THE ANNULAR CRACK PROBLEM FOR AN ISOTROPIC ELASTIC SOLID

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SUMMARY

This paper examines the problem of the internal loading of a flat annular crack located in an isotropic elastic solid. The surfaces of the annular crack are subjected to a uniform pressure. The analysis of the problem can be reduced to the solution of a system of triple integral equations. An approximate solution of this system of integral equations is used to generate the stress-intensity factors at the boundaries of the crack. The numerical results for the stress-intensity factors derived from the approximate solution are compared with equivalent results reported in the current literature.

1. Introduction

THE stress analysis of plane cracks located in elastic media is of interest to the examination of fracture and failure initiation in brittle solids. Considerable efforts have therefore been made in the investigation of three-dimensional problems involving plane cracks with a penny-shaped or elliptical planform (see e.g. Sneddon and Lowengrub (**1**), Liebowitz (**2**), Sih (**3**), Cherepanov (**4**)). The plane annular crack or the flat toroidal crack is also a defect which can be encountered at bonded material interfaces or homogeneous elastic solids. This paper considers the stress analysis of a flat annular crack which is located in an isotropic elastic solid. The surfaces of the crack are subjected to uniform pressure (Fig. 1). Several investigators have examined the plane annular crack problem by using diverse analytical techniques. Grinchenko and Ulitko (**5**) employ approximate methods for the solution of the annular crack problem. The analysis presented by Smetanin (**6**) uses an asymptotic method to solve the problem of an infinite space containing a flat toroidal crack, which is subjected to uniaxial tension. Moss and Kobayashi (**7**) have employed the solution technique proposed by Mossakovski and Rybka (**8**) to develop iterative approximate solutions for the stress-intensity factors at the crack boundaries. The analysis of the annular crack problem presented by Shibuya *et al.* (**9**) employs a technique whereby the governing integral equations are reduced to the solution of an infinite system of algebraic equations. Recently Choi and Shield (**10**) have presented a compact analysis of the problem of an annular crack located in an elastic solid which is subject to torsional and axial loads. These authors use Betti's reciprocal theorem to derive the integral equations governing the

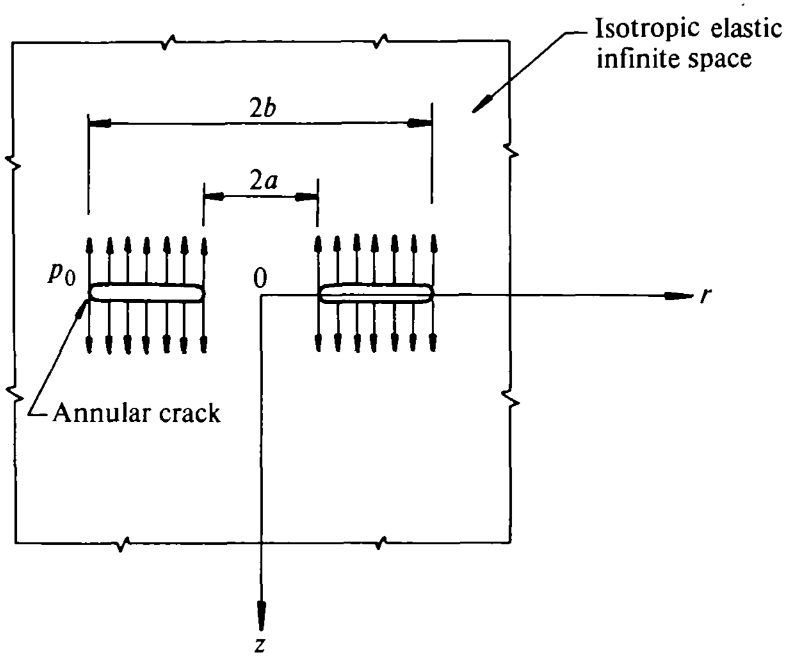

FIG. 1. Geometry of the annular inclusion

plane annular crack. The analysis provides an estimate of the accuracy of the solutions developed by Smetanin (6) and Moss and Kobayashi (7). Finally, the class of problems related to the stress analysis of ring-shaped cavities in elastic solids which are subjected to torsion and internal pressure are examined by Erdogan (11), Andreikev and Panasyuk (12) and Kanwal and Pasha (13).

This paper re-examines the problem related to the internal loading of an annular crack located in an isotropic elastic solid. The analysis develops a series solution for the system of triple integral equations governing the problem. The solutions developed for the stress-intensity factors at the inner and outer boundaries of the annular crack are expressed in terms of a series involving a non-dimensional parameter. This non-dimensional parameter corresponds to the ratio of the inner to the outer radius of the annular crack. The paper also presents a comparison between the stress-intensity factors derived from the approximate method of analysis and equivalent results presented in the current literature.

2. Formulation of the problem

For the solution of the axisymmetric problem related to an annular plane crack we employ the strain potential function approach of Love (14). Briefly, the solution of the displacement equations of equilibrium, for an elastic medium free of body forces, can be represented in terms of a single function $\Phi(r, z)$ which satisfies the equation

$$\nabla^2 \nabla^2 \Phi(r, z) = 0, \tag{1}$$

where ∇^2 is Laplace's operator referred to the cylindrical polar coordinate system. The displacement and stress components relevant to the analysis of the crack problem can be expressed in terms of $\Phi(r, z)$ in the following forms:

$$2Gu_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}, \tag{2}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right], \tag{3}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right], \tag{4}$$

where G and ν are the linear elastic shear modulus and Poisson's ratio respectively. We examine the annular crack problem in which the surfaces of the crack are subjected to a uniform compressive stress p_0. Owing to the symmetry of the problem about the plane $z = 0$ we can restrict our attention to the analysis of a single half-space region $(z \geqslant 0)$ of the infinite space. Referring to the half-space region occupying $z \geqslant 0$, the mixed boundary conditions relevant to the annular crack problem are

$$u_z(r, 0) = 0, \qquad 0 \leqslant r \leqslant a, \tag{5}$$

$$\sigma_{zz}(r, 0) = -p_0, \qquad a < r < b, \tag{6}$$

$$u_z(r, 0) = 0, \qquad b \leqslant r \leqslant \infty, \tag{7}$$

$$\sigma_{rz}(r, 0) = 0, \qquad r \geqslant 0. \tag{8}$$

In addition, the stresses and displacements derived from $\Phi(r, z)$ should reduce to zero as $(r, z) \to \infty$. For the integral-equation formulation of the annular crack problem we seek solutions of (1) which can be obtained from a Hankel transform development of this equation. The relevant solution of $\Phi(r, z)$ takes the form

$$\Phi(r, z) = \int_0^\infty [A(\xi) + zB(\xi)]e^{-\xi z} J_0(\xi r)\, d\xi, \tag{9}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions. For convenience we define the nth order Hankel operator as

$$H_n[f(\xi); r] = \int_0^\infty \xi f(\xi) J_n(\xi r)\, d\xi. \tag{10}$$

Using (5) to (8) and (9) it can be shown that the mixed boundary conditions reduce to a system of triple integral equations for a single unknown function

$R(\xi)$. (The functions $A(\xi)$ and $B(\xi)$ can be represented in terms of $R(\xi)$.) Thus

$$H_0[\xi^{-1}R(\xi); r] = 0, \qquad 0 \leqslant r \leqslant a, \tag{11}$$

$$H_0[R(\xi); r] = f(r), \qquad a < r < b, \tag{12}$$

$$H_0[\xi^{-1}R(\xi); r] = 0, \qquad b \leqslant r \leqslant \infty, \tag{13}$$

where $f(r) = -p_0$. The analysis of the internally pressurized annular crack is thus reduced to the solution of the system of triple integral equations (11) to (13).

3. Solution of the triple integral equations

Three-part boundary-value problems encountered in applied mathematics can be solved by employing a variety of approximate techniques. The methods outlined by Williams (**15**), Cooke (**16**), Noble (**17**), Tranter (**18**), Collins (**19**) and Jain and Kanwal (**20**) essentially reduce the three-part boundary-value problem to the solution of a Fredholm-type integral equation. Alternative techniques are presented in the articles (**6** to **9**). Detailed expositions of analytical techniques that can be adopted for the solution of triple integral equations are given by Sneddon (**21**) and Kanwal (**22**). In the present paper we shall adopt a method of analysis which is an extension of the basic techniques proposed by Cooke (**16**).

We assume that (12) admits a representation of the form

$$H_0[R(\xi); r] = \begin{cases} f_1(r), & 0 \leqslant r < a, \quad (14) \\ f_2(r), & b < r \leqslant \infty. \quad (15) \end{cases}$$

From the Hankel inversion theorem we obtain

$$R(\xi) = \int_0^a \lambda f_1(\lambda) J_0(\xi\lambda)\, d\lambda + \int_a^b \lambda f(\lambda) J_0(\xi\lambda)\, d\lambda + \int_b^\infty \lambda f_2(\lambda) J_0(\xi\lambda)\, d\lambda. \tag{16}$$

Substituting the value of $R(\xi)$ given by (16) into (11) and (13) we have

$$\int_0^a \lambda f_1(\lambda) L(\lambda, r)\, d\lambda + \int_a^b \lambda f(\lambda) L(\lambda, r)\, d\lambda + \int_b^\infty \lambda f_2(\lambda) L(\lambda, r)\, d\lambda = 0,$$
$$0 \leqslant r < a, \quad b < r \leqslant \infty, \tag{17}$$

where $L(\lambda, r) = \int_0^\infty J_0(\xi\lambda) J_0(\xi r)\, d\xi$. Using the representations

$$\int_s^a \frac{\lambda f_1(\lambda)\, d\lambda}{(\lambda^2 - s^2)^{\frac{1}{2}}} = F_1(s), \qquad 0 \leqslant s < a, \tag{18}$$

$$\int_b^s \frac{\lambda f_2(\lambda)\, d\lambda}{(s^2 - \lambda^2)^{\frac{1}{2}}} = F_2(s), \qquad b < s \leqslant \infty, \tag{19}$$

it can be shown that the functions $F_1(s)$ and $F_2(s)$ satisfy the system of coupled integral equations

$$\int_0^r \left[F_1(s)+\int_a^b \frac{\lambda f(\lambda)\,d\lambda}{(\lambda^2-s^2)^{\frac{1}{2}}}\right]\frac{ds}{(r^2-s^2)^{\frac{1}{2}}} = -\int_b^\infty \frac{ds}{(s^2-r^2)^{\frac{1}{2}}}\int_b^s \frac{\lambda F_2(\lambda)\,d\lambda}{(s^2-\lambda^2)^{\frac{1}{2}}}, \qquad 0\le r<a, \tag{20}$$

and

$$\int_r^\infty \left[F_2(s)+\int_a^b \frac{\lambda f(\lambda)\,d\lambda}{(s^2-\lambda^2)^{\frac{1}{2}}}\right]\frac{ds}{(s^2-r^2)^{\frac{1}{2}}} = -\int_0^a \frac{F_1(s)\,ds}{(r^2-s^2)^{\frac{1}{2}}}, \qquad b<r\le\infty. \tag{21}$$

By employing the series representations

$$F_1(s)=bp_0A(s_1)=bp_0\sum_{n=0}^\infty c^nA_n(s_1), \tag{22}$$

$$F_2(s)=bp_0B(s_1)=bp_0\sum_{n=0}^\infty c^nB_n(s_1), \tag{23}$$

where $c=a/b$ and $s_1=s/a$, the coupled system of integral equations (20) and (21) can be solved for $A_n(s_1)$ and $B_n(s_1)$. The details of the analysis will not be pursued here: we shall record only the final results for $A(s_1)$ and $B(s_1)$ (see also the Appendix). We have

$$\begin{aligned} A(s_1) = {} & -\frac{2}{\pi}+c\left\{(1-s_1^2)^{\frac{1}{2}}-\frac{8}{\pi^3}\right\}-c^2\left\{\frac{2}{\pi}\left(\frac{16}{\pi^4}-s_1^2\right)\right\} \\ & -c^3\left\{\frac{2}{\pi}\left(\frac{1}{24}-\frac{8}{9\pi^2}+\frac{64}{\pi^6}+\frac{4s_1^2}{3\pi^3}\right)\right\}+ \\ & +c^4\left\{-\frac{2s_1^4}{3\pi}-\frac{32s_1^2}{3\pi^5}-\frac{8}{\pi^3}\left(\frac{1}{24}-\frac{8}{9\pi^3}+\frac{64}{\pi^6}+\frac{8}{9\pi^2}\right)\right\}+ \\ & +c^5\left\{-\frac{2}{\pi}\left[\frac{2}{\pi}\left\{\frac{64}{9\pi^5}+\frac{4}{25\pi}+\frac{8}{\pi^3}\left(\frac{1}{24}+\frac{64}{\pi^6}+\frac{8}{9\pi^2}-\frac{8}{9\pi^3}\right)\right\}+\right.\right. \\ & \left.\left. +s_1^2\left\{\frac{1}{40}+\frac{2}{3\pi}\left(\frac{32}{\pi^5}-\frac{4}{15\pi^2}\right)\right\}+\frac{4s_1^2}{5\pi^2}\right]\right\}+O(c^6) \end{aligned} \tag{24}$$

and

$$\begin{aligned} B(s_1) = {} & (s_1^2-1)^{\frac{1}{2}}-s_1+c\left\{\left(\frac{4}{\pi^2 s_1}\right)+c^2\left(\frac{16}{\pi^4 s_1}\right)+c^3\left\{\frac{1}{8s_1^3}+\frac{2}{\pi^2 s_1}\left(-\frac{2}{3}+\frac{2}{3s_1^2}+\frac{32}{\pi^4}\right)\right\}\right. \\ & +c^4\left\{\frac{16}{3\pi^4 s_1^3}+\frac{4}{\pi^2 s_1}\left(\frac{1}{24}-\frac{8}{9\pi^2}+\frac{64}{\pi^6}+\frac{4}{9\pi^3}\right)\right\}+ \end{aligned}$$

$$+c^5\left\{\frac{4}{5\pi^2 s_1^5}+\frac{4}{\pi^2 s_1^3}\left(\frac{16}{3\pi^4}-\frac{1}{5}\right)+\frac{4}{15\pi^2 s_1}+\frac{64}{9\pi^6 s_1}+\right.$$
$$\left.+\frac{16}{\pi^4 s_1}\left(\frac{1}{24}+\frac{64}{\pi^6}-\frac{8}{9\pi^3}+\frac{8}{9\pi^2}\right)\right\}+O(c^6). \quad (25)$$

Using the solution of the Abel integral equation it can be shown that

$$f_1(\lambda_1 a)=-\frac{2p_0}{\pi\lambda_1 c}\frac{d}{d\lambda_1}\int_{\lambda_1}^{1}\frac{s_1 A(s_1)\,ds_1}{(s_1^2-\lambda_1^2)^{\frac{1}{2}}}, \qquad 0\leqslant\lambda_1<1, \quad (26)$$

$$f_2(\lambda_1 b)=\frac{2p_0}{\pi\lambda_1}\frac{d}{d\lambda_1}\int_{1}^{\lambda_1}\frac{s_1 B(s_1)\,ds_1}{(\lambda_1^2-s_1^2)^{\frac{1}{2}}}, \qquad 1<\lambda\leqslant\infty. \quad (27)$$

The stress-intensity factors at the boundaries of the annular crack are defined by

$$K_a=\lim_{r\to a^-}[2(a-r)]^{\frac{1}{2}}\sigma_{zz}(r,0), \quad (28)$$

$$K_b=\lim_{r\to b^+}[2(r-b)]^{\frac{1}{2}}\sigma_{zz}(r,0). \quad (29)$$

Using the decompositions (14) and (15) it can be shown that

$$K_a=\lim_{\lambda_1\to 1^-}\sqrt{2a}\,[1-\lambda_1]^{\frac{1}{2}}f_1(\lambda_1 a) \quad (30)$$

and

$$K_b=\lim_{\lambda_1\to 1^+}\sqrt{2b}\,[\lambda_1-1]^{\frac{1}{2}}f_2(\lambda_1 b). \quad (31)$$

Evaluating the above expressions we obtain the following series expansions for the stress-intensity factors at the boundary of the annular crack:

$$\frac{K_a}{p_0\sqrt{b}}=\frac{4}{\pi^2\sqrt{c}}\left[1+\frac{4c}{\pi^2}+c^2\left(\frac{16}{\pi^4}-1\right)+c^3\left(\frac{1}{24}-\frac{8}{9\pi^2}+\frac{4}{3\pi^3}+\frac{64}{\pi^6}\right)+\right.$$
$$+c^4\left\{\frac{1}{3}+\frac{16}{3\pi^4}+\frac{4}{\pi^2}\left(\frac{1}{24}-\frac{8}{9\pi^3}+\frac{64}{\pi^6}+\frac{8}{9\pi^2}\right)\right\}+$$
$$\left.+c^5\left\{\frac{1}{40}+\frac{16}{\pi^4}\left(\frac{1}{24}+\frac{64}{\pi^6}+\frac{8}{9\pi^2}-\frac{8}{9\pi^3}\right)+\frac{212}{225\pi^2}+\frac{310}{9\pi^6}\right\}+O(c^6)\right], \quad (32)$$

$$\frac{K_b}{p_0\sqrt{b}}=\frac{2}{\pi}\left[1-\frac{4c}{\pi^2}-\frac{16c^2}{\pi^4}-c^3\left(\frac{1}{8}+\frac{64}{\pi^6}\right)\right.$$
$$-c^4\left\{\frac{16}{3\pi^4}+\frac{4}{\pi^2}\left(\frac{1}{24}-\frac{8}{9\pi^2}+\frac{64}{\pi^6}+\frac{4}{9\pi^3}\right)\right\}$$
$$\left.-c^5\left\{\frac{16}{\pi^4}\left(\frac{1}{24}+\frac{64}{\pi^6}-\frac{8}{9\pi^3}+\frac{8}{9\pi^2}\right)+\frac{256}{9\pi^6}-\frac{4}{15\pi^2}\right\}+O(c^6)\right]. \quad (33)$$

The strain energy in the elastic medium due to the uniform pressurization of the annular crack is given by

$$W = p_0 \int_0^{2\pi} \int_a^b r u_z(r, 0)\, dr\, d\theta. \tag{34}$$

The results for $A(s_1)$ and $B(s_1)$ derived previously can be utilized to obtain a series expansion for W. Avoiding details of some rather lengthy algebraic manipulations it can be shown that the asymptotic expansion for the strain energy W is given by

$$\begin{aligned} W = \frac{8(1-\nu^2)p_0^2 b^3}{3E} \Bigg[& 1 - \frac{12}{\pi^2} c + c^2 \left\{ \frac{3}{2}\left(1 - \frac{\pi}{2}\right) - \frac{48}{\pi^4} \right\} \\ & - c^3 \left\{ \left(\frac{\pi}{4} - 1\right)\left(\frac{1}{8} + \frac{4}{3\pi^2}\right) + \frac{4}{\pi^2}\left(1 - \frac{\pi}{2}\right)\left(\frac{16}{\pi^5} - \frac{1}{5}\right) + 3\left(\frac{32}{\pi^5} - \frac{1}{\pi}\right) \right\} \\ & - 3c^4 \left\{ \left(\frac{\pi}{4} - 1\right)\left(\frac{16}{3\pi^4} + \frac{1}{8}\right) + \frac{1}{8} - \frac{\pi}{32} + \frac{4}{\pi^2}\left(\frac{1}{24} - \frac{8}{9\pi^2} + \frac{64}{\pi^6} + \frac{4}{9\pi^3}\right) \right\} + \\ & + 3c^5 \left\{ -\frac{8}{\pi^3}\left(\frac{1}{24} - \frac{8}{9\pi^3} + \frac{64}{\pi^6} + \frac{8}{9\pi^2}\right) + \frac{1}{2}\left(\frac{1}{24} - \frac{8}{9\pi^2} + \frac{64}{\pi^6}\right) + \right. \\ & + \frac{16}{9\pi^5} - \frac{17}{60\pi} + \frac{1}{6\pi^3} - \frac{4}{5\pi^2}\left(\frac{\pi}{16} - \frac{1}{3}\right) - \frac{4}{\pi^2}\left(\frac{\pi}{4} - 1\right)\left(\frac{16}{3\pi^4} - \frac{1}{5}\right) \\ & - \left(1 - \frac{\pi}{2}\right)\left[\left(\frac{4}{15\pi^2} + \frac{64}{9\pi^6}\right) + \frac{16}{\pi^4}\left(\frac{1}{24} + \frac{64}{\pi^6} - \frac{8}{9\pi^3} + \frac{8}{9\pi^2}\right)\right] \\ & \left. - \left[\frac{1}{48} + \frac{1}{\pi^2}\left(\frac{32}{\pi^4} - \frac{4}{9}\right)\right] - \frac{1}{6\pi^2} \right\} + O(c^6) \Bigg]. \end{aligned} \tag{35}$$

This formally completes the analysis of the internally loaded flat annular crack problem. The state of stress in the elastic medium can be obtained by making use of the solutions for $F_1(s)$ and $F_2(s)$ derived previously.

4. Numerical results

In the preceding sections we have developed approximate solutions for the stress-intensity factors at the boundaries of a flat annular crack which is subjected to uniform internal pressure p_0. The approximate nature of the solution stems from the techniques which are employed to solve the system of triple integral equations governing the crack problem. In particular, the solutions for the stress-intensity factors are expressed as power series in terms of the parameter c $(=a/b) \ll 1$.

In the limiting case when the inner radius of the crack a tends to zero, the result (33) yields

$$K_b = 2p_0\sqrt{b/\pi}, \tag{36}$$

which corresponds to the stress-intensity factor for a penny-shaped crack of radius b which is subjected to internal pressure p_0 (see e.g. Kassir and Sih (**23**)).

Also in the limit as $b \to \infty$, the analysis should yield the stress-intensity factor at the boundary of an external circular crack of radius a. In the limit when $b \to \infty$, it can be shown that

$$F_1(s) = -\int_a^\infty \frac{\lambda f(\lambda)\, ds}{(\lambda^2 - s^2)^{\frac{1}{2}}}, \qquad a < s. \tag{37}$$

Using the solution of the Abel integral equation we note that

$$f_1(\lambda) = -\frac{2}{\pi\lambda}\frac{d}{d\lambda}\int_\lambda^a \frac{sF_1(s)\, ds}{(s^2 - \lambda^2)^{\frac{1}{2}}}, \qquad \lambda < a, \tag{38}$$

$$f_2(\lambda) = \frac{2}{\pi\lambda}\frac{d}{d\lambda}\int_b^\lambda \frac{sF_2(s)\, ds}{(\lambda^2 - s^2)^{\frac{1}{2}}}, \qquad b < \lambda. \tag{39}$$

Substituting (37) in (38) and making use of the result

$$\frac{d}{d\lambda}\int_\lambda^a \frac{s\, ds}{(s^2 - \lambda^2)(u^2 - s^2)^{\frac{1}{2}}} = \frac{-a(u^2 - a^2)^{\frac{1}{2}}}{(u^2 - \lambda^2)(a^2 - \lambda^2)^{\frac{1}{2}}} \tag{40}$$

we find that

$$f_1(\lambda) = \frac{2a}{\pi\lambda(\lambda^2 - a^2)^{\frac{1}{2}}}\int_a^\infty \frac{u(u^2 - a^2)^{\frac{1}{2}} f(u)\, du}{(u^2 - \lambda^2)}. \tag{41}$$

Using (14), (39) and (41) it can be shown that

$$K_a = \frac{2}{\pi\sqrt{a}}\int_a^\infty \frac{uf(u)\, du}{(u^2 - a^2)^{\frac{1}{2}}}. \tag{42}$$

The result (42) is identical to the general expression derived by Kassir and Sih (**23**) for the stress-intensity factor for an external circular crack. It must, however, be noted that since

$$I = 2\pi\int_a^\infty rf(r)\, dr \tag{43}$$

is divergent the stress-intensity factor at the crack boundary is also divergent.

TABLE 1. *Stress-intensity factors for the uniformly loaded flat annular crack*

Reference	$K_a/p_0\sqrt{b}$		$K_b/p_0\sqrt{b}$	
	$c = 0{\cdot}06948$	$c = 0{\cdot}36788$	$c = 0{\cdot}06948$	$c = 0{\cdot}36788$
Moss and Kobayashi (**7**)	0·863	0·742	0·618	0·543
Smetanin (**6**)	1·574	0·697	0·618	0·526
Choi and Shield (**10**)	1·575	0·693	0·618	0·525
Present study	1·574	0·699	0·618	0·525

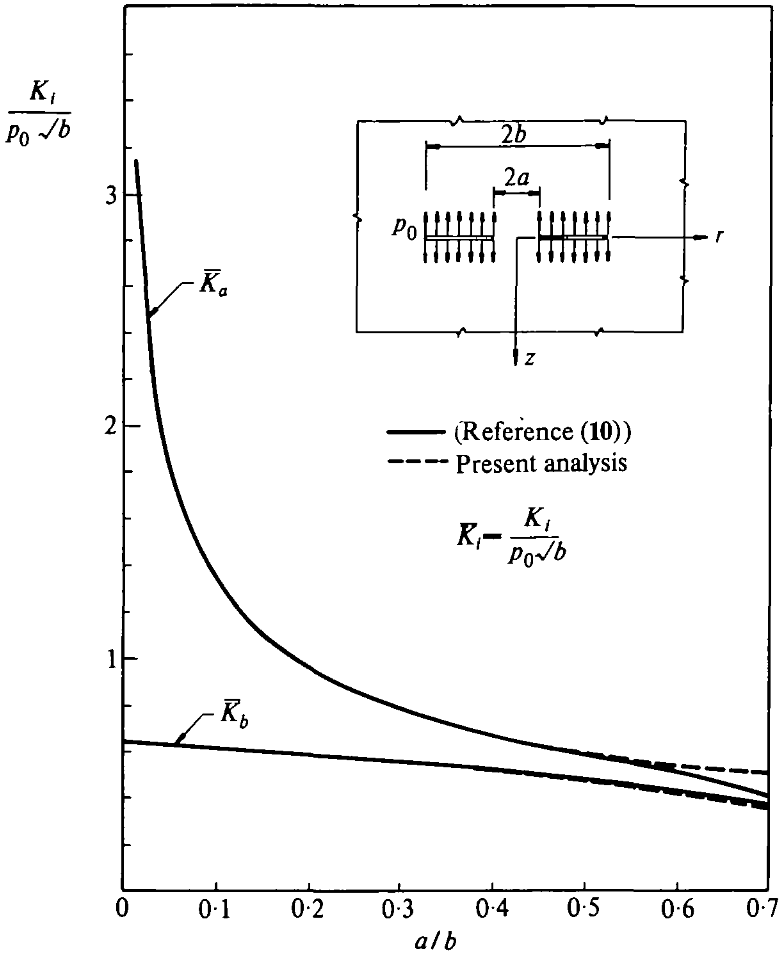

FIG. 2

The accuracy of the series solutions for the stress-intensity factors given by (32) and (33) can be ascertained by comparing the results of the present analysis with equivalent results given by Choi and Shield (**10**). These authors have also examined the accuracy of the results given by Smetanin (**6**) and Moss and Kobayashi (**7**) with the results derived from the application of the M-integral conservation law. Table 1 shows a comparison of the various analytical estimates for two specific values of c. It is evident that the results of the present investigation compare well with the results given by Smetanin (**6**) and Choi and Shield (**10**). Again, as has been observed by Choi and Shield (**10**), the agreement with the results derived from the solution given by Moss and Kobayashi (**7**) is somewhat poor especially in the estimation of

K_a. The maximum variation between the results derived from (**6**) and (**10**) and the present analysis is less than one per cent.

Figure 2 shows the comparison between the results for the stress-intensity factors derived from the present analysis and equivalent results given by Choi and Shield (**10**). As is evident the results show excellent agreement for annular crack geometries in the range $c \in (0, 0{\cdot}5)$. As c becomes greater than 0·5 the contributions from the higher-order terms become important.

REFERENCES

1. I. N. Sneddon and M. Lowengrub, *Crack Problems in the Classical Theory of Elasticity* (John Wiley, New York, 1969).
2. H. Liebowitz (ed.), *Fracture*, Vols. I–IV (Academic Press, New York, 1968).
3. G. C. Sih (ed.), *Mechanics of Fracture*, Vols I-V (Noordhoff International, Leyden, 1974).
4. G. P. Cherepanov, *Mechanics of Brittle Fracture* (translation edited by R. deWit and W. C. Cooley, McGraw-Hill, New York, 1979).
5. V. T. Grinchenko and A. F. Ulitko, *Prikl. Mekh.* **1** (1965) 265–270.
6. B. I. Smetanin, *Prikl. Math. Mekh.* **32** (1967) 461–466.
7. L. W. Moss and A. S. Kobayashi, *Int. J. Fracture Mechanics* **7** (1971) 89–99.
8. V. I. Mossakovskii and M. T. Rybka, *Prikl. Math. Mekh.* **28** (1964) 1061–1069.
9. T. Shibuya, I. Nakahara and T. Koizumi, *J. appl. Math. Mech.* (*ZAMM*) **55** (1975) 395–402.
10. I. Choi and R. T. Shield, *Int. J. Solids Struct.* **18** (1982) 479–486.
11. F. Erdogan, *J. appl. Mech.* **87** (1965) 829–836.
12. A. E. Andreikev and V. V. Panasyuk, *Prikl. Mekh.* **6** (1971) 124–128.
13. R. P. Kanwal and M. L. Pasha, *J. appl. Mech.* **41** (1974) 516–517.
14. A. E. H. Love, *A Treatise on the Mathematical Theory of Elasticity*, 4th edition (Dover, New York, 1944).
15. W. E. Williams, *Proc. Edin. math. Soc.* (2) **13** (1963) 317–323.
16. J. C. Cooke, *Q. Jl Mech. appl. Math.* **16** (1963) 193–203.
17. B. Noble, *J. Math. Phys.* **37** (1958) 128–136.
18. C. J. Tranter, *Proc. Glasgow math. Assoc.* **4** (1960) 200–203.
19. W. D. Collins, *Proc. Edin. math. Soc.* (2) **13** (1963) 235–246.
20. D. L. Jain and R. P. Kanwal, *J. D'Analyse Math.* **25** (1972) 107–158.
21. I. N. Sneddon, *Mixed Boundary Value Problems in Potential Theory* (North-Holland, Amsterdam, 1966).
22. R. P. Kanwal, *Linear Integral Equations: Theory and Technique* (Academic Press, New York, 1971).
23. M. K. Kassir and G. C. Sih, Three-Dimensional Crack Problems, *Mechanics of Fracture*, Vol. 2 (ed. G. C. Sih, Noordhoff, Leyden, 1975).

APPENDIX

The functions $A_n(s_1)$ and $B_n(s_1)$ take the following forms:

$$A_0(s_1) = -1 - \frac{2}{\pi}\int_1^\infty [(u_1^2-1)^{\frac{1}{2}} - u_1]\frac{du_1}{u_1} = -\frac{2}{\pi},$$

$$B_0(s_1) = (s_1^2-1)^{\frac{1}{2}} - s_1,$$

$$A_1(s_1)=(1-s_1^2)^{\frac{1}{2}}-\frac{2}{\pi}\int_1^\infty \frac{B_1(u_1)\,du_1}{u_1}=(1-s_1^2)^{\frac{1}{2}}-\frac{8}{\pi^3},$$

$$B_1(s_1)=-\frac{2}{\pi s_1}\int_0^1 A_0(u_1)\,du_1=\frac{4}{\pi^2 s_1},$$

$$A_2(s_1)=\frac{s_1^2}{2}-\frac{2s_1^2}{\pi}\int_1^\infty \frac{B_0(u_1)\,du_1}{u_1^3}-\frac{2}{\pi}\int_1^\infty \frac{B_2(u_1)\,du_1}{u_1}=-\frac{2}{\pi}\left[\frac{16}{\pi^4}-s_1^2\right],$$

$$B_2(s_1)=\frac{1}{2s_1}-\frac{2}{\pi s_1}\int_0^1 A_1(u_1)\,du_1=\frac{16}{\pi^4 s_1},$$

$$A_3(s_1)=-\frac{2}{\pi}\int_1^\infty\left\{B_3(u_1)+\frac{s_1^2B_1(u_1)}{u_1^2}\right\}\frac{du_1}{u_1}=-\frac{2}{\pi}\left[\frac{1}{24}-\frac{8}{9\pi^2}+\frac{64}{\pi^6}+\frac{4s_1^2}{3\pi^3}\right],$$

$$B_3(s_1)=\frac{1}{8s_1^3}-\frac{2}{\pi s_1}\int_0^1\left\{\frac{u_1^2}{s_1^2}A_0(u_1)+A_2(u_1)\right\}du_1=\frac{1}{8s_1^3}+\frac{2}{\pi^2 s_1}\left[-\frac{2}{3}+\frac{2}{3s_1^2}+\frac{32}{\pi^4}\right],$$

$$\begin{aligned}A_4(s_1)&=\frac{s_1^4}{8}-\frac{2}{\pi}\int_1^\infty\left\{\frac{B_4(u_1)}{u_1}+\frac{s_1^2B_2(u_1)}{u_1^3}+\frac{s_1^4}{u_1^5}B_0(u_1)\right\}du_1\\&=-\frac{2s_1^4}{3\pi}-\frac{32s_1^2}{3\pi^5}-\frac{8}{\pi^3}\left(\frac{1}{24}-\frac{8}{9\pi^3}+\frac{64}{\pi^6}+\frac{8}{9\pi^2}\right),\end{aligned}$$

$$\begin{aligned}B_4(s_1)&=-\frac{2}{\pi s_1}\int_0^1 A_3(u_1)\,du_1+\frac{1}{8s_1^3}-\frac{2}{\pi s_1^3}\int_0^1 u_1^2A_1(u_1)\,du_1\\&=\frac{16}{3\pi^3 s_1^3}+\frac{4}{\pi^2 s_1}\left(\frac{1}{24}-\frac{8}{9\pi^2}+\frac{64}{\pi^6}+\frac{4}{9\pi^3}\right),\end{aligned}$$

$$\begin{aligned}A_5(s_1)&=-\frac{2}{\pi}\int_1^\infty\left[\frac{s_1^4B_1(u_1)}{u^5}+\frac{s_1^2B_3(u_1)}{u_1^3}+\frac{B_5(u_1)}{u_1}\right]du_1\\&=-\frac{2}{\pi}\left[\frac{2}{\pi}\left\{\frac{64}{9\pi^5}+\frac{4}{25\pi}+\frac{8}{\pi^3}\left(\frac{1}{24}+\frac{64}{\pi^6}-\frac{8}{9\pi^3}+\frac{8}{9\pi^2}\right)\right\}+\right.\\&\qquad\left.+s_1^2\left\{\frac{1}{40}+\frac{2}{3\pi}\left(\frac{32}{\pi^5}-\frac{4}{15\pi}\right)\right\}+\frac{4s_1^2}{5\pi^2}\right],\end{aligned}$$

$$\begin{aligned}B_5(s_1)&=-\frac{2}{\pi s_1}\left[\int_0^1\frac{u_1^4}{s_1^4}A_0(u_1)\,du_1+\int_0^1\frac{u_1^2}{s_1^2}A_2(u_1)\,du_1+\int_0^1 A_4(u_1)\,du_1\right]\\&=\frac{2}{\pi s_1}\left[\frac{2}{5\pi s_1^4}+\frac{2}{\pi s_1^2}\left(\frac{16}{3\pi^4}-\frac{1}{5}\right)+\frac{2}{15\pi}+\frac{32}{9\pi^5}+\right.\\&\qquad\left.+\frac{8}{\pi^3}\left\{\frac{1}{24}+\frac{64}{\pi^6}-\frac{8}{9\pi^3}+\frac{8}{9\pi^2}\right\}\right].\end{aligned}$$