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On the Buckling of an Infinite Beam of Finite Width Embedded in an Isotropic Elastic Solid*

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ABSTRACT

The present paper considers the problem of buckling of a beam of finite width that is embedded in bonded contact with an isotropic elastic solid. Analysis of the buckling problem is restricted to the class of slender beams of narrow width that exhibit flexure only in the longitudinal direction. The governing integral equations are solved in an approximate fashion. Numerical results presented indicate the manner in which the buckling load is influenced by the relative flexibility of the beam-elastic medium system.

I. INTRODUCTION

The category of problems that deals with interaction between an embedded structural element such as a beam or a plate and an elastic medium has several useful engineering applications. These problems serve as models for the study of interaction between the matrix and fibers of fiber-reinforced materials

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[1, 2], kink band formation in fibrous composites, wood and geological media [3–5], and in the analysis of flexural interaction between pipelines and the surrounding soil mass [6, 7]. In particular, the work by Biot [8, 9] considers the plane strain problem of flexure of a thin beam or a layer embedded in bonded contact with two elastic half spaces. These analyses attempt to examine the development of folding in multilayered geological media.

In this paper the problem of buckling of an infinitely long beam of finite width, embedded in bonded contact with an isotropic elastic medium of infinite extent (Fig. 1), is examined. In particular, the category of beams of narrow width that experience flexure only in the longitudinal direction is considered. The infinite beam is essentially replaced by a flat (i.e., $h/b \ll 1$, where h is the thickness of the beam and b is its width) flexible ribbon-shaped inclusion. In the analysis presented here, the restriction related to no flexure in the transverse direction is satisfied in an approximate manner. The approximate techniques follow the analysis outlined by Biot [10], Rvachev [11], and Lekkerkerker [12], in connection with their treatment of the bending of an infinite beam of finite width that rests in smooth contact with an isotropic elastic half space. Numerical results presented in the paper examine the influence of approximate techniques and relative flexibility of the beam-elastic medium system on the buckling load for the embedded infinite beam.

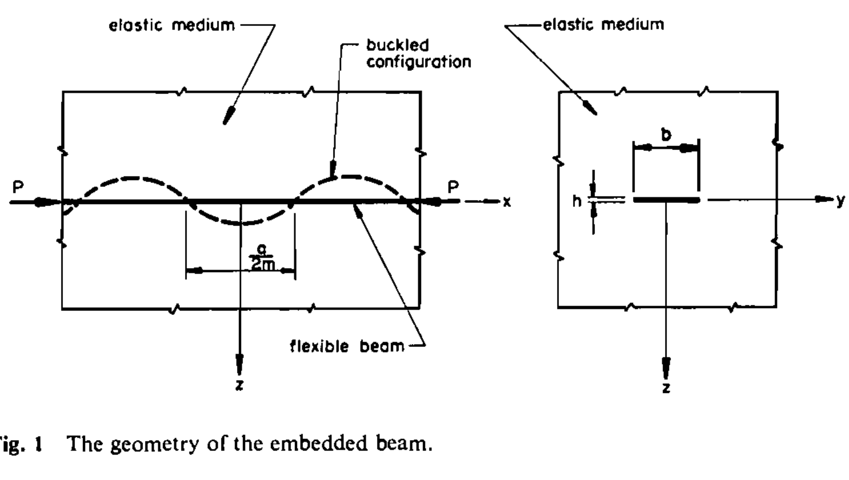

Fig. 1 The geometry of the embedded beam.

II. THE BUCKLING PROBLEM

The problem of an infinitely long flexible beam of finite width b and flexural rigidity $E_b I$, embedded in adhesive contact with an isotropic elastic medium of infinite extent, is examined. The cross-sectional geometry of the beam is assumed to be such that $h/b \ll 1$ and the beam itself can be idealized as a flat ribbon-shaped inclusion that occupies the plane $z = 0$ and exhibits flexure only in the longitudinal direction. Flexural behavior of the embedded beam is characterized by classical Bernoulli-Euler beam theory. According to this assumption, the beam exhibits inextensibility in the longitudinal direction. The embedded beam is subjected to a compressive force P that acts along the axis of the beam. Attention here is restricted to the category of beams in which the buckling mode is as shown in Fig. 1. It must be noted that since the beam is of infinite length, boundary conditions can be prescribed at the nodes of the deflected shape. Flexure of the beam takes place in the x–z plane. Normal contact stresses at the beam–elastic medium interfaces are denoted by $\tilde{q}^+(x, y)$ and $\tilde{q}^-(x, y)$, where $x \in (-\infty, \infty)$ and $y \in (-b/2, b/2)$. The positive and negative superscripts signify the components of the normal stresses at the faces of the beam in contact with the half-space regions $z > 0$ and $z < 0$, respectively. It should be noted that since $h/b \ll 1$, flexural effects of the shear stresses that act on the bonded surfaces $z = 0^+$ and $z = 0^-$ can be neglected. In the absence of flexure in the transverse plane (y–z), the general differential equation of beam buckling is given by

$$E_b I \frac{d^4 w(x, y)}{dx^4} + P \frac{d^2 w(x, y)}{dx^2} + \int_{-b/2}^{b/2} \{\tilde{q}^+(x, y) - \tilde{q}^-(x, y)\}\, dy = 0 \qquad (1)$$

Considering the one-dimensional nature of the problem, it is convenient to introduce average measures of $w(x, y)$ and $\tilde{q}(x, y)$, taken over the width of the beam. It is assumed that $w(x, y)$ and $\tilde{q}(x, y)$ admit representations of the form

$$w(x, y) = w(m, y) \cos\left[\frac{m\pi x}{a}\right] \qquad (2)$$

and

$$\tilde{q}(x, y) = \tilde{q}(m, y) \cos\left[\frac{m\pi x}{a}\right] \qquad (3)$$

The average measures of $w(x, y)$ and $\tilde{q}(x, y)$ are denoted by $w(x)$ and $q(x)$, respectively. These are defined by the relationships

$$w(x) = \frac{1}{b}\int_{-b/2}^{b/2} w(x, y)\, dy = \left\{\frac{1}{b}\int_{-b/2}^{b/2} w(m, y)\, dy\right\} \cos\left[\frac{m\pi x}{a}\right] \tag{4a}$$

$$q(x) = \int_{-b/2}^{b/2} \tilde{q}(x, y)\, dy = \left\{\frac{1}{b}\int_{-b/2}^{b/2} \tilde{q}(m, y)\, dy\right\} \cos\left[\frac{m\pi x}{a}\right] \tag{4b}$$

For conciseness, denote Eqs. 4a and 4b by

$$w(x) = \bar{w}(m) \cos\left[\frac{m\pi x}{a}\right] \tag{5a}$$

$$q(x) = \bar{q}(m) \cos\left[\frac{m\pi x}{a}\right] \tag{5b}$$

respectively. The representations Eqs. 5a and 5b are also necessitated by the state of symmetry that exists in the problem. Since the elastic medium is of infinite extent and since the beam is considered to be of infinite length, spatial symmetry will ensure equivalence in the harmonic distributions for $w(x)$ and $q(x)$.

Using these representations, the differential equation of buckling of Eq. 1 can be rewritten in the form

$$E_b I \frac{d^4 w(x)}{dx^4} + P\frac{d^2 w(x)}{dx^2} + b\{q^+(x) - q^-(x)\} = 0 \tag{6}$$

Note, from the spatial symmetry of the buckling problem, that

$$q^+(x) = -q^-(x)$$

In order to complete the description of the problem, it is necessary to define a kernel function or a load-displacement relationship that relates the contact forces $bq^+(x)$ and $bq^-(x)$ to the beam deflection $w(x)$. It will be assumed that this load-displacement behavior can be defined as a relationship between $\bar{w}(m)$ and $\bar{q}(m)$, i.e.,

$$b\bar{q}(m) = S\bar{w}(m) \tag{7}$$

In Eq. 7, S depends on the buckling wavelength parameter a, the width of the beam b, and the elastic constants of the surrounding elastic medium. A complete discussion of the evaluation of $S(m, a)$ will be presented in the ensuing section. Using Eqs. 4, 5, 6, and 7 in Eq. 1, one obtains

$$P = \frac{m^2\pi^2 E_b I}{a^2}\left[1 + \frac{2a^4 S(m, a)}{m^4\pi^4 E_b I}\right] \tag{8}$$

The critical wavelength of the buckled shape for the fundamental mode $m = 1$ is given by the minimization condition

$$\partial P/\partial a = 0 \tag{9}$$

This formally completes analysis of the buckling behavior of a flat beam that is embedded in bonded contact with an isotropic elastic solid. Explicit values for the critical load can be evaluated by making use of the relevant expressions for the stiffness parameter $S(m, a)$.

III. LOAD-DISPLACEMENT RESPONSE OF THE ELASTIC SOLID

The stiffness property $S(m, a)$ is a mathematical relationship between the displacements and interface normal tractions that act at the beam–elastic medium interfaces. The interfaces occupy the regions $x \in (-\infty, \infty)$, $y \in (-b/2, b/2)$, $z = 0^+$ and $x \in (-\infty, \infty)$, $y \in (-b/2, b/2)$, $z = 0^-$. The flexible beam experiences a constant displacement across its width at any cross section $x =$ constant. Owing to the assumed bonded conditions at the beam–elastic medium interface and the spatial symmetry of the buckling behavior, the displacements $u(x, y, z)$ and $v(x, y, z)$ and the stress component $\sigma_{zz}(x, y, z)$ related to the rectangular Cartesian coordinate system exhibit a state of asymmetry about the plane $z = 0$ (Fig. 1).

Prior to evaluating the expression for stiffness parameter $S(m, a)$, it is instructive to consider an auxiliary problem in which the infinite space is subjected to the internal loading

$$p(x, y) = 2\sigma_0 \cos\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{a}\right] \tag{10}$$

which acts on the plane $z = 0$ in the positive z-direction. This loading is transmitted as a compressive normal traction $\sigma_{zz}(x, y, 0^+) = -\sigma_0 \cos(m\pi x/a) \cos(n\pi y/a)$ on the half-space region $z > 0$ and as tensile normal traction on the boundary of the half-space region $z < 0$. In examination of the above problem, analysis is restricted to a single half-space region $z \geq 0$ in which the plane $z = 0$ is subjected to the boundary conditions

$$u(x, y, 0^+) = 0 \tag{11}$$

$$v(x, y, 0^+) = 0 \tag{12}$$

$$\sigma_{zz}(x, y, 0^+) = -\sigma_0 \cos\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{a}\right] \quad (13)$$

It must be noted that on the plane $z = 0$ the shear stresses are, in general, nonzero. If the shear stresses vanish on $z = 0$ (for $\nu < 1/2$), there will be a discontinuity of the displacement field and the two half-space regions will deform independently. Such a discontinuity will violate the requirement for a complete elastic solid occupying the region

$$(x, y, z) \in (-\infty, \infty)$$

The negative sign of Eq. 13 indicates a compressive stress and (a/m) and (a/n) are wavelengths. For analysis of the problem, a Galerkin stress function approach (see, e.g., Fung [13], Gladewell [14]) is selected. The displacement and stress fields in the elastic solid can be uniquely represented in terms of the Galerkin stress function $Z(x, y, z)$, which satisfies the biharmonic equation

$$\nabla^2\nabla^2 Z(x, y, z) = 0 \quad (14)$$

where ∇^2 is Laplace's operator. The displacement and stress components of specific interest to the present study can be expressed in terms of $Z(x, y, z)$ in the form

$$2Gu(x, y, z) = -\frac{\partial^2 Z}{\partial x \partial z} \quad (15)$$

$$2Gv(x, y, z) = -\frac{\partial^2 Z}{\partial y \partial z} \quad (16)$$

$$2Gw(x, y, z) = 2(1 - \nu)\nabla^2 Z - \frac{\partial^2 Z}{\partial z^2} \quad (17)$$

$$\sigma_{zz}(x, y, z) = \frac{\partial}{\partial z}\left[(2 - \nu)\nabla^2 Z - \frac{\partial^2 Z}{\partial z^2}\right] \quad (18)$$

respectively, where G is the linear elastic shear modulus and ν is Poisson's ratio. It can be shown that the stress function

$$Z(x, y, z) = -\frac{\sigma_0}{2(1 - \nu)}[2(1 - \nu) + cz]e^{-cz} \cos\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{a}\right] \quad (19)$$

where $c^2 = \pi^2(m^2 + n^2)/a^2$ satisfies Eq. 14 and gives displacement and stress fields that reduce to zero as $z \to \infty$. The expression for the displacement component $w(x, y, z)$ takes the form

$$w(x, y, z) = -\frac{\sigma_0}{4Gc(1-\nu)}[3 - 4\nu + cz]e^{-cz} \cos\left[\frac{m\pi x}{a}\right] \cos\left[\frac{n\pi y}{a}\right] \quad (20)$$

From Eqs. 13 and 20, one obtains the load-displacement response at the plane $z = 0^+$; i.e.,

$$w(x, y, 0^+) = \frac{(3-4\nu)a}{4\pi G(1-\nu)(m^2+n^2)^{1/2}}\sigma_{zz}(x, y, 0^+) \quad (21)$$

The displacement $w(x, y, 0^+)$ and the stress $\sigma_{zz}(x, y, 0^+)$ can be identified with deflection of the beam and contact stress at the beam–elastic medium interface, respectively. For the particular class of beams of narrow width, which are embedded in bonded contact with the elastic medium and do not exhibit flexural effects in the y–z plane,

$$w(x, y, 0^+) = w(x, y, 0^-) = w(x) \quad (22)$$

$$\sigma_{zz}(x, y, 0^+) = -\sigma_{zz}(x, y, 0^-) = q(x, y) \quad (23)$$

In order to develop the load displacement behavior for the beam region, the problem in which the internal load is applied only within the beam region $x \in (-\infty, \infty)$, $y \in (-b/2, b/2)$ is further investigated. In particular, the loading applied should be such that displacement induced in the beam region is constant across the width of the beam. In order to develop a solution for this problem, it is necessary to solve an integral equation of the form

$$w(x, y) = \frac{(3-4\nu)}{2\pi G(1-\nu)}\int_{-\infty}^{\infty}\int_{-b/2}^{b/2}\frac{q(\xi, \eta)\, d\xi d\eta}{[(x-\xi)^2 + (y-\eta)^2]^{1/2}} \quad (24)$$

Since Eq. 24 cannot be solved in an exact fashion, one can only attempt to develop approximate solutions in which the restriction $w(x, y) = w(x)$ is satisfied in an approximate fashion. Such approximate techniques were used by Biot [10], Rvachev [11], and Lekkerkerker [12] in connection with their analyses of beams of finite width resting in smooth contact with an isotropic elastic half space. Consider the problem in which the region of the infinite space occupied by the embedded beam is subjected to a harmonic strip load of the type

$$q(x, y) = \begin{cases} 2q^* \cos\left[\frac{m\pi x}{a}\right], & |y| \le b/2 \\ 0, & |y| \ge b/2 \end{cases} \quad (25)$$

The displacement field induced in the beam region takes the form

$$w(x, y) = \frac{aq^*(3 - 4\nu)}{4\pi^2 G(1 - \nu)} \left\{ \int_0^\infty \frac{R(n, y)\, dn}{n(m^2 + n^2)^{1/2}} \right\} \cos\left(\frac{m\pi x}{a}\right) \tag{26}$$

where

$$R(n, y) = \sin\left\{\frac{n\pi(2y + b)}{2a}\right\} - \sin\left\{\frac{n\pi(2y - b)}{2a}\right\} \tag{27}$$

The displacement field of Eq. 26 exhibits a variation across the width of the beam region. One can, however, superpose uniform load distributions of the type of Eq. 25 to achieve a nearly constant displacement across the width of the beam. It must be noted that since the cross section of the beam is assumed to be rigid, the stresses $q(x, y)$ at the beam–elastic medium interface should exhibit singular behavior as $|y| \to b/2$. Consequently, superposition of uniform loads only approximates the uniform displacement behavior across the width of the beam. Following procedures outlined by Biot [10], it can be shown that for a nearly uniform displacement across the width of the beam, the relationship between $\bar{w}$ and $\bar{q}$ is such that

$$S_1(m, a) = \frac{4\pi G m b(1 - \nu)\Psi(\xi)}{Ca(3 - 4\nu)} \tag{28}$$

where $1 \le C \le 1.13$, $\xi = \pi bm/2a$, and $\Psi(\xi)$ is a function that can be expressed as

$\xi =$	0.1	0.5	1.0	3	8	∞
$\Psi(\xi) =$	4.8	1.9	1.42	1.13	1.04	1.00

and for $\xi < 0.1$, $\Psi(\xi) \simeq (2/\pi\xi)\{\ln(1/\xi) + 0.923\}^{-1}$. The subscript of S in Eq. 28 indicates the first approximation for $S(m, a)$. Improved estimates for the result of Eq. 28 can also be obtained by using the approximate techniques proposed by Rvachev [11] and Lekkerkerker [12] for the analysis of the analogous half-space problem. Details of these techniques are given by Selvadurai [15]. It can be shown that the contact stress distribution $q(x, y)$ can be expressed in the form

$$q(x, y) = \tilde{q}(m, y) \cos\left(\frac{m\pi x}{a}\right) \tag{29}$$

where $\tilde{q}(m, y)$ is a solution of the integral equation

$$\bar{w}(m) = \frac{(3 - 4\nu)}{4\pi G(1 - \nu)} \int_{-b/2}^{b/2} \tilde{q}(m, t) K_0 \left[\frac{m\pi|y - t|}{a} \right] dt \tag{30}$$

and K_0 is the modified Bessel function of the second kind. From the definition of the finite Fourier transform and from the second of Eqs. 5,

$$b\bar{q}(m) = \int_{-b/2}^{b/2} \tilde{q}(m, y)\, dy \tag{31}$$

A solution of Eq. 30, therefore, formally completes analysis of the problem and generates a relationship between $\bar{w}(m)$ and $\bar{q}(m)$. The integral equation of Eq. 30 can be solved in an approximate manner by making use of the following methods. In the first method, an infinite series representation of $\tilde{q}(m, y)$, in terms of Mathieu functions, is used to solve Eq. 30. In the second method, an asymptotic series solution of Eq. 30 is derived by expanding the kernel function in terms of the small parameter ξ. The associated expressions for the stiffness parameter $S_i(m, a)$ $(i = 2, 3)$ reduce to the forms

$$S_2(m, a) = \frac{4G(1 - \nu)(a + \pi m b)}{(3 - 4\nu)a} \tag{32}$$

$$S_3(m, a) = \frac{4G(1 - \nu)}{(3 - 4\nu)} \left[\left\{ 2\xi + \frac{1}{2} \right\} \operatorname{erf}(\sqrt{2\xi}) + \left[\frac{2\xi}{\pi} \right]^{1/2} e^{-2\xi} - \xi \right] \tag{33}$$

respectively. The result of Eq. 32 can be obtained as a limiting case of Eq. 33 when terms of the order ξ^2 are neglected. In Eq. 33, erf (x) corresponds to the error function, which is defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{34}$$

Equations 28, 32, and 33 provide three estimates for $S(m, a)$ which can be used to compute the buckling load for the embedded beam.

VI. ESTIMATION OF THE BUCKLING LOAD: NUMERICAL RESULTS

The specific case of an embedded beam of finite width, in which deflection of the buckled shape is given by

$$w(x) = w^* \cos\left(\frac{\pi x}{a}\right) \tag{35}$$

is now examined. The corresponding estimate for the buckling load P is given by (cf. Eq. 8)

$$P = \frac{\pi^2 E_b I}{a^2}\left[1 + \frac{2a^4 S_i(a)}{\pi^4 E_b I}\right] \tag{36}$$

The result of Eq. 36 yields three estimates for the buckling load which are consistent with $S_i(a)$, $i = 1, 2, 3$. Using the load-displacement responses $S_i(a)$ derived previously, Eq. 36 can be written in the form

$$P = \frac{\pi^2 E_b I}{b^2}\left[\frac{1}{\lambda^2} + \frac{\lambda^2 \chi_i(\lambda)}{\pi^4 R^* \phi}\right] \tag{37}$$

where

$$\lambda = \frac{a}{b}, \qquad \phi = \frac{(1 + \nu)(3 - 4\nu)}{48(1 - \nu)}, \qquad R^* = \frac{E_b h^3}{E b^3} \tag{38}$$

and $\chi_i(\lambda)$ ($i = 1, 2, 3$) are defined by the expressions

$$\chi_1(\lambda) = \frac{\pi}{C\lambda}\Psi(\xi^*), \qquad \xi^* = \frac{\pi}{2\lambda} \tag{39}$$

$$\chi_2(\lambda) = \frac{\pi}{\lambda}\left(1 + \frac{\lambda}{\pi}\right) \tag{40}$$

$$\chi_3(\lambda) = \left\{\frac{1}{\lambda} + \frac{1}{2}\right\} \operatorname{erf}\left(\frac{1}{\sqrt{\lambda}}\right) + \left[\frac{1}{\pi\lambda}\right]^{1/2} e^{-1/\lambda} - \frac{1}{2\lambda} \tag{41}$$

The appropriate estimates for the buckling load can be obtained by a minimization of the result of Eq. 37, i.e., $\partial P/\partial\lambda = 0$.

Figures 2 and 3 show the variation of the nondimensional buckling load $\bar{P}_{cr}$ $(= P_{cr}/\{\pi^2 E_b I/b^2\})$, expressed as a function of the relative rigidity parameter R^* of the beam–elastic medium system. The length parameter in the problem is essentially the width b of the embedded beam. Since P_{cr} is normalized with respect to $\pi^2 E_b I/b^2$, any variation in R^* can only be interpreted as a change in the elastic modulus of the supporting medium. The limit $R \to 0$ represents a situation in which the beam is embedded in a relatively stiff elastic solid. Similarly, $R^* \to \infty$ represents a relatively soft elastic solid. Variations of $\bar{P}_{cr}$, derived from the three estimates for $S_i(a)$, are shown in Fig. 2. These results show trends that are consistent for the range of relative rigidities discussed previously.

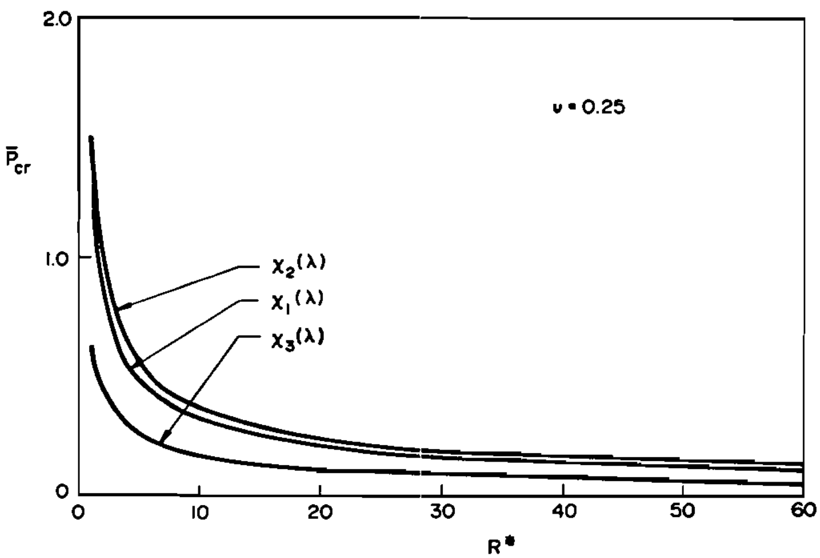

Fig. 2 Variation of buckling load with the relative rigidity of the beam-elastic medium system.

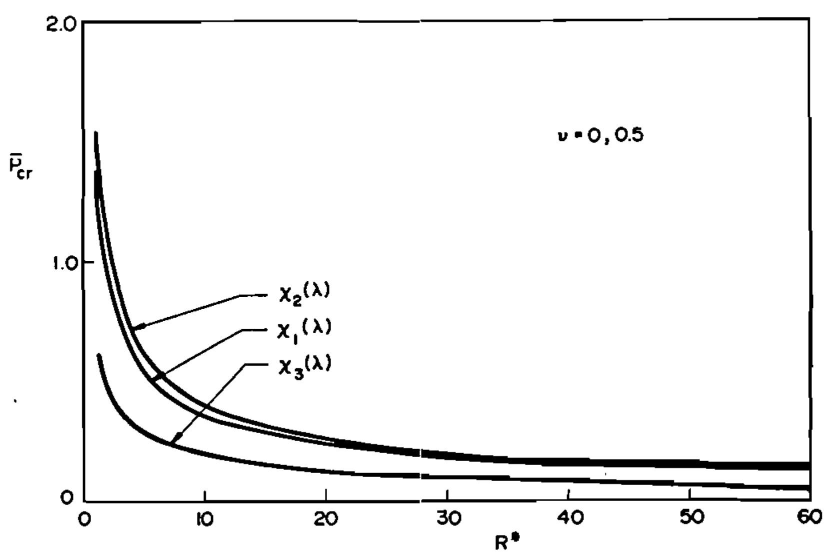

Fig. 3 Variation of the buckling load with the relative rigidity of the beam-elastic medium system.

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