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Bounds for the Elastic Stiffness of a Rigid Penny-shaped Inclusion Embedded at a Bi-material Interface

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ABSTRACT

The present paper examines the problem of a rigid penny-shaped inclusion embedded at a bi-material elastic interface. Certain solutions are developed for the elastostatic stiffness of the embedded inclusion by considering bonded inextensibility or complete smoothness at the interface. These solutions are proposed as bounds for the elastostatic stiffness of the rigid penny-shaped inclusion which is embedded in full bonded contact at the bonded bi-material interface.

INTRODUCTION

The class of problems which deals with defects such as inclusions or flaws located at the interface of two dissimilar elastic solids has important applications in the stress analysis of composite materials. Considerable attention has been devoted to the study of cracks which are located at the interface between bonded dissimilar elastic materials. The investigations by Mossakovskii and Rybka,[1] Willis,[2] Kassir and Bregman,[3] Erdogan and Arin,[4] and Lowengrub and Sneddon[5] examine problems related to the stress analysis of penny-shaped cracks located at the interface of two bonded dissimilar elastic materials. Recent studies by Keer *et al.*[6] re-examine the interface penny-shaped crack problem in which an annular zone of frictionless

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contact is incorporated to eliminate the oscillatory form of a stress singularity observed at the crack boundary.

The analogous problem related to the behaviour of a rigid penny-shaped inclusion embedded in bonded contact at an elastic bi-material interface has received little attention. Much of the literature on the embedded disc inclusion problem focuses on the behaviour of rigid or flexible disc inclusions which are embedded in bonded contact with an isotropic or anisotropic elastic infinite space. Comprehensive accounts of the subject of three-dimensional inclusions embedded in homogeneous elastic media are given by Eshelby,[7] Mura[8] and Willis.[9] Also articles by Selvadurai[10] and Selvadurai and Sing[16] give detailed accounts of the disc inclusion problem related to a homogeneous elastic solid. The problem of a penny-shaped rigid inclusion embedded at a bi-material elastic interface can be examined by employing procedures outlined in the references cited earlier in connection with the analogous penny-shaped crack problem. Such a formulation essentially reduces the problem to the solution of three simultaneous singular integral equations.[17] The numerical treatment of the problem is non-routine and is currently under investigation.

The purpose of this paper is to outline an alternative approach to the problem of a rigid penny-shaped inclusion embedded at a bonded bi-material interface. In particular the methodology focuses on the development of a set of bounds for the elastic stiffness of the embedded inclusion. The upper bound is derived by imposing an extensibility constraint at the bi-material interface. The lower bound assumes the presence of a frictionless interface. The bounds are developed in exact closed form.

FUNDAMENTAL EQUATIONS

We consider the axisymmetric problem in which a rigid penny-shaped inclusion is embedded in bonded contact at a bi-material interface region consisting of isotropic elastic materials with shear moduli G_i and Poisson's ratio $\nu_i (i=1, 2)$, respectively (Fig. 1). For the analysis of the axisymmetric problem we employ Love's strain potential $(\Phi_i(r, z),\ i=1, 2)$ development of the displacement equations[11] of

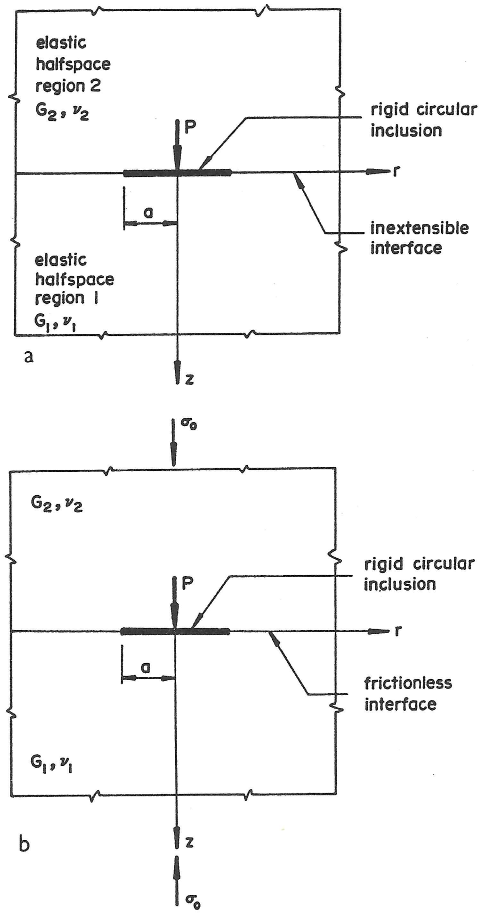

Fig. 1. Rigid circular inclusion embedded at a bi-material elastic interface. (a) Bonded inextensible interface, (b) frictionless interface.

equilibrium which satisfies

$$\nabla^2\nabla^2\Phi_i(r, z) = 0; \qquad i = 1, 2 \tag{1}$$

where ∇^2 is Laplace's operator referred to the cylindrical polar coordinate system. Also, the solutions of eqn. (1) appropriate for the halfspace regions '1' and '2' should satisfy regularity conditions pertaining to the stresses and displacements at infinity. A Hankel transform development of eqn. (1) yields the following solutions for

Φ_i:

$$\Phi_1(r, z) = \frac{1}{a^2}\int_0^\infty \xi[A_1(\xi) + zB_1(\xi)]\mathrm{e}^{-\xi z/a}J_0(\xi r/a)\,\mathrm{d}\xi \tag{2}$$

$$\Phi_2(r, z) = \frac{1}{a^2}\int_0^\infty \xi[A_2(\xi) + zB_2(\xi)]\mathrm{e}^{\xi z/a}J_0(\xi r/a)\,\mathrm{d}\xi \tag{3}$$

where $A_i(\xi)$ and $B_i(\xi)$ are arbitrary functions which are to be determined by satisfying the interface conditions applicable to the relevant bounds. The displacements and stress components in the elastic media can be expressed uniquely in terms of Φ_i. The relevant components of the stress tensor $\boldsymbol{\sigma}$ and the displacement vector $\mathbf{u}$ are given by

$$2G_i u_r^{(i)} = -\frac{\partial^2 \Phi_i}{\partial r \partial z} \tag{4}$$

$$2G_i u_z^{(i)} = 2(1-\nu_i)\nabla^2\Phi_i - \frac{\partial^2\Phi_i}{\partial z^2} \tag{5}$$

and

$$\sigma_{zz}^{(i)} = \frac{\partial}{\partial z}\left[(2-\nu_i)\nabla^2\Phi_i - \frac{\partial^2\Phi_i}{\partial z^2}\right] \tag{6}$$

$$\sigma_{rz}^{(i)} = \frac{\partial}{\partial r}\left[(1-\nu_i)\nabla^2\Phi_i - \frac{\partial^2\Phi_i}{\partial z^2}\right] \tag{7}$$

respectively.

AN UPPER BOUND

In order to develop an upper bound for the elastic stiffness of the rigid penny-shaped inclusion embedded in bonded contact with the bonded interface region we assume that the interface $z = 0$ exhibits inextensibility in the radial direction in the region $r \geq a$. Since the rigid inclusion is embedded in bonded contact in the region $r \leq a$, it is evident that the upper bound solution enforces an inextensibility constraint over the entire interface region $z = 0$; $r \geq 0$. The relevant interface conditions are

$$u_r^{(1)}(r,0)=u_r^{(2)}(r,0)=0; \qquad r\geq 0 \tag{8a}$$

$$u_r^{(1)}(r,0)=u_z^{(2)}(r,0)=\delta; \qquad r\leq a \tag{8b}$$

$$u_z^{(1)}(r,0)=u_z^{(2)}(r,0); \qquad r\geq 0 \tag{8c}$$

$$\sigma_{zz}^{(1)}(r,0)=\sigma_{zz}^{(2)}(r,0); \qquad r>a \tag{8d}$$

By making use of the interface conditions (8a) and (8c) it can be shown that

$$aB_1(\xi)=\xi A_1(\xi); \qquad aB_2(\xi)=-\xi A_2(\xi)=-\xi\frac{(3-4\nu_1)}{(3-4\nu_2)}\Gamma A_1(\xi) \tag{9}$$

where $\Gamma=G_2/G_1$. The mixed boundary conditions (8b) and (8d) yield the following set of dual integral equations for the unknown function $A_1(\xi)$:

$$H_0[\xi^2 A_1(\xi);r]=-\frac{2G_1\delta a^4}{(3-4\nu_1)}; \qquad r\leq a \tag{10a}$$

$$H_0[\xi^3 A_1(\xi);r]=0; \qquad r>a \tag{10b}$$

where H_0 is the Hankel operator of zero-order defined by

$$H_0[f(\xi);r]=\int_0^\infty \xi f(\xi)J_0(\xi r/a)\,\mathrm{d}\xi \tag{11}$$

The solution of the dual system is given by Sneddon[12] and the details of the method will not be pursued here. It is sufficient to note that the solution of eqn. (10) is given by

$$A_1(\xi)=-\frac{4G_1\delta a^4\sin\xi}{\pi(3-4\nu_1)\xi^4} \tag{12}$$

The upper bound elastic stiffness of the embedded inclusion can be evaluated by considering the normal tractions that act at the interfaces $z=0^{(1)}$ and $z=0^{(2)}$ in the region $r\leq a$. Evaluating the total load P we have

$$P=2\pi\int_0^a[\sigma_{zz}^{(1)}(r,0)-\sigma_{zz}^{(2)}(r,0)]r\,\mathrm{d}r \tag{13}$$

Assuming that the displacement of the embedded inclusion occurs in the direction of the applied force we have, from eqn. (13),

$$P=\frac{16\delta G_1(1-\nu_1)}{(3-4\nu_1)}\left[1+\Gamma\frac{(1-\nu_2)(3-4\nu_1)}{(1-\nu_1)(3-4\nu_2)}\right] \tag{14}$$

A LOWER BOUND

The lower bound estimate for the elastic stiffness of the rigid penny-shaped inclusion embedded at the bi-material interface is developed by assuming that the interface in the region $r>a$ is completely smooth. The rigid inclusion is also assumed to be embedded in smooth contact at the interface region. The smoothly embedded inclusion is subjected to a central concentrated force P which causes a rigid displacement δ in the z-direction. Furthermore, it is assumed that during the application of P the two halfspace regions remain in contact with each other. To physically realise this condition the interface can be subjected to a sufficiently large uniform compressive stress σ_0 (Fig. 1b). As long as no separation takes place at the bi-material interface the presence of σ_0 does not affect the elastostatic stiffness of the inclusion. The interface conditions associated with the problems are as follows

$$\sigma_{rz}^{(1)}(r,0)=\sigma_{rz}^{(2)}(r,0)=0; \qquad r\geq 0 \tag{15a}$$

$$u_z^{(1)}(r,0)=u_z^{(2)}(r,0)=\delta; \qquad r\leq a \tag{15b}$$

$$u_z^{(1)}(r,0)=u_z^{(2)}(r,0); \qquad r\geq 0 \tag{15c}$$

$$\sigma_{zz}^{(1)}(r,0)=\sigma_{zz}^{(2)}(r,0); \qquad r\geq a \tag{15d}$$

It may be noted that since $\sigma_{rz}^{(i)}=0$ for $r\geq 0$, there is no restriction on the radial displacements at the interface. Again by making use of the interface conditions (15a) and (15c) it can be shown that

$$2\nu_1 aB_1(\xi)=A_1(\xi); \qquad 2\nu_2 aB_2(\xi)=-\xi A_2(\xi)=-\frac{\xi\Gamma\nu_2(1-\nu_1)}{\nu_1(1-\nu_2)}A_1(\xi) \tag{16}$$

The mixed boundary conditions (15b) and (15d) yield the following dual system for the function $A_1(\xi)$:

$$H_0[\xi^2 A_1(\xi); r] = -\frac{2\delta G_1 \nu_1 a^4}{(1-\nu_1)}; \qquad r \leq a \tag{17a}$$

$$H_0[\xi^3 A_1(\xi); r] = 0; \qquad r > a \tag{17b}$$

The dual system (17) can be solved by employing the techniques outlined by Sneddon.[12] The result of primary importance to this paper, namely the elastic stiffness of the smoothly embedded inclusion, can be evaluated in the following form:

$$P = \frac{4G_1\delta a}{(1-\nu_1)}\left[1 + \frac{\Gamma(1-\nu_1)}{(1-\nu_2)}\right] \tag{18}$$

BOUNDS FOR THE ELASTIC STIFFNESS

Considering the developments presented in the preceding sections it is proposed that the elastic stiffness for the rigid penny-shaped inclusion embedded in bonded contact at a bonded bi-material interface can be presented in the form of the following set of bounds:

$$\frac{4\{(1-\nu_2)+\Gamma(1-\nu_1)\}}{(1-\nu_1)(1-\nu_2)(1+\Gamma)} \leq \frac{P}{(G_1+G_2)\delta a} \leq \frac{16\{(1-\nu_1)(3-4\nu_2)+\Gamma(1-\nu_2)(3-4\nu_1)\}}{(3-4\nu_1)(3-4\nu_2)(1+\Gamma)} \tag{19}$$

In the limit when $\nu_i = 0$ the bounds reduce to the following result

$$4 \leq \frac{P}{(G_1+G_2)\delta a} \leq \frac{16}{3} \tag{20}$$

Also when $\nu_1 = \frac{1}{2}$ the bounds (19) converge to the single result

$$P = 8(G_1+G_2)\delta a \tag{21}$$

The convergence of the bounds for $\nu_1 = \frac{1}{2}$ indicates that in the limit of material incompressibility the bi-material interface essentially behaves as an inextensible surface which transmits only normal stresses across the boundary. When $G_1 = G_2$ and $\nu_1 = \nu_2$ the asymmetry of the deformation imposes the inextensibility constraint at the interface

$z=0$; consequently the upper bound of eqn. (19) gives the exact solution (see, for example, Selvadurai[13]) for the stiffness of a rigid circular inclusion embedded in bonded contact with a homogeneous elastic solid.

Also, when G_2 and $\nu_2 \to 0$, eqn. (19) reduces to

$$\frac{4}{(1-\nu_1)} \leq \frac{P}{G_1 \delta a} \leq \frac{16(1-\nu_1)}{(3-4\nu)} \tag{22}$$

which represent the bounds for the stiffness of a rigid circular punch which is in adhesive contact with an isotropic elastic halfspace. The exact result for this problem is given by Mossakovskii[14] and Uflyand[15] as follows:

$$\frac{P}{G_1 \delta a} = \frac{4 \ln (3-4\nu_1)}{(1-2\nu_1)} \tag{23}$$

Again, as $\nu_i \to \frac{1}{2}$, the bounds of eqn. (22) converge to the exact result, eqn. (23). In the limit when $\nu_1 \to 0$, eqn. (22) yields $4 \leq P/G_1\delta a \leq \frac{16}{3}$, whereas eqn. (23) gives $P/G_1\delta a = 4 \ln 3 \simeq 4{\cdot}394$.

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