derived from (34) and (35) are

$$u_z^{(\alpha)} = 2(1-\nu_\alpha)\Phi^{(\alpha)} - z\frac{\partial\Phi^{(\alpha)}}{\partial z} , \tag{36}$$

$$\sigma_{zz}^{(\alpha)} = \frac{\partial\Phi^{(\alpha)}}{\partial z} - z\frac{\partial^2\Phi^{(\alpha)}}{\partial z^2} , \tag{37}$$

$$\sigma_{xz}^{(\alpha)} = -z\frac{\partial^2\Phi^{(\alpha)}}{\partial x\partial z} ; \quad \sigma_{yz}^{(\alpha)} = -z\frac{\partial^2\Phi^{(\alpha)}}{\partial y\partial z} . \tag{38}$$

Considering the boundary condition (31) and the result (36) it is evident that

$$\Phi^{(1)} = \frac{(1-\nu_2)}{(1-\nu_1)}\Phi^{(2)} = \tilde{\Phi} . \tag{39}$$

The mixed boundary conditions (32) and (33) on S yield the following:

$$2(1-\nu_1)\tilde{\Phi} = \Omega x ; \qquad (x,y) \in S_I , \tag{40a}$$

$$\frac{\partial\tilde{\Phi}}{\partial z} = 0 ; \qquad (x,y) \in S_E . \tag{40b}$$

The system (40) can be solved by employing the techniques outlined previously in connection with the evaluation of the upper bound estimate. Avoiding details of calculations it can be shown that the lower bound estimate for the moment-rotation relationship is given by

$$M = \frac{2\pi a^3 G_1 e_0^2\Omega}{3[K(e_0)-E(e_0)]}\left\{\frac{(1-\nu_2)+\Gamma(1-\nu_1)}{(1-\nu_1)(1-\nu_2)}\right\} . \tag{41}$$

5. BOUNDS FOR THE ROTATIONAL STIFFNESS

Considering the results derived in the previous sections it is proposed that the rotational elastic stiffness for the rigid elliptical disc shaped inclusion embedded in bonded contact at a bimaterial elastic interface can be presented in the following set of bounds:

$$\frac{\{(1-\nu_1)(3-4\nu_2)+\Gamma(1-\nu_2)(3-4\nu_1)\}}{(3-4\nu_1)(3-4\nu_2)(1+\Gamma)} \geq \frac{3M\{K(e_0)-E(e_0)\}}{8\pi a^3\Omega(G_1+G_2)e_0^2} \geq \frac{\{(1-\nu_2)+\Gamma(1-\nu_1)\}}{4(1-\nu_1)(1-\nu_2)(1+\Gamma)} \, . \tag{42}$$

In the ensuing, the accuracy of these bounds will be examined by appeal to certain limiting cases of material behaviour. In the limit when $\nu_\alpha = 0$, the bounds reduce to the following result:

$$\frac{1}{3} \geq \frac{3M\{K(e_0)-E(e_0)\}}{8\pi a^3\Omega(G_1+G_2)e_0^2} \geq \frac{1}{4} \, . \tag{43}$$

Also when $\nu_\alpha = \frac{1}{2}$, the bounds (42) converge to the single result:

$$\frac{3M\{K(e_0)-E(e_0)\}}{8\pi a^3\Omega(G_1+G_2)e_0^2} = \frac{1}{2} \, . \tag{44}$$

This result indicates that in the limit of material incompressibility the bi-material elastic interface behaves essentially as an inextensible surface which is capable of transmitting only normal surface tractions. When $G_\alpha = G$ and $\nu_\alpha = \nu$ the asymmetry of the deformation imposes an inextensibility constraint in the plane $z = 0$.

Consequently, the upper bound estimate (28) corresponds to the exact solution for the rotational stiffness of a rigid elliptical disc inclusion embedded in bonded contact with an isotropic elastic solid, i.e.,

$$M = \frac{16\pi G a^3 e_0^2\Omega(1-\nu)}{3[K(e_0)-E(e_0)](3-4\nu)} \, . \tag{45}$$

It can be shown that in the limit when $e_0 \to 0$, (45) reduces to the result given by Selvadurai [19] for the rotational stiffness of a rigid circular disc inclusion embedded in an isotropic elastic solid.

When G_2 and $\nu_2 \to 0$, the problem reduces to that of the rotation of an elliptical rigid disc which is bonded to the surface of an isotropic elastic halfspace. The bounds (42) give

$$\frac{(1-\nu_1)}{(3-4\nu_1)} \geq \frac{3M\{K(e_0)-E(e_0)\}}{8\pi a^3\Omega G_1 e_0^2} \geq \frac{1}{4(1-\nu_1)} \, . \tag{46}$$

The result (46) represents the limiting estimates for the rotational stiffness of an elliptical punch which is bonded to a halfspace region. To the writer's knowledge there does not appear to be an exact analytical solution to this particular elliptical punch problem. The limiting bounds of (46) for the bonded circular punch can be compared with the exact analytical result available in the literature (see e.g., Gladwell [20]). In the limit when $e_0 \to 0$, (46) gives

$$\frac{4(1-\nu_1)}{(3-4\nu_1)} \geq \frac{3M}{8\pi a^3 G_1 \Omega} \geq \frac{1}{(1-\nu_1)}. \tag{47}$$

From the results given by Gladwell [20], the moment-rotation response for the bonded circular rigid punch is

$$\frac{3M}{8\pi a^3 G_1 \Omega} = \frac{1}{(1-\nu_1)} \left\{1 + \frac{\ln(3-4\nu_1)}{2\pi^2}\right\} = \tilde{M} \ . \tag{48}$$

Again as $\nu_1 \to \frac{1}{2}$, the bounds coincide and agree with the exact result. When $\nu_1 = 0$, (47) gives $1.33 \geq \tilde{M} \geq 1$; whereas the exact result (48) gives $\tilde{M} = 1.16$.

The analysis of the bounds for the rotational stiffness presented here examines only the case where the rigid elliptical inclusion rotates about the y-axis. The analytical techniques can be extended to cover the situation in which the inclusion is subjected to a couple M^* which induces a rotation Ω^* about the x-axis. In this case the appropriate bounds are

$$\frac{\{(1-\nu_1)(3-4\nu_2)+\Gamma(1-\nu_2)(3-4\nu_1)\}}{(3-4\nu_1)(3-4\nu_2)(1+\Gamma)} \geq \frac{3M^*\{E(e_0)-(1-e_0^2)K(e_0)\}}{8\pi a^3\Omega^*(G_1+G_2)e_0^2(1-e_0^2)} \geq \frac{\{(1-\nu_2)+\Gamma(1-\nu_1)\}}{4(1-\nu_1)(1-\nu_2)(1+\Gamma)} \tag{49}$$

Figures 3 and 4 illustrate the manner in which the upper and lower bound estimates for the rotational stiffness vary with the modular ratio Γ and Poisson's ratios ν_i. The normalized value of the moment M is defined by

$$\overline{M} = \frac{3M\{K(e_0)-E(e_0)\}}{4\pi a^3\Omega(G_1+G_2)e_0^2} \ . \tag{50}$$

From the numerical results it is evident that the difference between the upper and lower bound estimates diminish either when ν_i approach limits of incompressibility or when the modular ratio Γ becomes large.

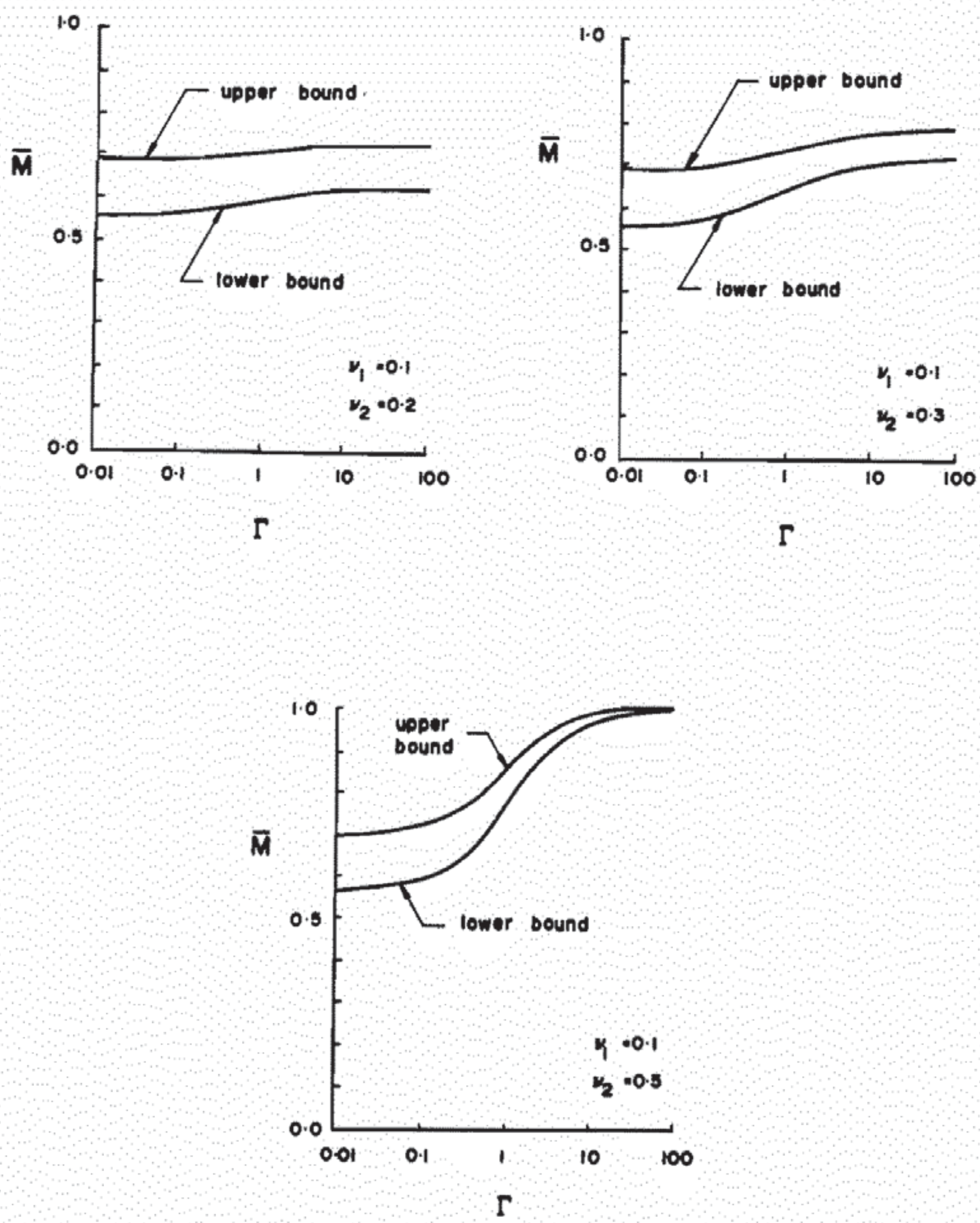

Figure 3 - Bounds for the Rotational Stiffness of a Rigid Elliptical Inclusion Embedded at a Bi-Material Interface

6. CONCLUSIONS

This paper develops a set of bounds which can be used to estimate the rotational stiffness of a rigid elliptical disc inclusion which is embedded in bonded contact at a bi-material elastic interface. These bounds are developed by imposing either an inextensibility constraint or a bi-lateral frictionless contact at the bi-material elastic interface.

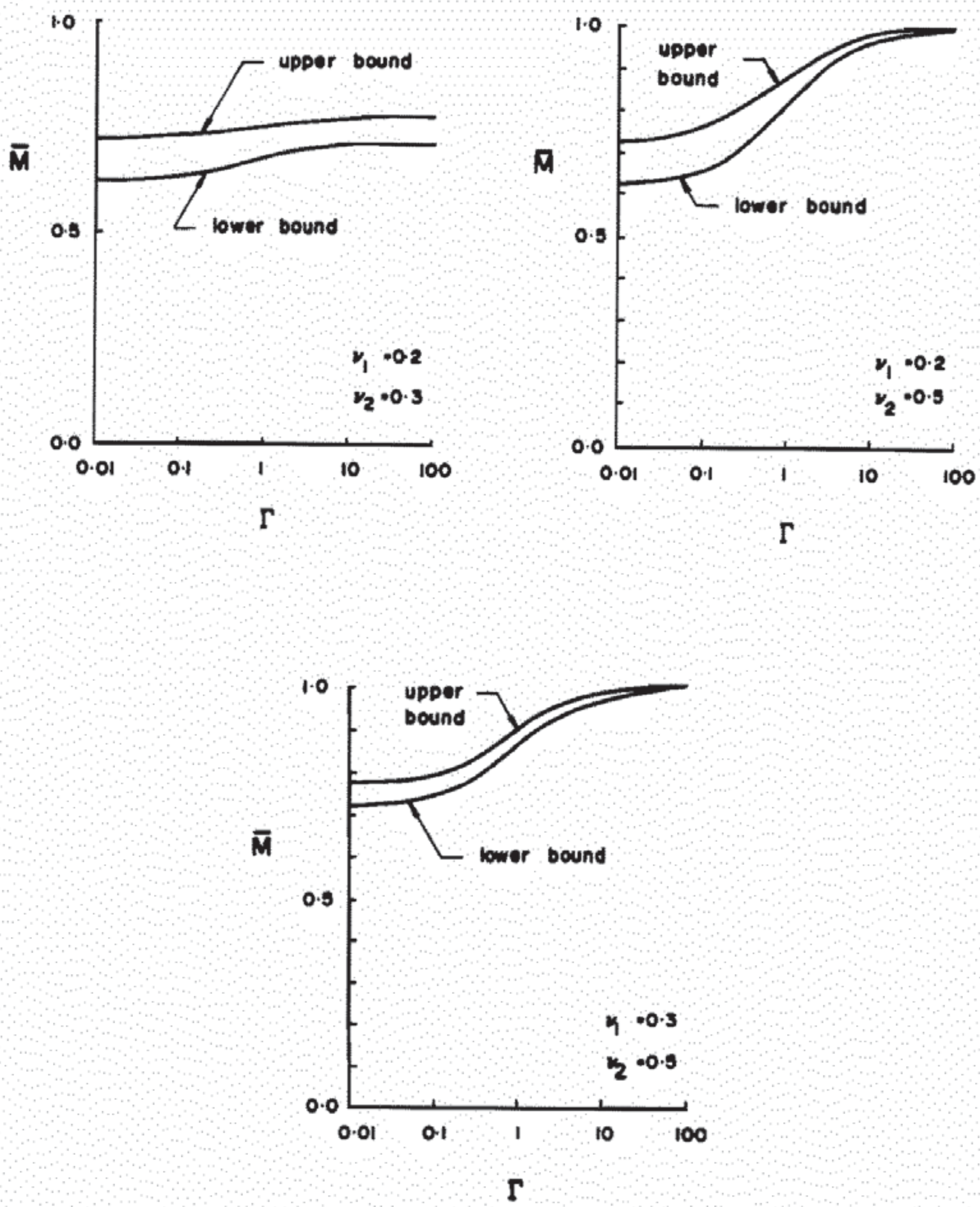

Figure 4 - Bounds for the Rotational Stiffness of a Rigid Elliptical Inclusion Embedded at a Bi-Material Interface

It is found that the bounds for the rotational stiffness can be evaluated in closed form. These bounds provide accurate engineering estimates for the elastostatic inclusion problem, the exact analysis of which requires the solution of a set of complicated integral equations. Owing to the imposed kinematic constraints, the analysis presented here cannot be used to determine additional displacements that can be experienced by the inclusion as a result of imposed asymmetric rotation. It may be noted that the couple M can induce a displacement δ_x in the x-direction. A change in the direction of application of M would indicate that δ_x is a first-order quantity which can be evaluated only from a complete analysis of the problem. On the other hand, any displacement δ_z induced by the couple M is a second-order contribution. Also by virtue of the symmetry of the problem the application of a couple M about the y-axis induces no lateral displacement of the inclusion in the y-direction. These additional displacements do not influence the bounds developed for the rotational stiffness of the embedded disc inclusion.

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