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CIRCULAR RAFT FOUNDATION WITH A RESTRAINED BOUNDARY

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SYNOPSIS

The present paper examines the axisymmetric flexure of a circular raft foundation resting in smooth contact with an isotropic elastic soil mass. In particular, the periphery of the circular raft is fully or partially restrained against rotation due to the presence of an interacting cylindrical shell. Such a condition can occur in structural foundation systems used for nuclear reactor vessels or silos used for the storage of bulk solids and fluids. A solution to this idealized soil foundation interaction problem is obtained by using a variational method. Numerical results presented in this paper illustrate the manner in which the settlements, contact stresses and flexural moments in a circular raft can be influenced by the degree of restraint offered by an interacting cylindrical shell.

INTRODUCTION

The class of problems which examines the behaviour of circular foundations resting on elastic soil media has received considerable attention. Solutions developed for the circular foundation problem have useful application in the analysis and design of structural foundations resting on soil and rock masses. The classical problem relating to the axisymmetric interaction between a circular plate and an isotropic elastic halfspace was examined by Borowicka (1936) who employed a power series expansion technique to represent the deflected shape of the raft. The investigations of Ishkova (1951) and Brown (1969) modify the power series expansion technique to incorporate effects of a singularity in the contact stress distribution at the interface. A host of other numerical techniques, such as finite difference techniques, finite element methods, boundary element methods and other discretization techniques have also been applied to the analysis of this interaction problem. A comprehensive account of the axisymmetric problem relating to the interaction between the circular plate and an elastic halfspace is given by Selvadurai (1979).

In existing treatments of the interaction problems related to the circular foundation it is explicitly assumed that the edge of the circular foundation is free from any edge restraint that may be provided by the

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superstructure. In engineering structures such as nuclear reactor vessels, as indicated in Figure 1, or storage silos, it is conceivable that a certain degree of edge restraint may be provided by the continuity that exists between the cylindrical shell and the circular raft foundation. The main objective of this paper is to investigate the manner in which such edge restraint can influence the interaction between the circular foundation and the supporting elastic soil medium.

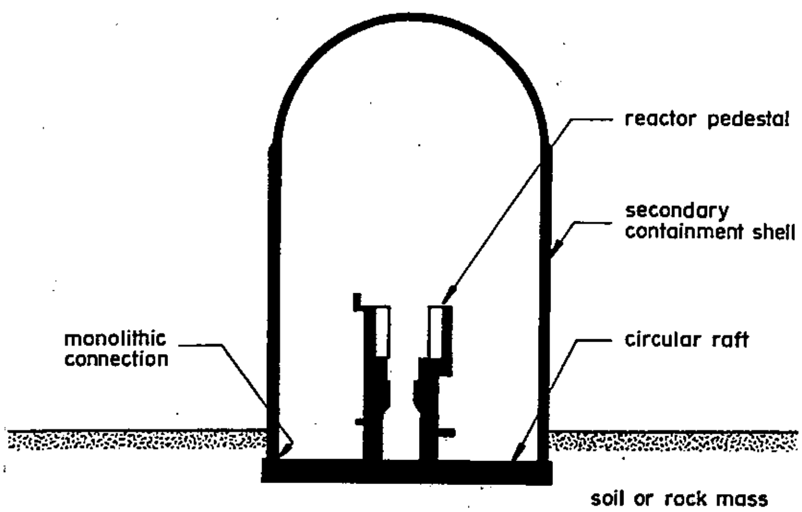

Fig. 1. Schematic cross section of a pressurized water reactor.

The method of analysis of the interaction problem employed in this paper is novel and presents a departure from the conventional procedures outlined in the references cited earlier. Basically the present paper employs an *energy method* for the analysis of interaction between the restrained circular raft and the isotropic elastic soil mass. The analysis assumes that the deflected shape of the restrained circular raft foundation can be represented in the form of a power series in terms of the radial coordinate r. This deflected shape is specified to within a set of undetermined constants and ordered in such a way that the kinematic constraints of the axisymmetric deformation and the conditions at the restrained edge are identically satisfied. Furthermore, the assumption of continuous smooth contact between the restrained raft and the soil medium ensures that this prescribed raft deflection corresponds to the surface deflection of the soil mass within the plate region. The energy method proposed here centres around the development of a *total potential energy functional* for the foundation-elastic soil mass system which consists of the following:

(i) the strain energy of the soil region,

(ii) the strain energy of the circular raft foundation,

(iii) the strain energy of the superstructure or shell element which provides the edge restraint,

(iv) the potential energy of the external loads.

The total potential energy functional thus developed is defined in terms of the constants which characterize the assumed form of the foundation deflection. These constants can be uniquely determined from the linearly independent equations generated from the minimization of the total potential energy functional. The general procedure outlined above is used to examine the flexural interaction of a circular foundation with an edge restraint resting on an isotropic elastic soil mass and subjected to uniform load of finite extent. The deflected shape of the foundation is represented by an even order polynomial in r (the radial coordinate) up to the sixth order. Using the energy method analytical solutions are developed for the deflected shape of the circular foundation, the contact stress at the interface and the central and edge moments in the foundation. These results depend on

(i) the relative rigidity of foundation—elastic soil mass system and

(ii) the degree of rotational restraint at the boundary.

It is observed that as the relative rigidity and edge restraint reduce to zero, the results of the energy solution compare accurately with the exact results for an isotropic elastic halfspace subjected to a flexible load. Similarly, Boussinesq's (1895) exact solution for the indentation of a halfspace by a rigid circular foundation occurs as a limiting case of the energy solution. The numerical results presented in this paper illustrate the manner in which the stiffness of the cylindrical superstructure can influence the flexural interaction effects in a restrained circular raft.

ANALYSIS OF THE INTERACTION PROBLEM

In order to examine the axisymmetric interaction of the circular raft foundation with a restrained boundary, resting in smooth contact with an isotropic elastic soil mass, it is convenient to idealize the structural system. Such an

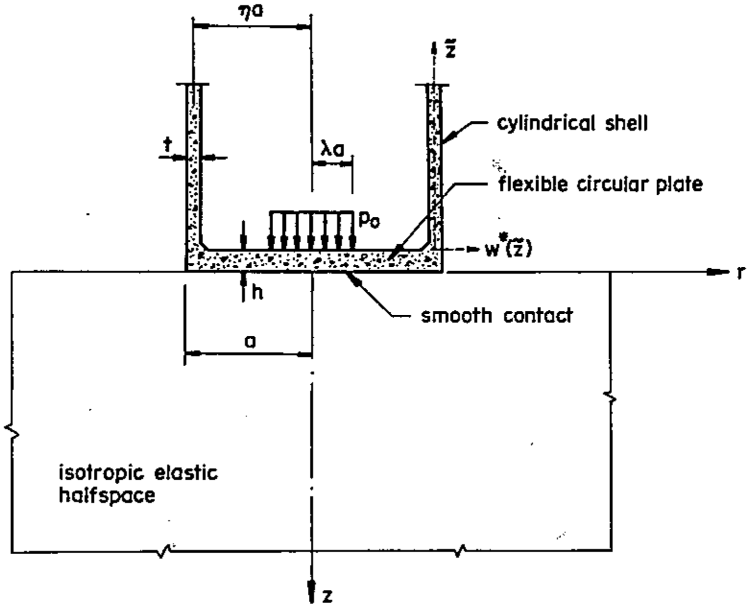

Fig. 2 (a). Geometry of the plate - shell system.

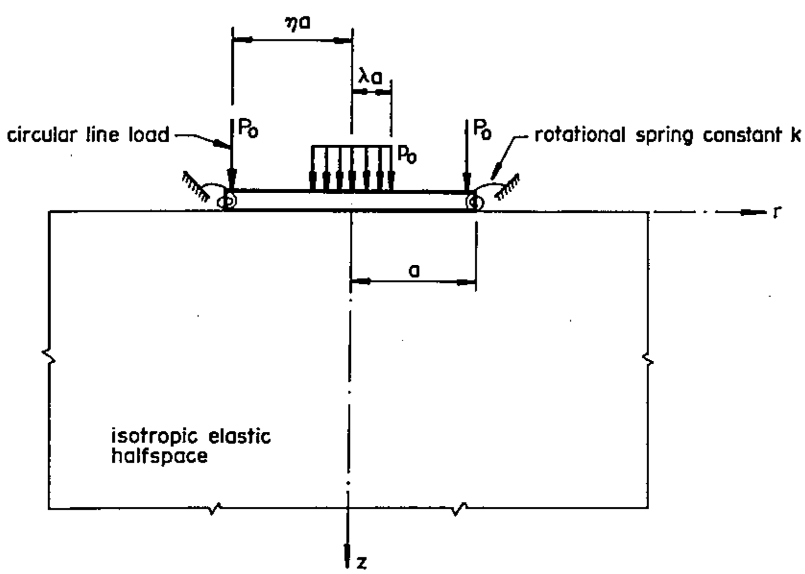

Fig. 2 (b). The circular plate — cylindrical shell system on an elastic halfspace.

idealized form is shown in Figure 2. The rotational stiffness element at the boundary of the foundation represents the interaction effect of the shell element. Similarly, the line load P_0 represents the weight of the shell element. The smooth contact at the raft-soil medium interface is assumed to be such that there is no loss of contact at the interface. The deflection of the circular foundation, w(r), therefore corresponds to the surface deflection of the elastic soil mass in the region $r \leq a$, where a is the radius of the plate. The deflection w(r) is prescribed in such a way that the kinematic constraints of the raft deformations and the soil deformations are satisfied. An expression for the total potential energy functional (appropriate to the idealized system shown in Figure 2) is developed by making use of w(r).

(a) *Flexural energy of the raft foundation*

The flexural behaviour of the raft foundation is described by the small deflection Poisson-Kirchhoff thin plate theory. The elastic energy of a thin circular plate subjected to an axisymmetric deflection w(r) is composed of only the flexural energy of the plate U_F, which is given by

$$U_F = \frac{D}{2}\int_0^{2\pi}\int_0^a \left[\left\{\nabla^2 w(r)\right\}^2 - \frac{2(1-\nu)}{r}\frac{dw(r)}{dr}\frac{d^2w(r)}{dr^2}\right] r dr d\theta \quad \text{....... (1)}$$

where

$$\nabla^2 = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}; \quad D = \frac{Eh^3}{12(1-\nu^2)} \quad \text{........................ (2)}$$

and E and ν are, respectively, the elastic modulus and Poisson's ratio for the plate material, with h corresponding to the plate thickness. The analysis of this problem can, of course, be extended to include effects of shear deformations and in-plane membrane stresses in the plate region.

(b) *Flexural energy of the shell*

The flexural energy of the cylindrical shell which is in continuous contact with the boundary of the circular foundation is given by

$$U_S = \int_0^{2\pi} \frac{k}{2}\left[\left\{\frac{dw(r)}{dr}\right\}_{r=a}\right]^2 a d\theta \quad \text{........................ (3)}$$

In Equation 3 the stiffness constant k depends on the geometry and elasticity characteristics of the shell element. Consider, for example, the behaviour of a thin shell of diameter 2a and thickness t and of semi-infinite length. The radial deflection of the shell, $w^*(\tilde{z})$, subjected to a concentrated moment of intensity M_0 acting along the circular edge is given by (see e.g. TIMOSHENKO and WOINOWSKY-KRIEGER, 1959)

$$w^*(\tilde{z}) = \frac{M_0 e^{-\beta\tilde{z}} \sin \beta\tilde{z}}{2\beta^2 D_0} \quad ; \quad \tilde{z} \geq 0 \qquad (4)$$

where

$$\beta^4 = \frac{3(1-\nu_0^2)}{a^2t^2} \quad ; \quad D_0 = \frac{E_0 t^3}{12(1-\nu_0^2)} \qquad (5)$$

and E_0, ν_0 and t are, respectively, the Young's modulus of the shell material, its Poisson's ratio and its thickness. (The distance $\tilde{z}$ is measured along the generator of the shell.) By considering Equation 4 and the relationship $M_0 = k\left[\frac{dw^*}{dz}\right]_{\tilde{z}=0}$ we have

$$k = \frac{E_0 t^2}{6(1-\nu_0^2)} \left[3(1-\nu_0^2)\frac{t^2}{a^2}\right]^{1/4} \qquad (6)$$

It may be noted that representations similar to Equation 5 can be developed to examine the rotational stiffness effects of thin shells of finite length.

(c) *Elastic energy of the soil region*

The third component of the total potential energy functional corresponds to the elastic strain energy of the isotropic elastic soil mass which is subjected to the displacement field w(r) (in the z-direction) in the region $r \leq a$. In the ensuing discussion it is assumed that the external loading configuration of the plate region is such that no tensile stresses act at the smooth plate-elastic medium interface. To physically realize continuous contact at the smooth interface region $r \leq a$, the contact stresses should be compressive. This assertion should be verified upon completion of the interaction analysis. From a practical point of view, the self weight may introduce sufficiently large contact stresses to prevent separation at the interface.

The distribution of normal contact stresses at the smooth interface associated with the imposed displacement field w(r) can be uniquely determined by making use of the integral equation methods developed by Green

(1949), Sneddon (1966) and others for the analysis of mixed boundary value problems in classical elasticity theory. The normal contact stress is given by

$$\sigma_{zz}(r, 0) = \frac{E_s}{2(1-\nu_s^2)r}\frac{d}{dr}\int_a^r \frac{tg(t)\,dt}{\sqrt{t^2-r^2}} \; ; \text{ (for } 0 \leq r \leq a) \qquad (7)$$

where

$$g(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{rw(r)\,dr}{\sqrt{t^2-r^2}} \; ; \text{ (for } 0 \leq t \leq a) \qquad (8)$$

and E_s, ν_s are the elastic constants of the elastic soil mass. From the above results, the elastic energy of the supporting soil region is given by

$$U_{HS} = \frac{E_s a^3}{2\pi(1-\nu_s^2)}\int_0^{2\pi}\int_0^a \frac{w(r)}{a}\left[\frac{d}{dr}\int_a^r \frac{t}{\sqrt{t^2-r^2}}\left\{\frac{d}{dt}\int_0^t \frac{rw(r)\,dr}{a^2\sqrt{t^2-r^2}}\right\}dt\right]drd\theta \qquad (9)$$

(d) *Potential energy of the external loads*

It is assumed that the idealized structural system (Figure 2) is subjected to the line load P_0 and the distributed load p_0. The former corresponds to a load such as the weight of the cylindrical shell. The distributed loads p_0 represent other structural loads. The total potential energy of the external loading is given by

$$U_L = -\int_0^{2\pi} aP_0[w(r)]_{r=\eta a}\,d\theta - \int_0^{2\pi}\int_0^{\lambda a} p_0 w(r) r\,dr\,d\theta \qquad (10)$$

The total potential energy functional for the shell-foundation-elastic soil mass system (U) is obtained by combining Equations 1, 3, 9 and 10; (i.e. $U = U_F + U_S + U_{HS} + U_L$). For the functional U to satisfy the principle of stationary potential energy we require

$$\delta U = 0 \qquad (11)$$

where δU is the variation of the functional. In order to apply this principle to the soil-raft foundation interaction problem, it is assumed that w(r) admits a representation in the form

$$w(r) = a\sum_{i=0}^{n} C_i\,\phi_i(r) \qquad (12)$$

where C_i are arbitrary constants and ϕ_i are functions which render the foundation deflections kinematically admissible. The accuracy of the energy solution is improved if, at the outset, the functions $\phi_i(r)$ are ordered in such a way that the boundary conditions at the restrained edge are identically satisfied. It should be noted that the boundary conditions related to the flexural moment at r = a is of a kinematic nature. The boundary conditions are

$$M_r(a) = -D\left[\frac{d^2w(r)}{dr^2} + \frac{\nu}{r}\frac{dw(r)}{dr}\right]_{r=a} = k\left[\frac{dw(r)}{dr}\right]_{r=a}$$

$$Q_r(a) = -D\left[\frac{d}{dr}\left\{\nabla^2 w(r)\right\}\right]_{r=a} = 0 \qquad \ldots\ldots\ldots\ldots\ldots (13)$$

By using the boundary conditions of Equation 13, the deflection in Equation 12 can be reduced to a representation in terms of (n–2) arbitrary constants. The principle of total potential energy requires that U can be an extremum with respect to the kinematically admissible displacement field characterized by C_i [see e.g. WASHIZU, (1975); MIKHLIN, (1969)]. Hence

$$\frac{\partial U}{\partial C_i} = 0 \quad (i = 0, 1, 2, \ldots, n-2) \qquad \ldots\ldots\ldots\ldots\ldots (14)$$

The above minimization procedure yields (n–2) linear equations for the undetermined constants C_i.

THE RESTRAINED RAFT PROBLEM

In order to examine the interaction between the circular foundation with an edge restraint and the isotropic elastic soil mass, it is assumed that complete contact is maintained in the plate region $0 \leq r \leq a$. As such the deflected shape can be represented in the form of a power series

$$w(r) = a \sum_{i=0}^{3} C_{2i}\left(\frac{r}{a}\right)^{2i} \qquad \ldots\ldots\ldots\ldots\ldots (15)$$

where C_{2i} are arbitrary constants. By examining Equations 12 and 15 it is evident that the particular choice of functions corresponding to $\phi_i(r)$ constitute a kinematically admissible set of functions which give finite and single value displacements and flexural moments in the plate region $0 \leq r \leq a$. By invoking the boundary conditions of Equation 13 corresponding to the restrained edge, Equation 15 can be reduced to the form

$$w(r) = a\left[C_0 + C_2\left\{\left(\frac{r}{a}\right)^2 + \rho_1\left(\frac{r}{a}\right)^4 + \rho_2\left(\frac{r}{a}\right)^6\right\}\right] \qquad \ldots\ldots\ldots\ldots (16)$$

where

$$\left\{\rho_1 ; \rho_2\right\} = \frac{(1+\nu-k_0)}{(2+\nu-k_0)} \left\{-\frac{3}{4} ; \frac{1}{6}\right\} \qquad \text{............... (17)}$$

and

$$k_0 = \frac{ka}{D} = \frac{2(1-\nu^2)}{(1-\nu_0^2)} \frac{E_0}{E} \left(\frac{t}{h}\right)^2 \left(\frac{a}{h}\right) \left[3(1-\nu_0^2) \frac{t^2}{a^2}\right]^{1/4} \qquad \text{............ (18)}$$

By making use of (16) the expression for the total potential energy functional can be reduced to the explicit form

$$U = \frac{E_s a^3}{(1-\nu_s^2)} \left[C_0^2 + \chi_1 C_0 C_2 + C_2^2 \chi_2\right] + \pi D C_2^2 \chi_3$$

$$- \pi p_0 a^3 \lambda^2 (1+N) \left[C_0 + \chi_4 C_2\right] \qquad \text{............... (19)}$$

where the constants χ_n ($n = 1, 2, 3, 4$) are defined in Appendix II. Also, the line load P_0 has been expressed as a multiple of the applied stress p_0 i.e.

$$P_0 = \frac{N p_0 \lambda^2 a}{2\eta}$$

The constants C_0 and C_2 are uniquely determined from the minimization conditions

$$\frac{\partial U}{\partial C_0} = 0 \; ; \; \frac{\partial U}{\partial C_2} = 0 \qquad \text{............... (20)}$$

The explicit form for the deflection of the circular plate with a restrained edge is given by

$$w(r) = \frac{\pi a \lambda^2 p_0 (1+N)(1-\nu_s^2)}{E_s} \left[c_0^* + c_2^* \left\{\left(\frac{r}{a}\right)^2 + \rho_1 \left(\frac{r}{a}\right)^4 + \rho_2 \left(\frac{r}{a}\right)^6\right\}\right] \text{... (21)}$$

where

$$\left\{c_0^* ; c_2^*\right\} = \frac{\left\{(\chi_1\chi_4 - 2\chi_2 - R\chi_3) ; (\chi_1 - 2\chi_4)\right\}}{\left[\chi_1^2 - 4\chi_2 - 2R\chi_3\right]} \qquad \text{............... (22)}$$

and R is a relative rigidity parameter defined by

$$R = \frac{\pi}{6} \frac{(1-\nu_s^2)}{(1-\nu^2)} \frac{E}{E_s} \left(\frac{h}{a}\right)^3 \qquad \text{............... (23)}$$

In addition, the contact stress at the soil-raft foundation interface is given by

$$\sigma_{zz}(r,0) = \frac{\lambda^2 p_0(1+N)}{\sqrt{1-(r/a)^2}}\left[c_0^* + c_2^*\left(-2-\frac{8}{9}\rho_1-\frac{16}{25}\rho_2\right)\right.$$
$$+\left(\frac{r}{a}\right)^2\left(4-\frac{32}{9}\rho_1-\frac{32}{25}\rho_2\right)+\left(\frac{r}{a}\right)^4\left(\frac{64}{9}\rho_1-\frac{128}{25}\rho_2\right)$$
$$\left.+\left(\frac{r}{a}\right)^6\left(\frac{256}{25}\rho_2\right)\right] \qquad \ldots\ldots\ldots\ldots\ldots \quad (24)$$

The flexural moments in the restrained circular foundation due to its interaction with the isotropic elastic soil mass can be obtained by one of two methods. In the first method, the distribution developed for w(r) can be directly used in the relationships for the flexural moments

$$M_r = -D\left[\frac{d^2w(r)}{dr^2} + \frac{\nu}{r}\frac{dw(r)}{dr}\right]$$
$$M_\theta = -D\left[\frac{1}{r}\frac{dw(r)}{dr} + \nu\frac{d^2w(r)}{dr^2}\right] \qquad \ldots\ldots\ldots\ldots\ldots \quad (25)$$

As has been observed by Dym and Shames (1973) and others, any inaccuracies that may be present in the energy estimate for w(r) are greatly magnified in the computation of M_r and M_θ, owing to the presence of derivatives up to the second-order. It can be shown that flexural moments in the unrestrained circular foundation as determined from Equations 21 and 25 are somewhat lower than those predicted by Brown (1969) using the numerical method involving a power series technique. In the second method, the flexural moments in the restrained circular raft are computed by considering the action of the external load distribution and the contact stress distribution. For this purpose it is convenient to utilize basic solutions developed for a circular plate simply supported along its boundary. Using this technique, we obtain the following result for the flexural moment at the centre of the restrained circular raft ($M_r(0) = M_\theta(0) = M_0$):

$$\frac{M_0}{\lambda^2 p_0 a^2(1+N)} = \frac{(3+\nu)}{16} - \frac{(1-\nu)}{4}m_1 + \frac{(1+\nu)}{2}m_2 - m_3 \ldots\ldots\ldots \quad (26)$$

where m_1, m_2 and m_3 are defined in Appendix I. Also, the flexural moment at the edge of the restrained raft ($M_r(a) = M_e$) is given by

$$\frac{M_e}{\lambda^2 p_0 a^2(1+N)} = m_3 \qquad \ldots\ldots\ldots\ldots\ldots \quad (27)$$

NUMERICAL RESULTS

In order to develop numerical results for the interaction of the restrained circular raft foundation, attention is restricted to the particular case where the raft is subjected only to a uniformly distributed load over the entire surface area. The effect of the line load is neglected. For this particular form of external loading $N = 0$ and $\lambda = 1$.

(a) *Flexural deflections*

The accuracy of the energy estimate for the flexural deflections of the raft (Equation 21) can be examined by first assigning suitable limiting values for the relative rigidity parameter R. As $R \rightarrow \infty$ the circular raft resembles an infinitely rigid foundation resting in smooth contact with an isotropic elastic halfspace. Taking the appropriate limit of Equation 21 we obtain the following expression for the rigid settlement (w_0) of the circular raft:

$$w_0 = \frac{\pi p_0(1-\nu_s^2)a}{2E_s} \qquad \ldots\ldots\ldots\ldots\ldots \quad (28)$$

This result is in agreement with the exact solution obtained by Boussinesq (1885) and Harding and Sneddon (1945) for the rigid indentation of an isotropic elastic halfspace by a circular foundation. These investigations are based on the use of potential function techniques and integral equation methods, respectively. It may be noted that for the rigid raft foundation $[dw(r)/dr]_{r=a} = 0$; as such, the degree of edge restraint has no influence on the settlement.

As the relative rigidity R reduces to zero (i.e. $D \rightarrow 0$), the interaction problem reduces to that of the surface loading of an isotropic elastic soil mass by a uniform flexible circular load of radius a and stress intensity p_0. A comparison of the energy estimate for the central deflection with the corresponding exact solution yields the following:

$$\frac{\left[\left\{w(0)\right\}_{\text{Energy}} : \left\{w(0)\right\}_{\text{Exact}}\right]}{p_0(1-\nu_s^2)a/E_s} = [2.09 : 2.00] \qquad \ldots\ldots\ldots\ldots\ldots \quad (29)$$

Similarly, the following results are obtained for the comparison of the differential deflection $[w_d = w(0) - w(a)]$ within the uniformly loaded region:

$$\frac{\left[\left\{w_d\right\}_{\text{Energy}} : \left\{w_d\right\}_{\text{Exact}}\right]}{p_0(1-\nu_s^2)a/E_s} = [0.730 : 0.727] \qquad \ldots\ldots\ldots\ldots\ldots \quad (30)$$

Again, since the flexural rigidity D is assigned a value zero, the edge restraint has no effect on the deflection of the perfectly flexible plate.

The influence of the edge restraint can, however, be observed for finite values of the relative rigidity parameter R. To observe such effects it is instructive to examine the non-dimensional parameter k_0 which influences the degree of restraint. Assuming that the cylindrical shell is composed of the same material as the circular raft (i.e. $E_0 = E$; $\nu_0 = \nu$) Equation 18 reduces to

$$k_0 = 2\,[3\,(1-\nu_0^2)\,]^{1/4} \quad \theta^2 \left\{\frac{\theta}{\mu}\right\}^{1/2} \qquad (31)$$

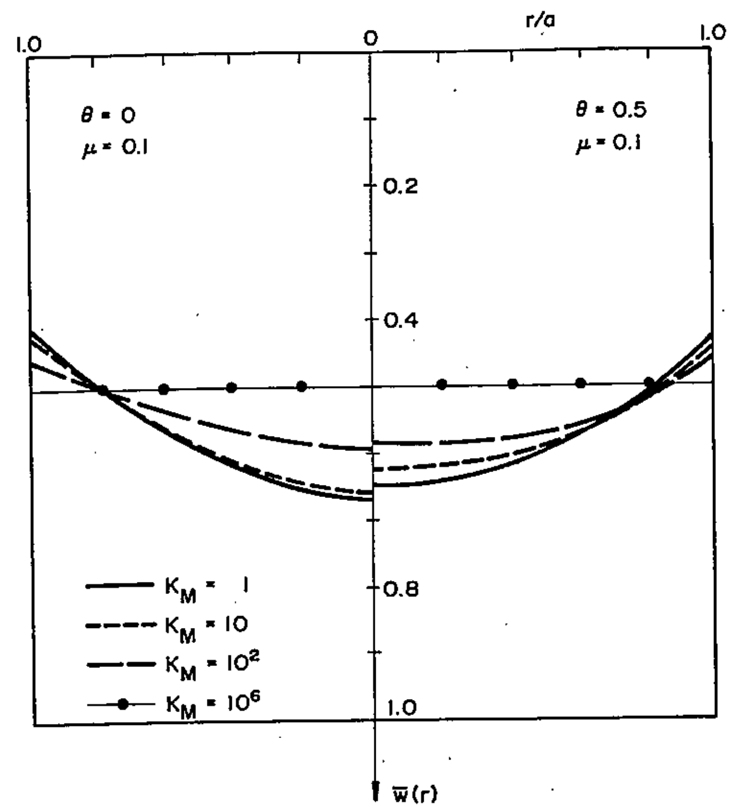

Fig. 3. Deflections of the flexible circular raft $\bar{w}(r) = w(r)/a$.

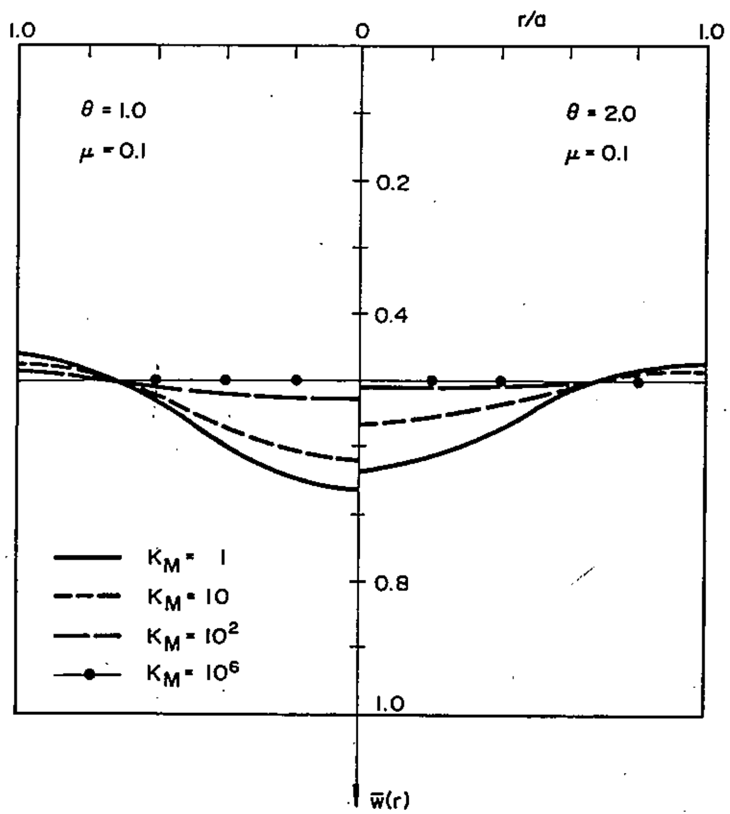

Fig. 4. Deflections of the flexible raft $\bar{w}(r) = w(r)/a$.

where $\theta = t/h$ and $\mu = h/a$.

It is evident that k_0 incorporates the geometric characteristics of both the raft foundation and the cylindrical shell. As such, to observe the influences of the shell and plate dimensions on the raft performance it is convenient to treat θ and μ as separate variables. Also, the relative rigidity parameter R can be written in the form

$$R = K_M \mu^3 \qquad \text{(32a)}$$

where K_M is a modular ratio defined by

$$K_M = \frac{\pi(1-\nu_s^2)}{6(1-\nu^2)} \frac{E}{E_s} \qquad \text{(32b)}$$

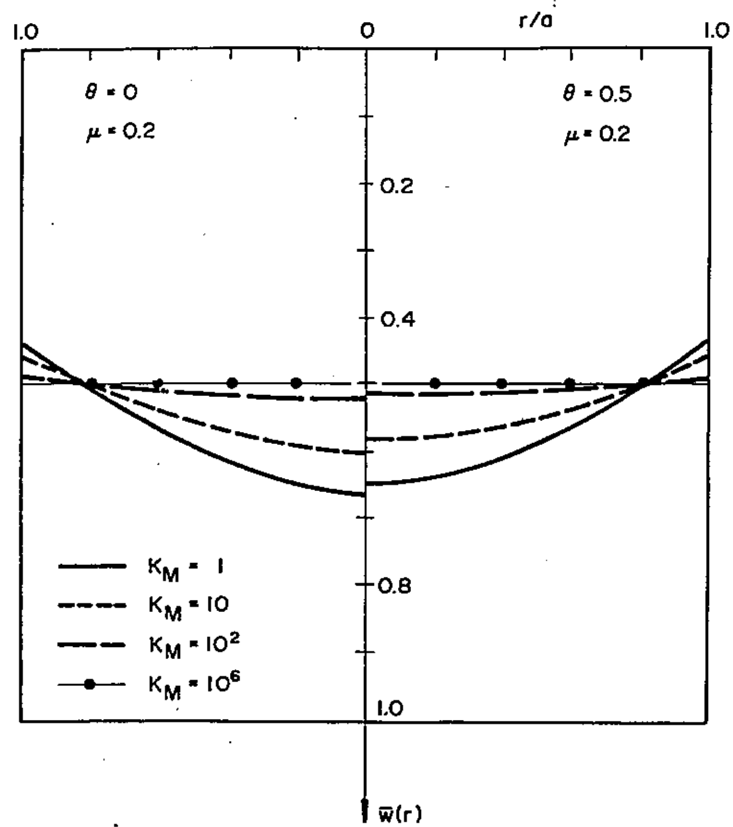

Fig. 5. Deflections of the flexible circular raft $\bar{w}(r) = w(r)/a$.

Figures 3-6 illustrate the manner in which the deflection of the circular raft is influenced by the degree of restraint offered by a monolithic cylindrical shell. The vaue $\theta = 0$ corresponds to the unrestrained circular raft. The value $\theta \doteq 2$ represents, from a practical point of view, a somewhat artificial situation where the cylindrical shell imposes considerable restraint on the edge rotation. Alternatively, the case of a rigid shell can be depicted by retaining the ratio (E_0/E) which occurs in Equation 18. This ratio can then be assigned a sufficiently large value to achieve conditions consistent with a rigid shell. From the results given in Figures 3-6 it is evident that the flexural deflections of the circular raft are considerably influenced by the degree of restraint that is offered by the stiffness of the monolithic thin cylindrical shell.

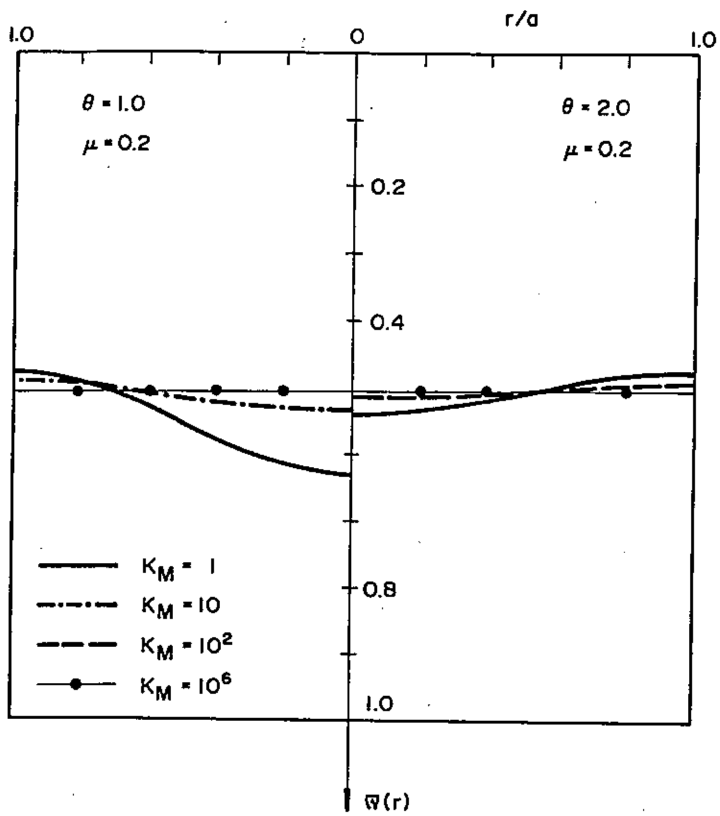

Fig. 6. Deflections of the flexible circular raft $\bar{w}(r) = w(r)/a$.

(b) *Contact stresses*

The analytical solution presented for the restrained raft problem assumes that the contact between the supporting medium and the circular raft is perfectly smooth and that no separation takes place in this region. It is therefore important to examine the nature of this contact stress distribution (Equation 24) and in particular investigate the manner in which the edge restraint influences the contact stresses. The contact stress (Equation 24) has been evaluated for the same set of representative values of θ and μ considered earlier. Again it should be noted that λ and N are set equal to unity and zero respectively. Figures 7-10 illustrate the contact stress distributions evaluated for various degrees of edge restraint. It is evident that edge stresses are substantially altered by the presence of the restraint.

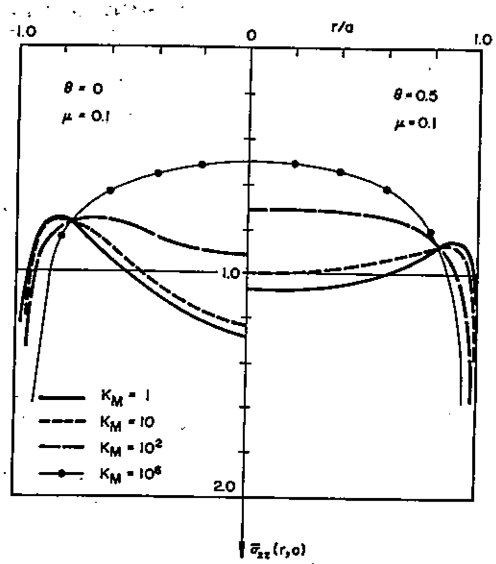

Fig. 7. Contact stresses at the raft - elastic medium interface $\bar{\sigma}_{zz}(r,o) = \sigma_{zz}(r,o)/p_0$.

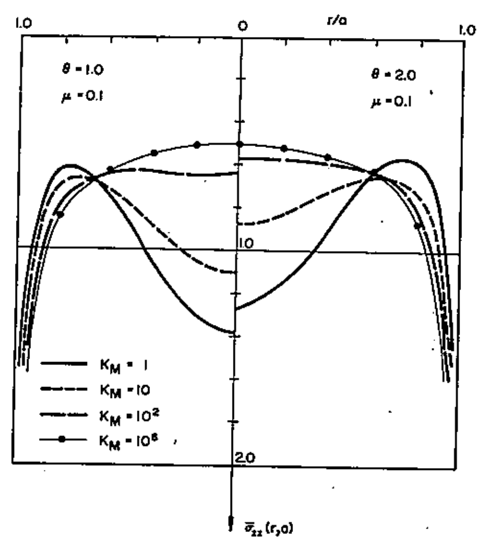

Fig. 8. Contact stresses at the raft - elastic medium interface $\bar{\sigma}_{zz}(r,o) = \sigma_{zz}(r,o)/p_0$.

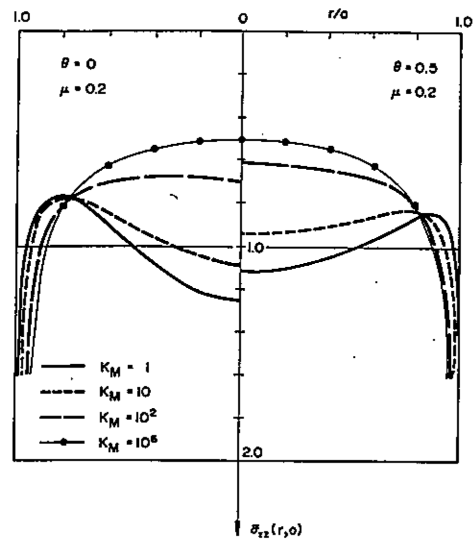

Fig. 9. Contact stresses at the raft - elastic medium interface $\bar{\sigma}_{zz}$ (r,o) = σ_{zz}(r,o)/p_0.

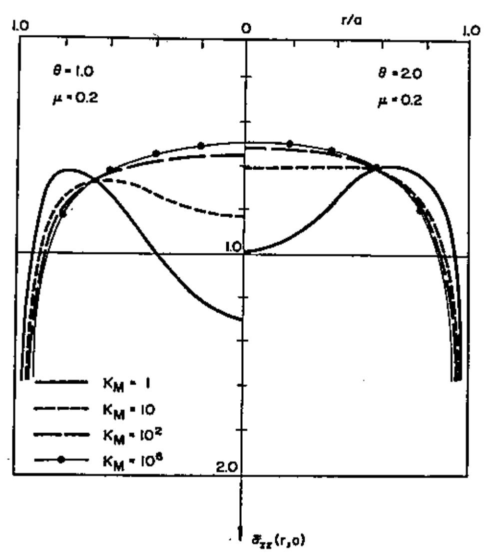

Fig. 10. Contact stresses at the raft - elastic medium interface $\bar{\sigma}_{zz}$ (r,o) = σ_{zz} (r,o)/p_0.

CONCLUDING REMARKS

This paper presents an analytical solution to the problem of the frictionless axisymmetric interaction between a flexible circular raft foundation and an underlying elastic soil mass. In contrast to classical treatments of this interaction problem, here we consider the influence of partial fixity at the boundary of the circular raft. Such effects can be introduced by the monolithic action between a foundation raft and a containment shell. The interaction problem is examined by using a variational approach. It is shown that the variational method yields compact results of engineering interest particularly for the deflections and flexural moments in the raft and for the contact stresses at the raft-elastic medium interface. Numerical results presented in the paper indicate that the degree of edge restraint has a significant influence on the total and differential settlements experienced by the circular raft. Similar conclusions apply to the normal contact stresses that are developed at the raft-elastic medium interface.

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APPENDIX I

The functions χ_n (n = 1, . . . , 4) are given by

$$\chi_1 = \left\{\xi_0 + \frac{2}{3}(1+\xi_2) + \frac{8}{15}(\rho_1+\xi_4) + \frac{16}{35}(\rho_2+\xi_6)\right\}$$

$$\chi_2 = \left\{\frac{2}{3}\xi_0 + \frac{8}{15}(\rho_1\xi_0+\xi_2) + \frac{16}{35}(\rho_2\xi_0+\rho_1\xi_2+\xi_4) + \frac{128}{315}(\xi_2\rho_2+\xi_4\rho_1+\xi_6) + \frac{256}{693}(\xi_4\rho_2+\xi_6\rho_1) + \frac{1024}{3003}\xi_6\rho_2\right\}$$

$$\chi_3 = \left\{8 + 32\rho_1 + \frac{144}{3}\rho_2 + \frac{128}{3}\rho_1^2 + \frac{1188}{5}\rho_2^2 + 144\rho_1\rho_2 - (1-\nu)(4 + 16\rho_1 + 24\rho_2 + 16\rho_1^2 + 36\rho_2^2 + 48\rho_1\rho_2)\right\} + \frac{ka}{D}\left\{2 + 4\rho_1 + 6\rho_2\right\}^2$$

$$\chi_4 = \frac{\left\{12N(\eta^2+\rho_1\eta^4+\rho_2\eta^6) + 6\lambda^2 + 4\rho_1\lambda^4 + 3\rho_2\lambda^6\right\}}{12(1+N)}$$

where

$$\xi_0 = -2 - \frac{8}{9}\rho_1 - \frac{16}{25}\rho_2 \quad ; \quad \xi_4 = \frac{64}{9}\rho_1 - \frac{128}{25}\rho_2$$

$$\xi_4 = 4 - \frac{32}{9}\rho_1 - \frac{32}{25}\rho_2 \quad ; \quad \xi_6 = \frac{256}{25}\rho_2$$

Also,

$$m_1 = \frac{c_0^*}{3} + c_2^*\left\{\frac{\xi_0}{3} + \frac{2}{15}\xi_2 + \frac{8}{105}\xi_4 + \frac{16}{315}\xi_6\right\}$$

$$m_2 = \left[c_0^* + c_2^*\left\{\xi_0 + \frac{2}{3}\xi_2 + \frac{8}{15}\xi_4 + \frac{16}{35}\xi_6\right\}\right]\ln 2 - \left[c_0^* + c_2^*\left\{\xi_0 + \frac{5}{9}\xi_2 + \frac{94}{225}\xi_4 + \frac{1276}{3675}\xi_6\right\}\right]$$

$$m_3 = K_M\,[3(1-\nu^2)]^{1/4}\quad \theta^2\mu^2\sqrt{\theta\mu}\,\frac{2c_2^*}{\pi}\left\{1 + 2\rho_1 + 3\rho_2\right\}$$

APPENDIX II

Notation

r, θ, z	=	cylindrical polar coordinate system
$\tilde{z}$	=	distance along the generator of the shell
$w(r)$	=	raft deflection in the z-direction
$w^*(\tilde{z})$	=	shell deflection in the r-direction
∇^2	=	$\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr}$
D	=	$\frac{Eh^3}{12(1-\nu^2)}$
D_0	=	$\frac{E_0 t}{12(1-\nu_0^2)}$
h	=	thickness of raft foundation
t	=	thickness of shell wall
E	=	Young's modulus of raft material
ν	=	Poisson's ratio of raft material
E_0	=	Young's modulus of shell material
ν_0	=	Poisson's ratio of shell material
E_s	=	Young's modulus of the soil material
ν_s	=	Poisson's ratio of the soil material
P_0	=	line load due to shell
p_0	=	distributed load
a	=	radius of raft foundation
ηa	=	radius of circular line load
λa	=	radius of distributed load
N	=	$\frac{2P_0\eta}{p_0\lambda^2 a}$
M_0	=	concentrated edge moment in shell
U_F	=	flexural energy of raft foundation
U_S	=	flexural energy of shell
U_{HS}	=	elastic energy of soil mass
U_L	=	potential energy of external loads
U	=	$U_F + U_S + U_{HS} + U_L$
δU	=	first variation of U
σ_{zz}	=	contact stresses at the raft - soil mass interface
M_r	=	radial bending moment in raft
M_θ	=	circumferential bending moment in raft

M_e	=	edge moment in raft
Q_r	=	shearing force in raft
C_0, C_2	=	arbitrary constants
ρ_1, ρ_2	=	substitution parameters defined by Equation 17
k	=	rotational stiffness of shell defined by Equation 6
k_0	=	$\frac{ka}{D}$
R	=	$\frac{\pi}{6}\frac{(1-\nu_s^2)}{(1-\nu^2)}\frac{E}{E_s}\left[\frac{h}{a}\right]^3$
w_0	=	central displacement of raft
K_M	=	$\frac{\pi}{6}\frac{(1-\nu_s^2)}{(1-\nu^2)}\frac{E}{E_s}$
θ	=	$\frac{t}{h}$
μ	=	$\frac{h}{a}$
m_1, m_2, m_3	=	substitution parameters
$\chi_1, \chi_2, \chi_3, \chi_4$	=	substitution parameters
$\xi_0, \xi_2, \xi_4, \xi_6$	=	substitution parameters
c_0^*, c_2^*	=	substitution parameters