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SOME GENERALIZED RESULTS FOR AN ORTHOTROPIC ELASTIC QUARTER-PLANE

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Abstract

This paper presents some generalized results for the stress analysis of an orthotropic elastic quarter-plane which may be subjected to localized or distributed loads on the boundary or at the interior of the quarter-plane. Formal analytical results are presented for the special cases where the orthotropic elastic quarter-plane is subjected, separately, to a concentrated force applied on the surface and in the interior. Numerical results are also presented for those concentrated force problems where the quarter-plane is composed of typical unidirectional fibre reinforced materials such as boron-epoxy and graphite-epoxy composites.

§ 1. Introduction

The theory of anisotropic elasticity has been employed quite extensively in the stress analysis of natural and artificial laminated structural materials such as wood-laminates and fibre reinforced composites [1–5]. In such treatments, the laminated material which usually exhibits heterogeneous properties on the microscale is represented by a homogeneous material whose macroscopic properties are predominantly anisotropic. The anisotropic properties can then be determined from the various laws of mixtures and allied theories [4–10] applicable to composite media.

In the present paper we develop some generalized results for the stress analysis of an orthotropic elastic quarter-plane, in which the principal directions of the orthotropic elastic material coincide with the rectangular cartesian directions of the spatial coordinate system. From the particular results developed here it is possible to determine the state of stress in an orthotropic elastic quarter-plane which may be subjected to either concentrated or distributed force systems which act on the boundary or at the interior of the orthotropic quarter-plane. The elastic quarter-plane constitutes a special case of the more general class of elastic wedge problems which have received considerable attention. The two dimensional problems of the stress distribution in isotropic elastic wedges loaded on its sides by uniformly distributed loads and at its apex by concentrated loads, have been analysed by Carothers, Levy [11], and Sternberg and Koiter [12]. References to further work on isotropic wedge problems are given by Flügge [13]. Also, integral transform techniques, such as Mellin transforms, have been employed by Tranter [14], Sneddon [15], Godfrey [16] and Bogy [17, 18] to obtain solutions to isotropic wedge problems where a more general type of uniform and concentrated loads are applied on the *surface* of the wedge. Similar techniques have been adopted by Conway [19, 20], Lekhnitsky [21], Benthem [22], and Baker [23] for the analysis of *surface* loading conditions associated with anisotropic elastic wedges. A Fourier integral solution of the isotropic elastic quarter-plane has also been obtained by Iyengar [24].

The generalized results for the orthotropic elastic quarter-plane presented in this paper are obtained as a direct consequence of applying a method of superposition similar to that employed by Hetenyi [25] for the analysis of an isotropic elastic quarter-plane, which is subjected to loads that are applied on its surface. The solution to an appropriate half-plane problem forms the basis of Hetenyi's method of solution of the quarter-plane. For example, the solution to the elastic quarter-plane subjected to a concentrated normal force at its surface is approached via a modified Flamant-Boussinesq solution [11] which consists of an elastic half-plane subjected to two equal, equidistant (from the origin) concentrated normal forces. A repeated superposition of known solutions of the elastic half-plane is then employed to satisfy the traction boundary conditions of the problem. Such a procedure then leads to a sequence of infinite integrals of a recursive pattern. This convergent sequence [25] of infinite integrals may be then continued to obtain the solution to the quarter-plane problem, to any required accuracy. Hetenyi [25] has applied this technique to obtain solutions to the isotropic quarter-plane subjected to concentrated normal and tangential forces and a partially distributed uniform load located at the surface. Craft and Richardson [26] have also applied Hetenyi's method to obtain the state of stress in an isotropic quarter-plane containing a circular inclusion. The superposition procedure has also been extended by Hetenyi [27] to obtain solutions to the elastic quarter-space subjected to concentrated forces.

Hetenyi's method for the solution of the quarter-plane problem is not general in character, in the sense that for each particular problem one has to start with the solution to the appropriate half-plane problem and superpose a series of infinite integrals at each stage to satisfy the traction boundary conditions. Nevertheless the efficiency of this method lies in the fact that the recurrent infinite integrals encountered are readily amenable to programmed numerical computation. Alternative methods of solution of the quarter-plane problem which employ either the Fourier integral approach or the Mellin transform technique remove this objection but only at the expense of reducing the problem to either the solution of complex integral equations or the inversion of complicated infinite integrals. The mathematical rigour necessary for such treatments is clearly illustrated in [17, 18] and in a recent paper by Harrington and Ting [28] who apply Mellin transform techniques to investigate the existence and uniqueness of the state of stress in some isotropic elastic wedges. Also in the context of the present paper, it should be mentioned that traditional numerical methods of stress analysis such as relaxation techniques or finite element methods, without refinement, do not generally yield accurate information about the state of stress in the vicinity of highly localized loads. In orthotropic media such as most fibre reinforced materials, a detailed knowledge of such local states of stress is of course essential to establish safe loads which would prevent brittle fracture and failure of structural elements.

The fundamental equations of orthotropic elasticity are developed in section 2. In section 3 we present the generalized analysis of the orthotropic quarter-plane and present results for the particular case of an orthotropic quarter-plane which is subjected to a concentrated load applied normal to the surface, and a concentrated force applied at the interior. In section 4 we present numerical results to the problem of the interior load which acts in one of the coordinate directions. Such a highly localized loading represents a situation which can occur in the vicinity of a metal connector in timber or fibre reinforced structural elements. In the special case when the concentrated force migrates to the outer boundary of the quarter-plane we have a condition that may be encountered at the support of a structural element. The special cases of orthotropic materails considered here are assumed to be typical of fibre reinforced elastic materials such as boron-epoxy, and graphite-epoxy composites. Using these generalized results it is also possible to obtain the state of stress in orthotropic elastic quarter-planes which are subjected to oblique loads, concentrated couples or finite distributed loads which act on the boundary or at the interior of the quarter-plane.

§ 2. Basic equations

We consider an orthotropic elastic material whose principal axes coincide with the rectangular cartesian spatial coordinate directions X and Y. The constitutive equations for orthotropic plane stress can then be written in the form

$$\begin{aligned} \varepsilon_{xx} &= c_{11}\sigma_{xx} + c_{12}\sigma_{yy}, \\ \varepsilon_{yy} &= c_{12}\sigma_{xx} + c_{22}\sigma_{yy}, \\ \varepsilon_{xy} &= \frac{c_{44}}{2}\,\sigma_{xy}, \end{aligned} \tag{1}$$

where σ_{xx}, σ_{yy}, σ_{xy} are the components of the symmetric stress tensor σ_{ij} referred to the X–Y plane

$$\varepsilon_{xx} = \frac{\partial u}{\partial X}, \quad \varepsilon_{yy} = \frac{\partial v}{\partial Y}, \quad \varepsilon_{xy} = \tfrac{1}{2}\left(\frac{\partial u}{\partial Y} + \frac{\partial v}{\partial X}\right), \tag{2}$$

are the components of the infinitesimal strain tensor ε_{ij} and u and v are the components of the displacement vector in the X and Y directions respectively.

For the integrability of the kinematic system (2) the strain components should satisfy the compatibility condition

$$\frac{\partial^2 \varepsilon_{xx}}{\partial Y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial X^2} = 2\,\frac{\partial^2 \varepsilon_{xy}}{\partial X \partial Y}. \tag{3}$$

In the absence of body forces, the equations of force equilibrium are

$$\frac{\partial \sigma_{xx}}{\partial X} + \frac{\partial \sigma_{xy}}{\partial Y} = 0, \quad \frac{\partial \sigma_{yy}}{\partial Y} + \frac{\partial \sigma_{yx}}{\partial X} = 0. \tag{4}$$

By introducing an Airy stress function $\varphi(X, Y)$ such that

$$\sigma_{xx} = \frac{\partial^2\varphi}{\partial Y^2}, \quad \sigma_{yy} = \frac{\partial^2\varphi}{\partial X^2}, \quad \sigma_{xy} = -\frac{\partial^2\varphi}{\partial X\partial Y}, \tag{5}$$

we observe that the equations of equilibrium (4) are identically satisfied and using (1), the compatibility condition (3) can be reduced to the form

$$\left(\frac{\partial^2}{\partial X^2} + k_1^2\frac{\partial^2}{\partial Y^2}\right)\left(\frac{\partial^2}{\partial X^2} + k_2^2\frac{\partial^2}{\partial Y^2}\right)\varphi = 0, \tag{6}$$

where

$$\left.\begin{matrix} k_1^2 \\ k_2^2 \end{matrix}\right\} = \frac{1}{2c_{22}}\,[2c_{12} + c_{44} \pm (4c_{12}^2 + c_{44}^2 + 4c_{12}c_{44} - 4c_{11}c_{22})^{\frac{1}{2}}]. \tag{7}$$

The solution of the plane stress problem in orthotropic elasticity is reduced to the solution of (6) subject to appropriate boundary conditions. In the case of traction boundary conditions we have

$$T_x = n_x\sigma_{xx} + n_y\sigma_{xy}, \quad T_y = n_x\sigma_{xy} + n_y\sigma_{yy}, \tag{8}$$

where T_x, T_y are the components of the traction vector on a surface $F(X, Y) = 0$, and n_x, n_y are the components of the unit normal to this surface such that

$$n_x^2 + n_y^2 = 1, \quad \frac{n_x}{n_y} = \frac{\partial F/\partial X}{\partial F/\partial Y}. \tag{9}$$

§ 3. The orthotropic elastic quarter-plane

Before proceeding with the analysis of the orthotropic quarter-plane we shall first consider, for future reference, the solution to the problem of an orthotropic half-plane which is subjected to a concentrated force at the origin (Fig. 1a). It may be verified that the stress function

$$\varphi = \frac{Pa}{\pi(k_1 - k_2)}\int_0^\infty \frac{1}{\zeta^2}\,\{k_2\,\mathrm{e}^{-\zeta y/k_2} - k_1\,\mathrm{e}^{-\zeta y/k_1}\}\cos\zeta x\,\mathrm{d}\zeta, \tag{10}$$

which satisfies the governing differential equation (6) and traction boundary conditions (8), on the surface $Y = 0$ and at infinity, provides the complete solution of this problem. The stress compo-

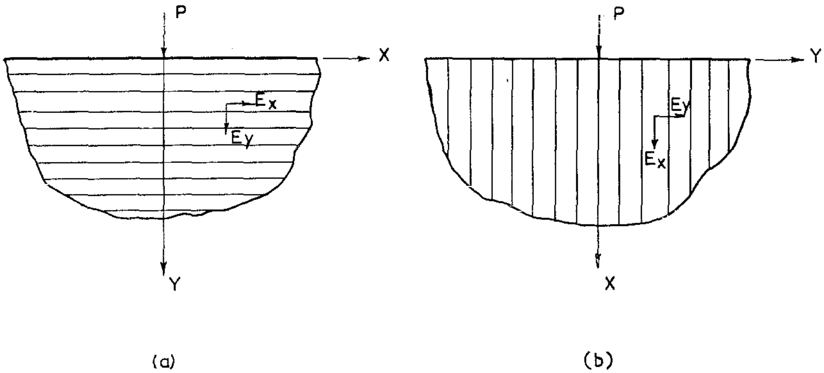

Fig. 1. Orthotropic elastic half-plane subjected to a concentrated force.

nents derived from (10) and (5) can be written in the form

$$[\sigma_{xx}; \sigma_{yy}; \sigma_{xy}] = \frac{P(k_1 + k_2)}{\pi a[k_1^2 x^2 + y^2][k_2^2 x^2 + y^2]} [x^2 y; y^3; xy^2] \qquad (11)$$

where

$$x = X/a, \qquad y = Y/a,$$

are the non-dimensional spatial coordinates and a is a typical length parameter. Similarly, in the particular case of an orthotropic half-plane occupying the region $X > 0$ (Fig. 1b) the state of stress due to a concentrated force acting at the origin in the X-direction is given by the stress function

$$\varphi = \frac{Pa}{\pi(k_1 - k_2)} \int_0^\infty \frac{1}{\zeta^2} \{k_2 \,\mathrm{e}^{-k_1 \zeta x} - k_1 \,\mathrm{e}^{-k_2 \zeta x}\} \cos \zeta y \,\mathrm{d}\zeta. \qquad (12)$$

The stress components corresponding to (12) are

$$[\sigma_{xx}; \sigma_{yy}; \sigma_{xy}] = \frac{P k_1 k_2 (k_1 + k_2)}{\pi a[k_1^2 x^2 + y^2][k_2^2 x^2 + y^2]} [x^3; xy^2; x^2 y]. \qquad (13)$$

Let us now restrict our attention to a simply connected orthotropic half-plane (Fig. 1a) occupying the region $-\infty < X < \infty, 0 < Y < \infty$. We assume that this half-plane is subjected to force systems, that may be applied on the boundary or at the interior, in such a manner that the resulting state of stress is symmetric about the Y-axis. We shall refer to this state of stress as the "basic state of stress". By virtue of the symmetry of this basic state of stress, the shear stresses are zero on the plane of symmetry. The plane $X = 0$ is therefore subjected to only a purely normal stress, $F_0(\bar{y})$. If an additional state of stress can be found such that in relation to the quarter-plane occupying the first quadrant (Fig. 1a), the shear tractions on planes $X = 0$ and $Y = 0$ are identically zero and the normal tractions on planes $X = 0$ and $Y = 0$ are $-F_0(\bar{y})$ and zero respectively, then this additional state of stress together with the basic state of stress constitutes the solution of the particular quarter-plane problem represented by either half of the initial half-plane problem. The technique proposed by Hetenyi [25] for the elimination of $F_0(\bar{y})$ on the plane $X = 0$ can be generalized as follows; consider first the problem of a half-plane which is subjected to the symmetric basic state of stress

$$\sigma_{ij} = \sigma_{ij}^{(0)}. \tag{14}$$

On the plane of symmetry $X = 0$ we have

$$\sigma_{xy}^{(0)}(0, Y) = 0, \quad \sigma_{xx}^{(0)}(0, Y) = F_0(\bar{y}). \tag{15}$$

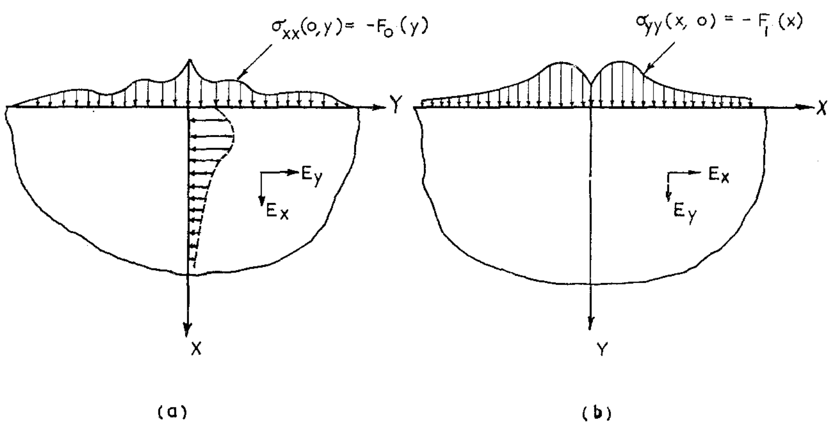

Fig. 2. Orthotropic elastic half-plane subjected to corrective stress systems.

Now consider the problem of the half-plane $X > 0$ (Fig. 2a), which is subjected to a symmetric distribution of $\sigma_{xx} = -F_0(\bar{y})$ on the plane $X = 0$ (Step 1). The resulting stress field $(\sigma_{xx}^{(1)}, \sigma_{yy}^{(1)}, \sigma_{xy}^{(1)})$ can be determined by an integration of the stress components (12). We have

$$\sigma_{ij}^{(1)} = -\frac{k_1 k_2 (k_1 + k_2)}{\pi} \int_0^\infty F_0(\bar{y})\, J_{ij}(\bar{y})\, d\bar{y} \tag{16}$$

where

$$J_{ij}(\bar{y}) = J_{ij}^{+}(\bar{y}) + J_{ij}^{-}(\bar{y})$$

and

$$[J_{xx}^{\pm}(\bar{y}); J_{yy}^{\pm}(\bar{y}); J_{xy}^{\pm}(\bar{y})] = \frac{[x^3; x(y \pm \bar{y})^2; x^2(y \pm \bar{y})]}{[k_1^2x^2 + (y \pm \bar{y})^2][k_2^2x^2 + (y \pm \bar{y})^2]}. \tag{17}$$

Thus, combining the stress components derived from Step I with those of the basic state of stress renders the plane $X = 0$ free of normal traction but gives rise to a non-zero normal traction $F_1(\bar{x})$ on the plane $Y = 0$. To eliminate $F_1(\bar{x})$ we consider the symmetric state of external normal stress $-F_1(\bar{x})$ on the plane $Y = 0$ for the half-plane $Y > 0$ (Step 2). Again, the complete stress components $\sigma_{xx}^{(2)}$, $\sigma_{yy}^{(2)}$, $\sigma_{xy}^{(2)}$ can be determined by an integration of the results (11) for the concentrated force problem. We now write

$$\sigma_{ij}^{(2)} = \frac{(k_1 + k_2)}{\pi} \int_0^{\infty} F_1(\bar{x})\, K_{ij}(\bar{x})\, \mathrm{d}\bar{x}, \tag{18}$$

where

$$K_{ij}(\bar{x}) = K_{ij}^{+}(\bar{x}) + K_{ij}^{-}(\bar{x}),$$

and

$$[K_{xx}^{\pm}(\bar{x}); K_{yy}^{\pm}(\bar{x}); K_{xy}^{\pm}(\bar{x})] = \frac{[(x \pm \bar{x})^2 y; y^3; (x \pm \bar{x})\, y^2]}{[k_1^2(x \pm \bar{x})^2 + y^2][k_2^2(x \pm \bar{x})^2 + y^2]}. \tag{19}$$

We note that (16) and (18) have been derived for $P = 1$.

We may verify that the state of stress represented by (17) eliminates the normal stress $F_1(\bar{x})$ on the plane $Y = 0$ but in doing so gives rise to a normal traction $F_2(\bar{y})$ on the plane $X = 0$. It is now evident that the techniques outlined in steps 1 and 2 have to be repeatedly applied in order to satisfy traction boundary conditions on the plane surfaces $X = 0$ and $Y = 0$. This procedure leads to a set of integrals of a recursive pattern and the combination of these individual states of stress gives

$$\sigma_{ij}^{(c)} = \sum_{n=1}^{\infty} \sigma_{ij}^{(n)}, \tag{20}$$

which, in the orthotropic quarter-plane satisfies the boundary conditions

$$\sigma_{xx}^{(c)}(0, Y) = -F_0(\bar{y}),\ \sigma_{yy}^{(c)}(X, 0) = \sigma_{xy}^{(c)}(X, 0) = \sigma_{xy}^{(c)}(Y, 0) = 0. \quad (21)$$

Using (16) and (18), (20) can be written in the form

$$\sigma_{ij}^{(c)} = \frac{(k_1 + k_2)}{\pi}\left[-k_1 k_2 \int_0^\infty \{J_{ij}(\bar{y}) \sum_{m=0,2,4}^{\infty} F_m(\bar{y})\} \, \mathrm{d}\bar{y} + \right.$$
$$\left. + \int_0^\infty \{K_{ij}(\bar{x}) \sum_{m=1,3,5}^{\infty} F_m(\bar{x})\} \, \mathrm{d}\bar{x}\right]. \quad (22)$$

The functions $F_m(\bar{x})$ and $F_m(\bar{y})$ are given by the recurrence relations

$$F_{m+1}(\bar{x}) = \frac{k_1 k_2(k_1 + k_2)}{\pi}\int_0^\infty \frac{2F_m(\bar{y})\bar{x}\bar{y}^2 \, \mathrm{d}\bar{y}}{[k_1^2\bar{x}^2 + \bar{y}^2][k_2^2\bar{x}^2 + \bar{y}^2]}, \quad (23a)$$

$$F_{m+1}(\bar{y}) = \frac{(k_1 + k_2)}{\pi}\int_0^\infty \frac{2F_m(\bar{x})\, \bar{x}^2\bar{y} \, \mathrm{d}\bar{x}}{[k_1^2\bar{x}^2 + \bar{y}^2][k_2^2\bar{x}^2 + \bar{y}^2]}. \quad (23b)$$

The complete solution of the orthotropic quarter-plane problem is obtained by combining the basic state of stress (14) with the corrective combination stress system (22). The complete expression for the state of stress in the orthotropic quarter-plane is given by

$$\sigma_{ij} = \sigma_{ij}^{(0)} + \sigma_{ij}^{(c)}. \quad (24)$$

By considering the resultant of normal tractions on the plane $X = 0$ due to the complete stress system (24) and obtaining the relationships between the maxima of two successive F_m functions, such as $F_{m+1}(\bar{x})$ and $F_m(\bar{y})$, it can be established [25] that the procedure outlined here leads to a convergent solution for the problem of the orthotropic elastic quarter-plane provided the elastic constant, c_{11}, c_{22}, c_{12} and c_{44} satisfy the consistency conditions

$$2c_{12} + c_{44} > 0; \qquad (2c_{12} + c_{44})^2 > 4c_{11}c_{22}. \quad (25)$$

We shall now consider the solution to the particular basic states of stress from which we may determine the state of stress within an orthotropic elastic quarter-plane subjected separately to a concentrated force on the plane $Y = 0$ and a concentrated force at a point $(a, \eta a)$ within the interior.

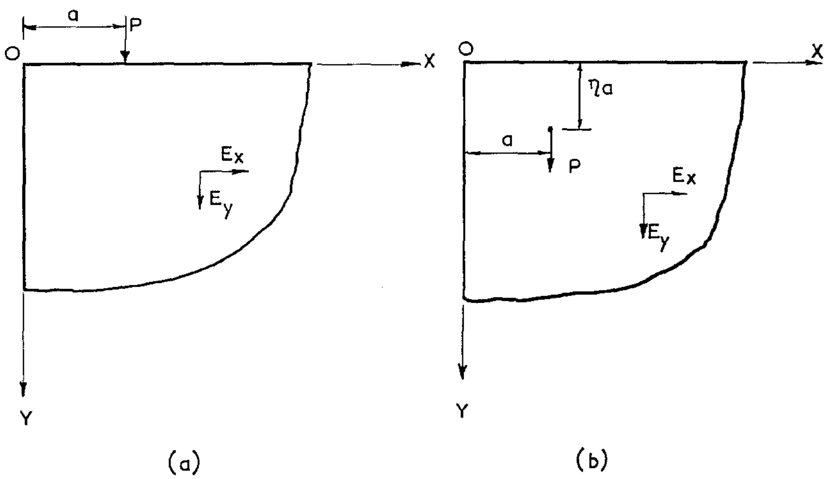

Fig. 3. The orthotropic quarter-plane.

a) Concentrated force acting normal to the boundary

Consider the problem of a concentrated force, P, applied on the boundary of the orthotropic quarter-plane at a distance a from the origin (Fig. 3a). The basic state of stress can be obtained by combining the results for two concentrated normal forces, acting equidistant from the origin, on the surface of the half-plane. The basic state of stress can be written as

$$\sigma_{ij}^{(0)} = \frac{P(k_1 + k_2)}{\pi a} S_{ij}, \tag{26}$$

where

$$S_{ij} = S_{ij}^{+} + S_{ij}^{-},$$

and

$$[S_{xx}^{\pm}; S_{yy}^{\pm}; S_{xy}^{\pm}] = \frac{[(x \pm 1)^2 y; y^3; (x \pm 1)\, y^2]}{[k_1^2(x \pm 1)^2 + y^2][k_2^2(x \pm 1)^2 + y^2]}, \tag{27}$$

Also the distribution of normal stress on the plane of symmetry is

$$F_0(\bar{y}) = \frac{2P(k_1 + k_2)\, \bar{y}}{\pi a(k_1^2 + \bar{y}^2)(k_2^2 + \bar{y}^2)}. \tag{28}$$

b) Concentrated force acting at the interior of the quarter-plane

We now refer to the problem of an orthotropic elastic quarter-plane which is subjected to a concentrated force applied at its interior. The basic state of stress can be obtained from the results for the interior concentrated force solution for the orthotropic half-plane problem given by Conway [19], and it can be expressed in the form

$$\begin{aligned}[\sigma_{xx}^{(0)}; \sigma_{yy}^{(0)}; \sigma_{xy}^{(0)}] = \frac{P}{\pi a(k_1^2 - k_2^2)} \{(\alpha_1 + \alpha_3)\, \tau_1[l_1;\, -l_1 k_1^2;\, -k_1^2 x] + \\ + (\alpha_2 - \alpha_4)\, \tau_3[l_1;\, -l_1 k_2^2;\, -k_2^2 x] + \alpha_1 \tau_2[-l_2; l_2 k_1^2; k_1^2 x] + \\ + \alpha_2 \tau_4[-l_2; l_2 k_2^2;\, k_2^2 x] + \alpha_3 \tau_5[-l_3; l_3 k_2^2; k_1^2 k_2^2 x] + \\ + \alpha_4 \tau_6[l_4;\, -l_4 k_1^2;\, k_1^2 k_2^2 x]\},\end{aligned} \tag{29}$$

where

$$\tau_n = \tau_n^+ + \tau_n^-, \qquad (n = 1, 2, \ldots, 6),$$

and

$$\tau_1^\pm = [k_1^2(x \pm 1)^2 + (y+\eta)^2]^{-1}, \qquad \tau_2^\pm = [k_1^2(x \pm 1)^2 + (y-\eta)^2]^{-1},$$

$$\tau_3^\pm = [k_2^2(x \pm 1)^2 + (y+\eta)^2]^{-1}, \qquad \tau_4^\pm = [k_2^2(x \pm 1)^2 + (y-\eta)^2]^{-1},$$

$$\tau_5^\pm = [k_1^2 k_2^2(x \pm 1)^2 + (k_2\eta + k_1 y)^2]^{-1},$$

$$\tau_6^\pm = [k_1^2 k_2^2(x \pm 1)^2 + (k_1\eta + k_2 y)^2]^{-1},$$

$$\left.\begin{matrix} l_1 \\ l_2 \end{matrix}\right\} = y \pm \eta, \qquad \left.\begin{matrix} l_3 \\ l_4 \end{matrix}\right\} = k_1 k_2 \eta + \begin{matrix} k_1^2 y, \\ k_2^2 y \end{matrix} \tag{30}$$

$$\alpha_1 = \frac{k_1}{2}(1 - \lambda k_2^2), \qquad \alpha_3 = \frac{-k_1^2(1 - \lambda k_2^2)}{(k_1 - k_2)},$$

$$\alpha_2 = -\frac{k_2}{2}(1 - \lambda k_1^2), \qquad \alpha_4 = \frac{k_2^2(1 - \lambda k_1^2)}{(k_1 - k_2)},$$

$$\lambda = \frac{c_{12}}{c_{11}}.$$

§ 4. Numerical results

In order to evaluate the stresses in the orthotropic elastic quarter-plane numerically, it is convenient to adopt the following procedure. The boundary stresses applied on the surfaces of the two overlapping half-planes are first calculated by numerical integration in the logarithmic scale using Simpson's rule. The reversal procedure of these boundary stresses were carried out up to forty cycles which provided in the resulting boundary stress components an accuracy of at least ten correct decimals. The corrective state of stress $\sigma_{ij}^{(c)}$ at a point in the quarter-plane is calculated by combining the stresses induced in the respective half-planes due to these boundary stresses. In order to evaluate the stresses $\sigma_{ij}^{(c)}$, the boundary stresses are represented as a series of uniformly distributed loads of finite but variable width. It is found that this particular uniform load representation of the boundary stresses leads to a better convergence of results when evaluating the stress components in the vicinity of the boundary of the quarter-plane, than the representation of the boundary stresses in terms of a series of concentrated forces. The basic state of stress $\sigma_{ij}^{(0)}$, which is evaluated separately, when combined with $\sigma_{ij}^{(c)}$ yields the complete solution for the orthotropic quarter-plane.

We present numerical results for the state of stress in the quarter-plane due to the two concentrated force problems mentioned earlier. The properties of the orthotropic materials considered here, namely unidirectional graphite-epoxy and boron-epoxy composites [29], are listed in table 1. We restrict our attention to the interior force acting at the point (a, a).

TABLE I

Type of material	c_{11}	c_{22}	c_{12}	c_{44}	k_1	k_2	λ
Boron Epoxy	3.624×10^{-3}	36.247×10^{-3}	-0.906×10^{-3}	96.665×10^{-3}	1.6055	0.1969	-0.25
Graphite Epoxy	3.624×10^{-3}	90.62×10^{-3}	-0.906×10^{-3}	181.257×10^{-3}	1.3998	0.1428	-0.25

[All dimensional quantities are in mm^2/kN]

The variation of stress components σ_{xx}, σ_{yy} and σ_{xy} in the two composites due to the two loading conditions are shown in Figs. 4–15. From the distributions for σ_{yy} vs x it is evident that for both surface and interior loading conditions, the stress concentration effects of the concentrated forces are restricted to regions in the vicinity ($x < 4$) of the point of application of the load. The decay

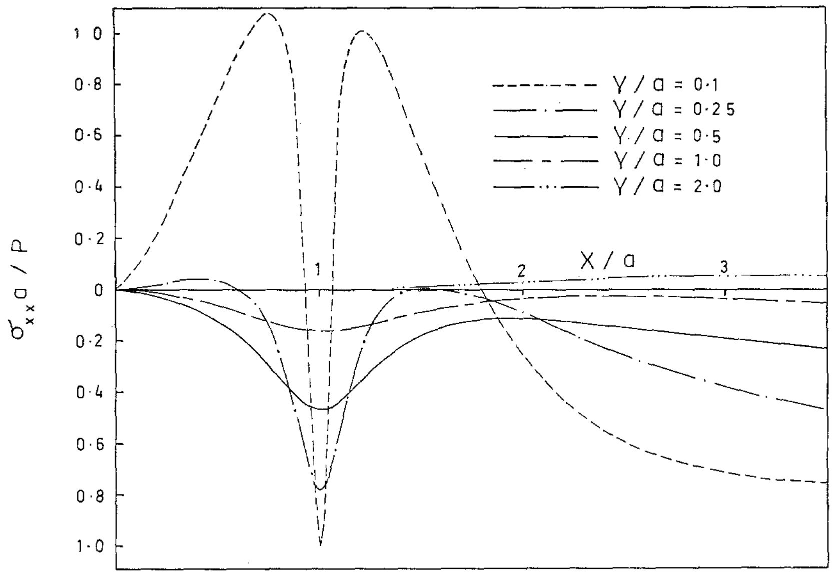

Fig. 4. Surface force – Variation of σ_{xx} with X/a – Graphite-epoxy composite.

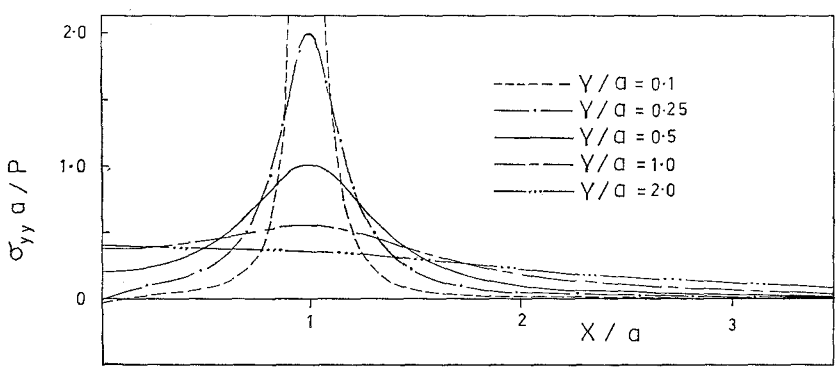

Fig. 5. Surface force – Variation of σ_{yy} with X/a – Graphite-epoxy composite.

of the stresses is monotonic and the presence of material orthotropy tends to alter the shape and magnitude of these distributions. Similar effects are evident in the graphs for the shear stresses σ_{xy}. The normal stresses σ_{xx} on the other hand possess a double maxima in their variation along the X-direction. This effect becomes more pronounced as the degree of orthotropy c_{22}/c_{11} increases.

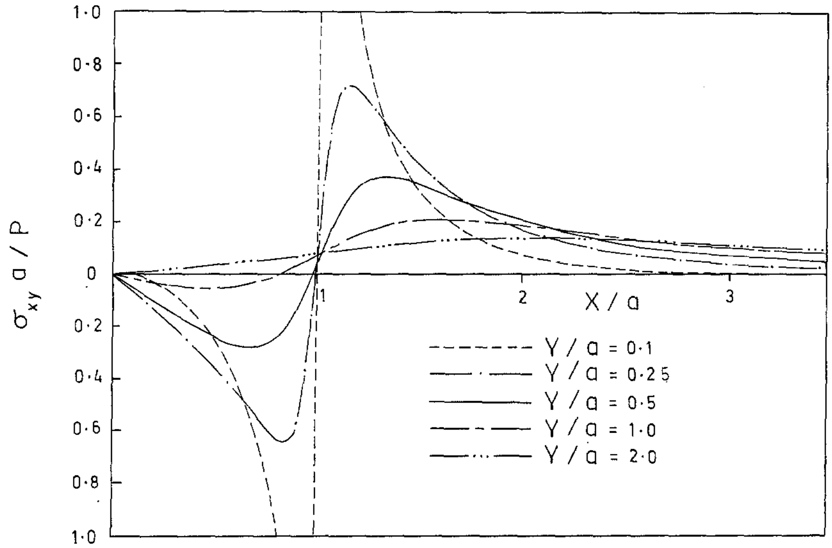

Fig. 6. Surface force – Variation of σ_{xy} with X/a – Graphite-epoxy composite.

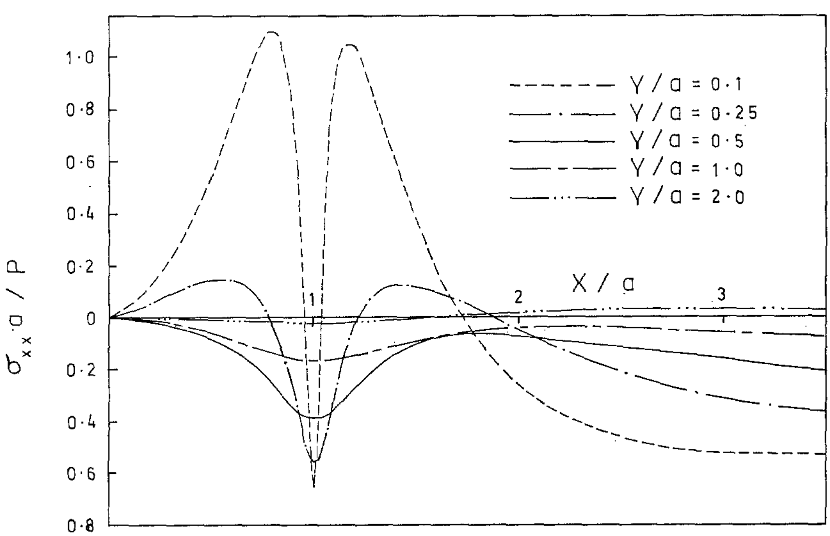

Fig. 7. Surface force – Variation of σ_{xx} with X/a – Boron-epoxy composite.

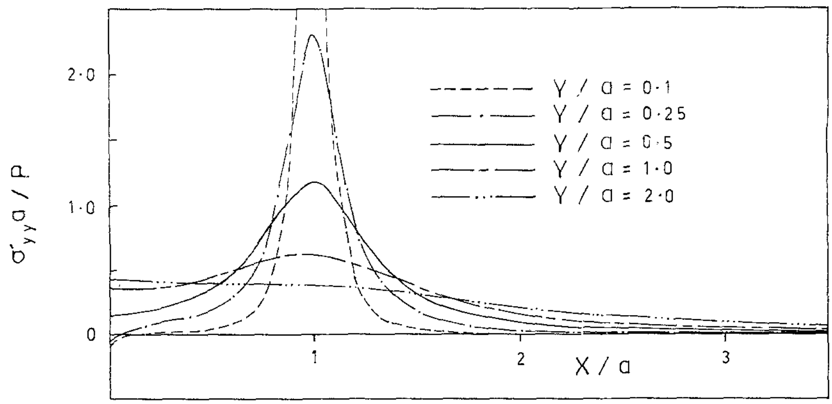

Fig. 8. Surface force – Variation of σ_{yy} with X/a – Boron-epoxy composite.

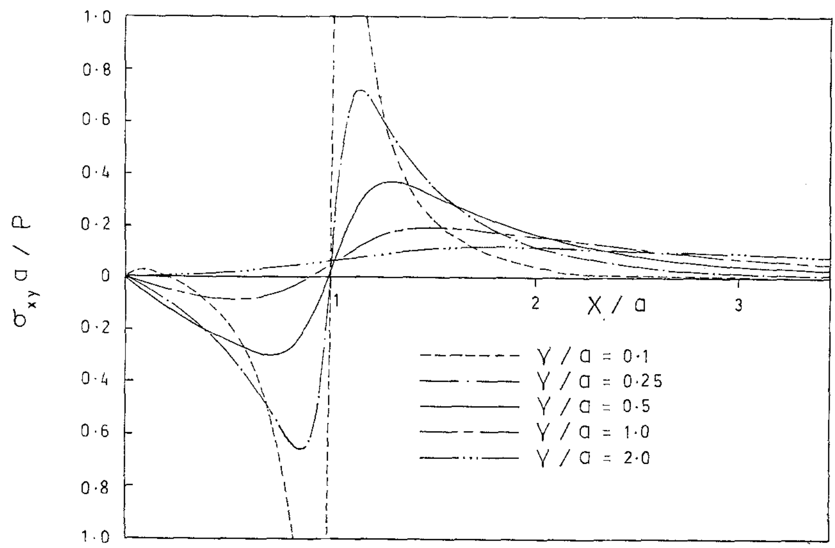

Fig. 9. Surface force – Variation of σ_{xy} with X/a – Boron-epoxy composite.

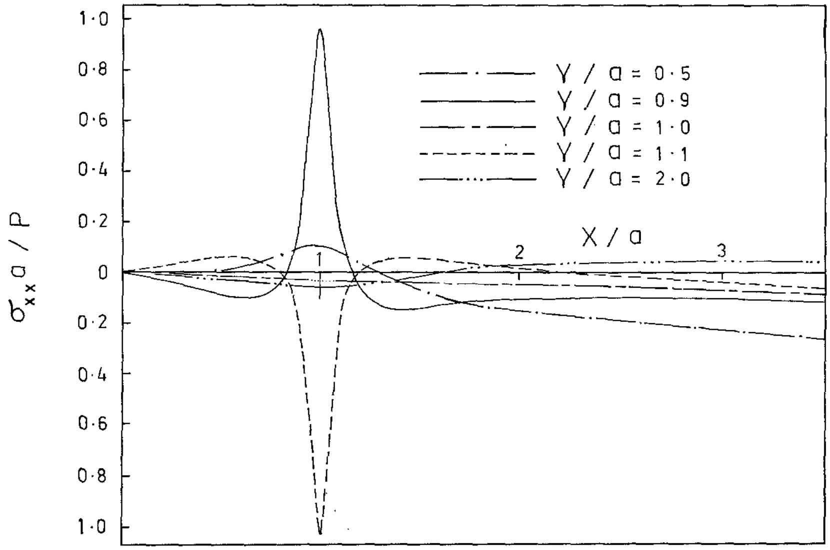

Fig. 10. Interior force – Variation of σ_{xx} with X/a – Graphite-epoxy composite.

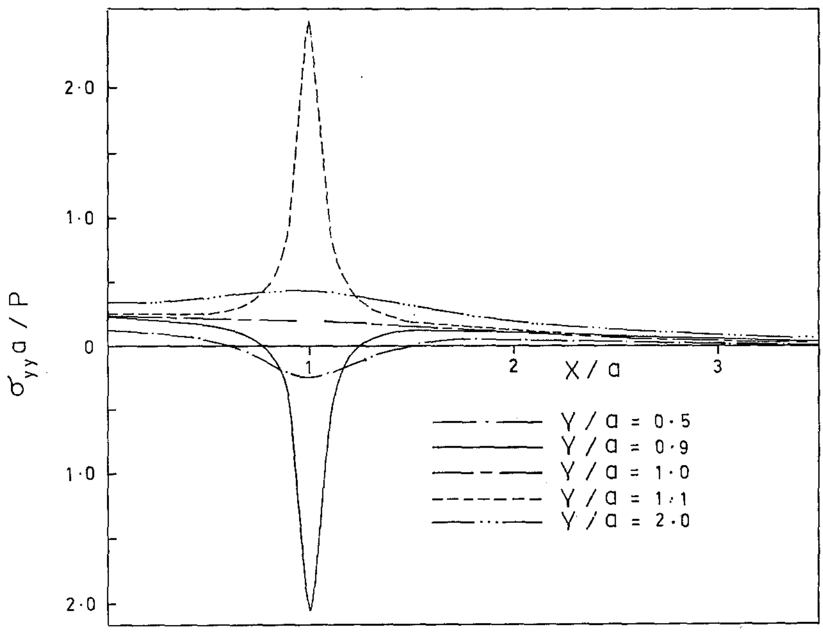

Fig. 11. Interior force – Variation of σ_{yy} with X/a – Graphite-epoxy composite.

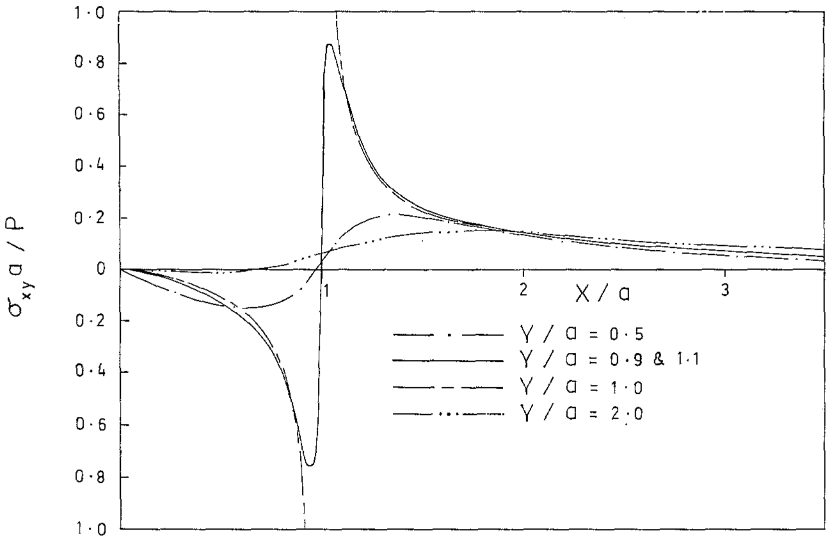

Fig. 12. Interior force – Variation of σ_{xy} with X/a – Graphite-epoxy composite.

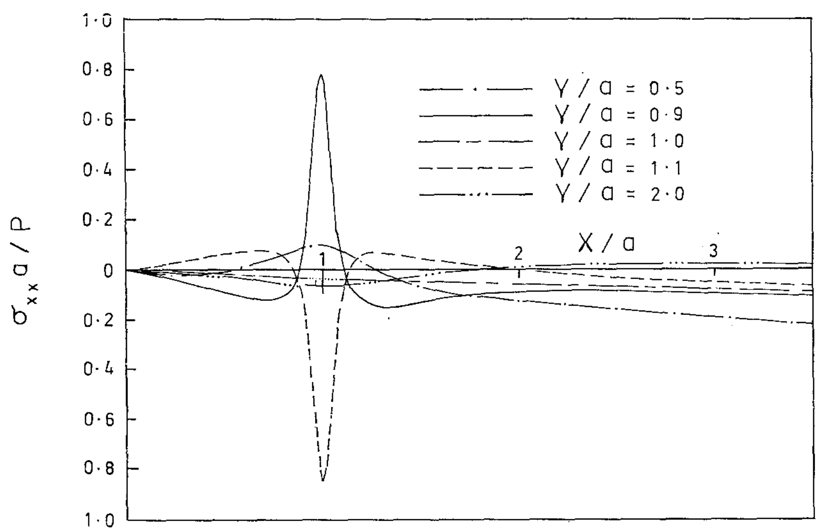

Fig. 13. Interior force – Variation of σ_{xx} with X/a – Boron-epoxy composite.

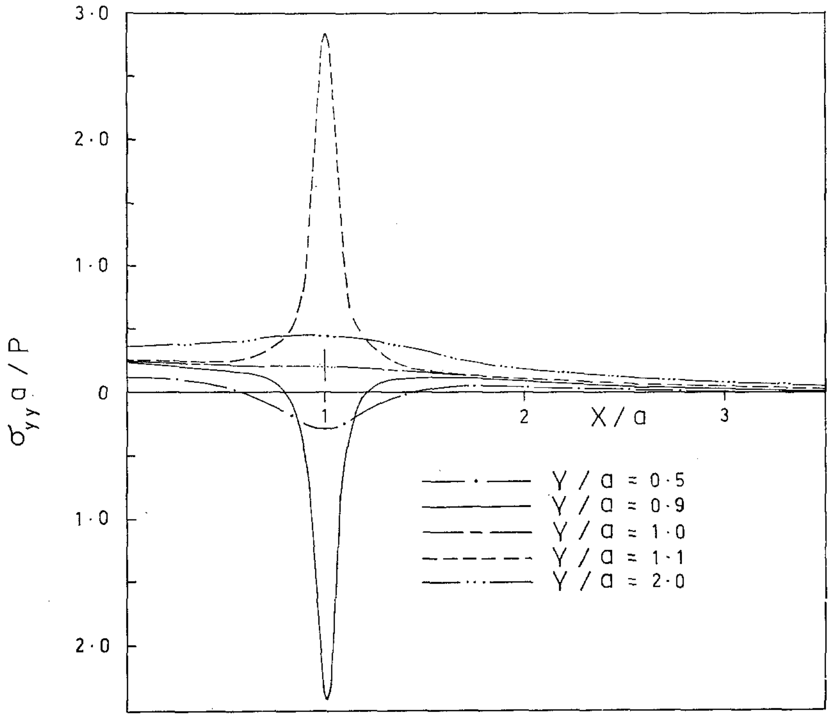

Fig. 14. Interior force – Variation of σ_{yy} with X/a – Boron-epoxy composite.

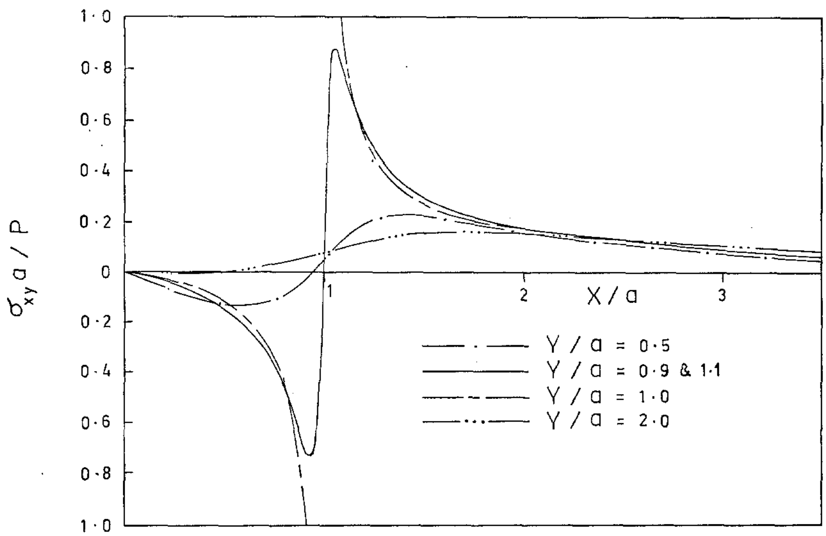

Fig. 15. Interior force – Variation of σ_{xy} with X/a – Boron-epoxy composite.

§ 5. Concluding remarks

The analysis of the orthotropic elastic quarter-plane presented in this paper examines the influence of material orthotropy on the state of stress in highly localized loaded corner regions of structural elements. The method of analysis described here is applicable to both surface and internal loading conditions of the quarter-plane for the particular instance where the spatial coordinate directions coincide with the principal elastic axes of the orthotropic material. The results obtained for the two concentrated force problems indicate that the degree of orthotropy of the material has a significant influence on both the magnitude and distribution of stresses within the quarter-plane.

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