Diffraction of torsional wave or plane harmonic compressional wave by an annular rigid disc

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In this paper we have considered the following two problems. Firstly the diffraction of normally incident *SH* waves by a rigid annular disc situated at the interface of two elastic half-spaces is considered. The solution of the problem is reduced into the solution of triple integral equations involving Bessel functions. The solution of the triple integral equations is reduced into Fredholm integral equations of th second kind. By finding the solution of the Fredholm integral equation, the numerical values for the moment required to produce the rotation of disc are obtained. Secondly, the problem of diffraction of plane harmonic compressional wave by an annular circular disc embedded in an infinite elastic space is considered. The annular disc is assumed to be perfectly welded with the infinite solid. The solution of the problem is reduced into the solution of Fredholm integral equation of the second kind. The Fredholm integral equation is solved numerically and the numerical values for the couple applied on the disc are obtained.

INTRODUCTION

It is well known that the problems of cracks or inclusions are of considerable interest in seismology and geophysics. If the inclusions are located at the interface of layered media, the study becomes more important. Loeber and Sih[1,2] have studied the scattering of torsional waves by line crack and a penny-shaped crack laying on a bimaterial interface.

Diffraction of torsional waves or normal compressional waves by a flat annular crack in an infinite elastic medium has been studied by Shindo.[3,4] Diffraction of torsional waves by a flat annular crack at the interface of two bonded dissimilar elastic solids has been discussed by Singh, Dhaliwal and Vrbik.[5] The problem of diffraction of torsional waves by a circular rigid disc situated at the interface of two dissimilar elastic half-spaces has been considered by Singh, Rokne and Dhaliwal.[6]

In this paper we have considered the problem of diffraction of torsional waves by an annular rigid disc situated at the interface of two dissimilar elastic half-spaces. Using Hankel transforms the problem has been reduced to the solution of triple integral equations. The solution of triple integral equations is reduced into the solution of the Fredholm integral equation of the second kind. Numerical solution of the Fredholm integral equation is used to find the shear stress component on the disc. The solution of such a type of problem may give some information of building foundations in the event of earthquake waves. In the second problem, a rigid annular disc embedded in an infinite, isotropic elastic medium is assumed to be excited by a normally incident plane, harmonic compressional wave. The disc is assumed to be perfectly welded with the surrounding solid. The solution of the problem is reduced into the solution of the Fredholm integral equation of the second kind. By finding numerical solution of the integral equation the numerical values of the couple applied on the disc are obtained.

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FORMULATION AND SOLUTION OF PROBLEM (A)

Consider a cylindrical coordinate system (r, θ, z) with origin at the centre of the annular disc and let the disc occupy the region $a \leqslant r \leqslant b$, $z = 0$ at the interface of two half-spaces $z > 0$ and $z < 0$. The rigid disc is assumed to be excited by a torsional wave originating at $z = -\infty$. Now the problem is to find the stress distribution subject to the following boundary conditions:

$$u_\theta(r, 0^+) = u_\theta(r, 0^-) = u_0(r)\exp(-i\omega t), \quad a \leqslant r \leqslant b \quad (1)$$

$$u_\theta(r, 0^+) = u_\theta(r, 0^-), \quad 0 < r < a, \ r > b \quad (2)$$

$$\sigma_{\theta z}(r, 0^+) = \sigma_{\theta z}(r, 0^-), \quad 0 < r < a, \ r > b \quad (3)$$

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where ω is circular frequency. In what follows, the factor $\exp(-i\omega t)$ shall be suppressed.

The problem of determining the stress distribution reduces to that of obtaining the solution of the displacement equation:

$$\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}+\frac{\partial^2 u_\theta}{\partial z^2}+k^2 u_\theta=0 \qquad (4)$$

where $k^2=\rho/\mu$, μ is Lame's constant, ρ is the density of the elastic material and k is the wave number. The solution of equation (4) can be written in the following form:

$$u_\theta(r,z)=\begin{cases}\displaystyle\int_0^\infty \xi A_1(\xi)\exp(-\beta_1 z)J_1(\xi r)\,d\xi, & z\geqslant 0\\[2ex] \displaystyle\int_0^\infty \xi A_2(\xi)\exp(\beta_2 z)J_1(\xi r)\,d\xi, & z\leqslant 0\end{cases} \qquad (5)$$

$$\beta_j=\begin{cases}(\xi^2-k_j^2)^{1/2}, & \xi>k_j\\ -i(k_j^2-\xi^2)^{1/2}, & \xi<k_j\end{cases} \qquad (6)$$

$$k_j^2=\frac{\rho_j\omega^2}{\mu_j},\ (j=1,2) \qquad (7)$$

where suffixes 1 and 2 correspond to the half-spaces $z>0$ and $z<0$, respectively. In equations (5), $A_1(\xi)$ and $A_2(\xi)$ are unknown functions which are to be determined by using boundary conditions (1)-(3). With the help of the following equations:

$$\sigma_{z\theta}=\mu\frac{\partial u_\theta}{\partial z},\quad \sigma_{r\theta}=\mu\left(\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r}\right) \qquad (8)$$

we find that

$$\sigma_{\theta z}(r,z)=\begin{cases}\displaystyle -\mu_1\int_0^\infty \xi\beta_1 A_1(\xi)\exp(-\beta_1 z)J_1(\xi r)\,d\xi, & z\geqslant 0\\[2ex] \displaystyle \mu_2\int_0^\infty \xi\beta_2 A_2(\xi)\exp(\beta_2 z)J_1(\xi r)\,d\xi, & z\leqslant 0\end{cases} \qquad (9)$$

$$\sigma_{r\theta}(r,z)=\begin{cases}\displaystyle -\mu_1\int_0^\infty \xi^2 A_1(\xi)\exp(-\beta_1 z)J_2(\xi r)\,d\xi, & z\geqslant 0\\[2ex] \displaystyle -\mu_2\int_0^\infty \xi^2 A_2(\xi)\exp(\beta_2 z)J_2(\xi r)\,d\xi, & z\leqslant 0\end{cases} \qquad (10)$$

From the continuity conditions (1) and (2) of the displacement field $u_\theta(r,0^+)=u_\theta(r,0^-)$ for all values of r we find that:

$$A_1(\xi)=A_2(\xi) \qquad (11)$$

From conditions (1) and (3) we find that:

$$\int_0^\infty \xi A_1(\xi)J_1(\xi r)\,d\xi=u_0(r),\quad a<r<b \qquad (12)$$

$$\int_0^\infty \xi A_1(\xi)\,[\mu_1\beta_1+\mu_2\beta_2]\,J_1(\xi r)\,d\xi=0,\quad 0<r<a,\ r>b \qquad (13)$$

Let

$$\xi\,[\mu_1\beta_1+\mu_2\beta_2]\,A_1(\xi)=(\mu_1+\mu_2)B(\xi) \qquad (14)$$

then the triple integral equations (12) and (13) can be written in the following form:

$$\int_0^\infty \xi^{-1}B(\xi)\,[1+M(\xi)]\,J_1(\xi r)\,d\xi=u_0(r),\quad a<r<b \qquad (15)$$

$$\int_0^\infty B(\xi)J_1(\xi r)\,d\xi=0,\quad 0<r<a,\ r>b \qquad (16)$$

where

$$M(\xi)=\left[\frac{\xi(\mu_1+\mu_2)}{(\beta_1\mu_1+\beta_2\mu_2)}-1\right] \qquad (17)$$

We can easily see that $M(\xi)\to 0$ when $\xi\to\infty$. To find the solution of triple integral equations (15) and (16) we follow Cooke[7] and assume

$$\int_0^\infty B(\xi)J_1(\xi r)\,d\xi=g(r),\quad a<r<b \qquad (18)$$

Making use of the inversion theorem for Hankel transforms we get from equations (16) and (18) that

$$B(\xi)=\xi\int_a^b r g(r)J_1(\xi r)\,dr \qquad (19)$$

Substituting the value of $B(\xi)$ from equation (19) into (15) we get

$$\int_a^b u g(u)L(u,r)\,du=F(r),\quad a<r<b \qquad (20)$$

where

$$F(r)=u_0(r)-\int_0^\infty \xi^{-1}B(\xi)M(\xi)J_1(\xi r)\,d\xi,\quad a<r<b \qquad (21)$$

$$L(u,r)=\frac{2}{\pi u r}\int_0^{\min(u,r)}\frac{s^2\,ds}{[(u^2-s^2)(r^2-s^2)]^{1/2}}$$

Cooke[7] has shown that the solution of integral equation (20) can be written in the following form:

$$s^2G(s)=\frac{d}{ds}\int_a^s\frac{x^2F(x)\,dx}{(s^2-x^2)^{1/2}}-\frac{4s}{\pi^2\sqrt{s^2-a^2}}\times\int_a^b\frac{tG(t)K(s,t)\,dt}{\sqrt{t^2-a^2}},\quad a<s<b \qquad (22)$$

where

$$K(s,t)=\int_0^a \frac{y^2(a^2-y^2)\,dy}{(s^2-y^2)(t^2-y^2)} \qquad (23)$$

$$G(s)=\int_s^b \frac{g(r)\,dr}{(r^2-s^2)^{1/2}} \qquad (24)$$

If $u_0(r)=u_0 r$, where u_0 is constant, then we find that:

$$\frac{d}{ds}\int_a^s \frac{x^2F(x)\,dx}{(s^2-x^2)^{1/2}}=\frac{u_0 s(2s^2-a^2)}{\sqrt{s^2-a^2}}-\int_0^\infty \xi^{-1}B(\xi)M(\xi)I(s,\xi)\,d\xi \qquad (25)$$

where

$$I(s,\xi)=\frac{d}{ds}\int_a^s \frac{r^2J_1(\xi r)\,dr}{\sqrt{s^2-r^2}} \qquad (26)$$

Equation (24) is of the Abel type and hence its solution can be written in the following form:

$$g(r)=-\frac{2}{\pi}\frac{d}{dr}\int_r^b \frac{sG(s)\,ds}{(s^2-r^2)^{1/2}} \qquad (27)$$

We can easily show that

$$\int_0^\infty \xi^{-1}B(\xi)M(\xi)I(s,\xi)\,d\xi$$

$$=\frac{2}{\pi}\int_a^b G(t)\,dt\int_0^\infty M(\xi)I(t,\xi)I(s,\xi)\,d\xi \qquad (28)$$

Making use of the equations (25) and (28), we can write the integral equation (22) in the following form:

$$s^2G(s)=\frac{u_0 s(2s^2-a^2)}{\sqrt{s^2-a^2}}$$

$$-\int_a^b G(t)\,[K_1(s,t)+K_2(s,t)]\,dt,\quad a<s<b \qquad (29)$$

where

$$K_1(s,t)=\frac{4st}{\pi^2\sqrt{(s^2-a^2)(t^2-a^2)}}\int_0^a \frac{y^2(a^2-y^2)\,dy}{(s^2-y^2)(t^2-y^2)}$$

$$=\frac{2st}{\pi^2\sqrt{(s^2-a^2)(t^2-a^2)}\,[t^2-s^2]}$$

$$\times\left[s(a^2-s^2)\log\left|\frac{s+a}{s-a}\right|-t(a^2-y^2)\log\left|\frac{t+a}{t-a}\right|\right] \qquad (30)$$

$$K_2(s,t)=\frac{2}{\pi}\int_0^\infty M(\xi)I(t,\xi)I(s,\xi)\,d\xi \qquad (31)$$

If $a\to 0$, we find that $K_1(s,t)\to 0$ and the kernel $K_2(s,t)$ is the same as obtained by Singh, Rokne and Dhaliwal.[6] In this way we can find solution of the problem of diffraction of a torsional wave by a circular rigid disc at the interface of two bonded dissimilar elastic solids.

We can easily write that:

$$I(s,\xi)=s\xi e_1(s,\xi)+\frac{sa}{\sqrt{s^2-a^2}}J_1(\xi a) \qquad (32a)$$

where

$$e_1(s,\xi)=\int_a^s \frac{rJ_0(\xi r)\,dr}{(s^2-r^2)^{1/2}} \qquad (32b)$$

Now we find that:

$$\sigma_{z\theta}(r,0^+)-\sigma_{z\theta}(r,0^-)=(\mu_1+\mu_2)g(r),\quad a<r<b \qquad (33)$$

The moment required to produce the rotation of the disc is given by:

$$T=-2\pi\int_a^b r^2\,[\sigma_{\theta z}(r,0^+)-\sigma_{\theta z}(r,0^-)]\,dr$$

$$=-2\pi(\mu_1+\mu_2)\int_a^b r^2g(r)\,dr \qquad (34)$$

Making use of equations (19) and (27) we find that

$$B(\xi)=\frac{2}{\pi}\left[-\xi aJ_1(\xi a)\int_a^b \frac{sG(s)\,ds}{\sqrt{s^2-a^2}}+\int_a^b sG(s)\,e_1(s,\xi)\,ds\right] \qquad (35a)$$

Let

$$s=a\sec\theta,\quad t=a\sec\phi,$$
$$G(a\sec\phi)\sec^2\phi=u_0H_1(\phi) \qquad (35b)$$

then the integral equation (29) can be written in the following form:

$$\sin\theta\cos^2\theta H_1(\theta)$$

$$=(1+\sin^2\theta)-\sin\theta\cos^2\theta\int_0^{\sec^{-1}(b/a)}$$

$$\times\,[K_1(\theta,\phi)+K_2(\theta,\phi)]\sin\phi H_1(\phi)\,d\phi,$$

$$0<\theta<\sec^{-1}\left(\frac{b}{a}\right) \qquad (36)$$

$$K_1(\theta,\phi)=-\frac{4\sec\theta\sec\phi}{\pi^2\tan\theta\tan\phi\,[\sec^2\phi-\sec^2\theta]}$$

$$\times\left[\sec\phi\tan^2\phi\log\tan\left(\frac{\phi}{2}\right)-\sec\theta\tan^2\theta\log\tan\left(\frac{\theta}{2}\right)\right] \qquad (37)$$

$$K_2(\theta,\phi)=\frac{2}{\pi a}\int_0^\infty M(\xi)I(a\sec\phi,\xi)I(a\sec\theta,\xi)\,d\xi \qquad (38)$$

Making use of equations (27), (35b) we find from equation (34) that:

$$\frac{T}{u_0\mu_1} = -4a^3(1+\alpha)\int_0^{\sec^{-1}(b/a)} [1+2\tan^2\phi]\, H_1(\phi)\,d\phi \quad (39)$$

We can easily find that:

$$k_2 = k_1\sqrt{\frac{\rho_2\mu_1}{\rho_1\mu_2}} \quad (40)$$

The numerical values of $|T/u_0\mu_1|$ have been obtained from (39) where numerical values of $H_1(\phi)$ have been obtained from (36). The numerical values of $|T/u_0\mu_1|$ have been graphed in Figs. 1, 2 for $\rho_1/\rho_2 = 1$ and for various values of μ_1/μ_2 and for $\mu_1/\mu_2 = 1$ and for various values of ρ_1/ρ_2, respectively.

FORMULATION AND SOLUTION OF PROBLEM (B)

Consider a cylindrical polar coordinate system (r, θ, z) with origin at the centre of the disc and z-axis perpendicular to the plane of the annular disc. Let a plane harmonic compressional wave propagating along the z-axis be incident normally on the disc. The displacement vector associated with incident wave alone may be expressed as:

$$\exp(-i\omega t)\, u^0(r, z) = \{0, 0, u_0 \exp[ik_1(z - c_1 t)]\}$$

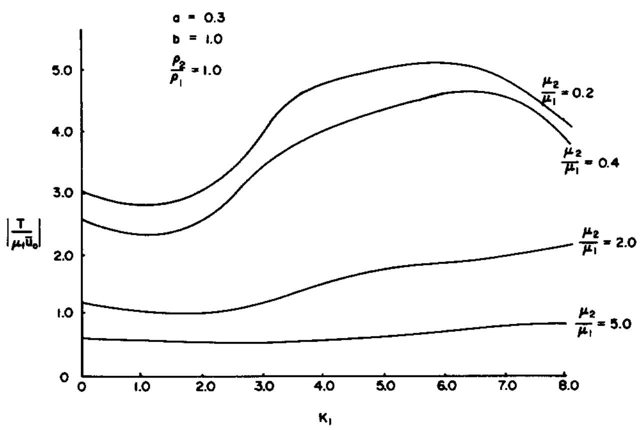

Figure 1. Variation of K_1 with $|T/\mu_1\bar{u}_0|$ for the annular disc. $[\bar{u}_0 = (1+\alpha)^2 u_0; \alpha = \mu_2/\mu_1]$

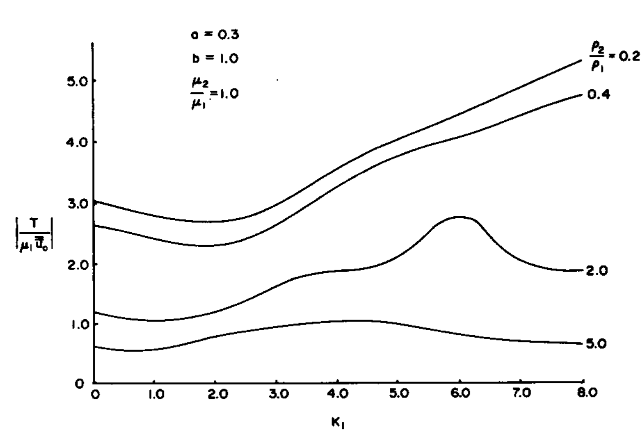

Figure 2. Variation of K_1 with $|T/\mu_1\bar{u}_0|$ for the annular disc. $[\bar{u}_0 = (1+\alpha)^2\{1 + (\rho_2/\rho_1)\}u_0; \alpha = \mu_2/\mu_1]$

where u_0 is the amplitude, $k_1 = \omega/c_1$ is the wave number in the incident wave and c_1 is the P-wave velocity in the solid. The time factor $\exp(-i\omega t)$ will be omitted in the subsequent analysis.

Since the problem is axisymmetric, the displacement components in the scattered field $u_r(r, z)$, $u_z(r, z)$ may be conveniently expressed in terms of two scalar potential functions $\phi(r, z)$, $\psi(r, z)$ by means of the equations:

$$u_r = \frac{\partial}{\partial r}\left(\phi + \frac{\partial\psi}{\partial z}\right)$$
$$u_z = \frac{\partial\phi}{\partial z} + \left(\frac{\partial^2}{\partial z^2} + k_2^2\right)\psi \quad (41)$$

where $k_2 = \omega/c_2$, c_2 being the shear wave velocity in the solid and ϕ, ψ satisfy Helmholtz equations

$$\nabla^2\phi + k_1^2\phi = 0$$
$$\nabla^2\psi + k_2^2\psi = 0 \quad (42)$$

The stress components on the $z = 0$ plane are given by

$$\sigma_{rz} = \mu\frac{\partial}{\partial r}\left\{2\frac{\partial\phi}{\partial z} + \left(2\frac{\partial^2}{\partial z^2} + k_2^2\right)\psi\right\}$$
$$\sigma_{zz} = \mu\left\{\left(2k_1^2 - k_2^2 + 2\frac{\partial^2}{\partial z^2}\right)\phi + 2\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial z^2} + k_2^2\right)\psi\right\} \quad (43)$$

where μ is the shear modulus of the elastic solid.

Anticipating that the presence of the disc will introduce discontinuities in the field quantities across $z = 0$, the solutions of (42) are postulated as follows following Mal, Ang and Knopoff:[8]

$$\phi(r, z) = \int_0^\infty k\left\{-\delta P(k) + \frac{k}{\nu_1}P_1(k)\right\} J_0(kr)\exp(-\nu_1|z|)\,dk \quad (44)$$

$$\psi(r, z) = \int_0^\infty \left\{\frac{k}{\nu_2}Q(k) - \delta Q_1(k)\right\} J_0(kr)\exp(-\nu_2|z|)\,dk \quad (45)$$

where

$$\nu_j = \begin{cases} \sqrt{k^2 - k_j^2}, & k > k_j \quad (46) \\ -i\sqrt{k_j^2 - k^2}, & k < k_j, \quad j = 1, 2 \quad (47) \end{cases}$$

$$\delta = \mathrm{sgn}(z)$$

$P(k)$, $P_1(k)$, $Q(k)$, $Q_1(k)$ are unknown functions to be determined from the boundary conditions. Since there is no discontinuity in the physical properties of the medium inside the region $a < r < b$, $z = 0$, which is occupied by the annular disc, all the field quantities must be continuous across $z = 0$ for $r > b$. The remaining boundary conditions are as follows:

$$\lim_{z\to 0} u_r(r, z) = 0, \quad a < r < b \quad (48)$$

$$\lim_{z\to 0} u_z(r, z) + u_0 = 0, \quad a < r < b \quad (49)$$

The continuity of u_r and u_z across $z = 0$ for all r implies that $P_1(k) + Q_1(k) = 0$ and $P(k) + Q(k) = 0$. The remaining boundary conditions can be shown to be satisfied provided $P_1(k) = 0$ and $P(k)$ is the solution of the following triple integral equations:

$$\int_0^\infty kP(k)\,J_0(kr)\,\mathrm{d}k = 0, \quad 0<r<a \tag{50}$$

$$\int_0^\infty k\left(\nu_1 - \frac{k^2}{\nu_2}\right)P(k)\,J_0(kr)\,\mathrm{d}k = -u_0, \quad a<r<b \tag{51}$$

$$\int_0^\infty kP(k)\,J_0(kr)\,\mathrm{d}k = 0, \quad b<r \tag{52}$$

The normal traction on the annular disc is given by:

$$\sigma_{zz}(r,0) = \mu k_2^2 \int_0^\infty kP(k)\,J_0(kr)\,\mathrm{d}k, \quad a<r<b \tag{53}$$

The integral equation (51) can be written in the following form:

$$\int_0^\infty P(k)\,J_0(kr)\,\mathrm{d}k = \frac{2u_0}{k_1^2+k_2^2} + \int_0^\infty G_1(k)\,P(k)\,J_0(kr)\,\mathrm{d}k, \quad a<r<b \tag{54}$$

where

$$G_1(k) = 1 + \frac{2k}{k_1^2+k_2^2}\left(\nu_1 - \frac{k^2}{\nu_2}\right) \tag{55}$$

It is to be noted that:

$$G_1(k) = 0(k^{-3}) \text{ as } k \to \infty$$

Let us assume that

$$\int_0^\infty kP(k)\,J_0(kr)\,\mathrm{d}k = h(r), \quad a<r<b \tag{56}$$

Making use of the inversion theorem for Hankel transforms we get from equations (50) and (56) that:

$$P(k) = \int_a^b rh(r)\,J_0(kr)\,\mathrm{d}r \tag{57}$$

Substituting the value of $P(k)$ from equation (57) into (54) we get:

$$\int_a^b uh(u)\,M(u,r)\,\mathrm{d}u = F(r), \quad a<r<b \tag{58a}$$

where

$$F(r) = \frac{2u_0}{k_1^2+k_2^2} + \int_0^\infty G_1(k)\,P(k)\,J_0(kr)\,\mathrm{d}k \tag{58b}$$

$$M(u,r) = \int_0^\infty J_0(kr)\,J_0(ku)\,\mathrm{d}k \tag{58c}$$

Following Cooke[7] the solution of integral equation (58a) can be written in the following form:

$$H_2(s) = \frac{2su_0}{(k_1^2+k_2^2)\sqrt{s^2-a^2}} - \frac{4s}{\pi^2\sqrt{s^2-a^2}}\int_a^b \frac{tH_2(t)\,K_1(s,t)}{\sqrt{t^2-a^2}}\,\mathrm{d}t + \int_0^\infty G_1(k)\,P(k)\,I_1(s,k)\,\mathrm{d}k, \quad a<s<b \tag{59}$$

where

$$I_1(s,\xi) = \frac{\mathrm{d}}{\mathrm{d}s}\int_a^s \frac{rJ_0(\xi r)\,\mathrm{d}r}{(s^2-r^2)^{1/2}} \tag{60}$$

$$K_1(s,t) = \int_0^a \frac{(a^2-y^2)\,\mathrm{d}y}{(s^2-y^2)(t^2-y^2)} = \frac{1}{2(t^2-s^2)} \times \left[\frac{(a^2-s^2)}{s}\log\left|\frac{s+a}{s-a}\right| - \frac{(a^2-t^2)}{t}\log\left|\frac{t+a}{t-a}\right|\right] \tag{61}$$

$$H_2(s) = \int_s^b \frac{h(r)\,\mathrm{d}r}{(r^2-s^2)^{1/2}}, \quad a<s<b \tag{62}$$

Equation (62) is of Abel type, hence its solution may be written as:

$$h(r) = -\frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}r}\int_r^b \frac{sH_2(s)\,\mathrm{d}s}{(s^2-r^2)^{1/2}} \tag{63}$$

We can easily prove that:

$$\int_0^\infty G_1(k)\,P(k)\,I_1(s,k)\,\mathrm{d}k = \int_a^b H_2(t)\,L_2(s,t)\,\mathrm{d}t \tag{64}$$

where

$$L_2(s,t) = \frac{2}{\pi}\int_0^\infty I_1(s,k)\,I_1(t,k)\,G_1(k)\,\mathrm{d}k \tag{65}$$

Hence the integral equation (59) can be written in the following form:

$$H_2(s) = \frac{2su_0}{\sqrt{s^2-a^2}(k_1^2+k_2^2)} + \int_a^b H_2(t)\,[L_1(s,t) + L_2(s,t)]\,\mathrm{d}t, \quad a<s<b \tag{66}$$

where

$$L_1(s,t) = \frac{-2}{\pi^2\sqrt{(s^2-a^2)(t^2-a^2)}\,[t^2-s^2)} \times \left[t(a^2-s^2)\log\left|\frac{s+a}{a-a}\right| - s(a^2-t^2)\log\left|\frac{t+a}{t-a}\right|\right] \tag{67}$$

If $a \to 0$, the integral equation (66) reduces to the corresponding equation of Mal.[9]

We can write equation (60) in the following form:

$$I(s, k) = \left[\frac{sJ_0(ka)}{\sqrt{s^2 - a^2}} \quad sk \int_a^s \frac{J_1(kr)\,\mathrm{d}r}{\sqrt{(s^2 - r^2)}}\right] \tag{68}$$

Let

$$s = a\sec\theta, \quad t = a\sec\phi,$$

$$\sin\theta H_2(a\sec\theta) = \frac{2u_0 R(\theta)}{(k_1^2 + k_2^2)} \tag{69}$$

then we can write the integral equation (66) in the following form:

$$\sin\theta R(\theta) = 1 + \sin\theta \int_0^{\sec^{-1}(b/a)} \sec^2\phi R(\phi) \times \{L_1(\phi, \theta) + L_2(\phi, \theta\}\,\mathrm{d}\phi, \quad 0 < \theta < \sec^{-1}\left(\frac{b}{a}\right) \tag{70}$$

where

$$L_1(\theta, \phi) = -\frac{4\cos^2\theta\cos^2\phi}{\pi^2\sin\theta\sin\phi\,[\cos^2\phi - \cos^2\theta]} \times \left[\cos\phi\tan^2\phi\log\left(\tan\frac{\phi}{2}\right) - \cos\theta\tan^2\theta\log\left(\tan\frac{\theta}{2}\right)\right] \tag{71}$$

$$L_2(\theta, \phi) = \frac{2}{\pi}\int_0^\infty G_1(k)\,I(a\sec\phi, k)\,I(a\sec\theta, k)\,\mathrm{d}k \tag{72}$$

$$I(a\sec\phi, k) = \left[\sec\phi\cot\phi J_0(ka) + k\tan\phi J_0(ka) + ka\sec\phi \int_a^{a\sec\phi} \frac{J_1(kr)\,\mathrm{d}r}{\sqrt{a^2\sec^2\phi - r^2}}\right] \tag{73}$$

The couple on the disc is given by the equation:

$$M = 2\pi\int_a^b r^2\sigma_{zz}(r, 0)\,\mathrm{d}r$$

$$= 4\mu k_2^2\left[\int_a^b \frac{a^2 s H_2(s)\,\mathrm{d}s}{\sqrt{s^2 - a^2}} + \int_a^b 2s\sqrt{s^2 - a^2}\,H_2(s)\,\mathrm{d}s\right]$$

$$= \frac{8\mu u_0 k_2^2 a^3}{(k_1^2 + k_2^2)}\int_0^{\sec^{-1}(b/a)} \frac{\sec^2\theta(1 + 2\tan^2\theta)}{\sin\theta} R(\theta)\,\mathrm{d}\theta \tag{74}$$

We know that:

$$\frac{k_1}{k_2} = \frac{c_2}{c_1} = \left(\frac{\mu}{\lambda + 2\mu}\right)^{1/2} = \left[\frac{1 - 2\nu}{2(1 - \nu)}\right]^{1/2} \tag{75}$$

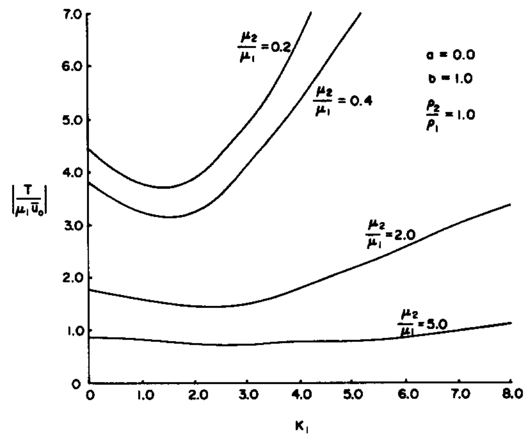

Figure 3. Variation of K_1 with $|T/\mu_1\bar{u}_0|$ for the solid disc. $[\bar{u}_0 = (1 + \alpha)^2 u_0; \alpha = \mu_2/\mu_1]$

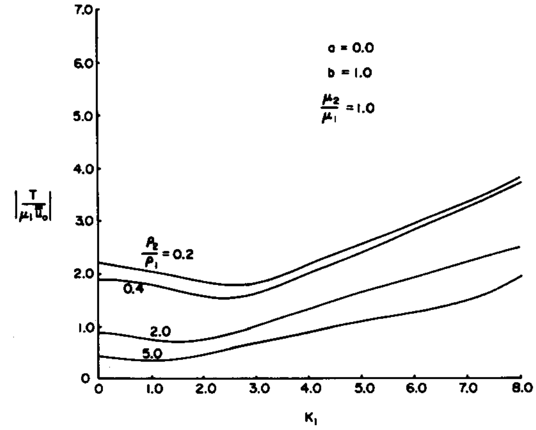

Figure 4. Variation of K_1 with $|T/\mu_1\bar{\bar{u}}_0|$ for the solid disc. $[\bar{\bar{u}}_0 = (1 + \alpha)^2\{1 + (\rho_2/\rho_2)\}u_0; \alpha = \mu_2/\mu_1]$

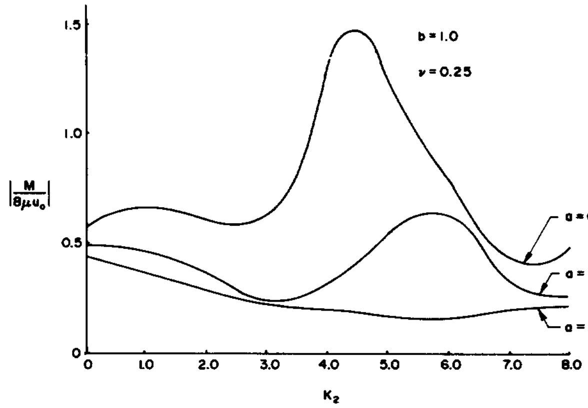

Figure 5. Variation of K_2 with $|M/8\mu u_0|$ for the annular disc

where μ and λ are Lame constants and c_1 is the P-wave velocity in the solid and c_2 is the velocity of the S-wave associated with the displacement and ν is the Poisson's ratio.

The numerical values of $|M/8u_0\mu|$ have been graphed in Fig. 3 against K_1 for various values of a and $\nu = 1/4$.

NUMERICAL RESULTS

The numerical values for $|T/\mu_1\bar{u}_0|$ and $|T/\mu_1\bar{\bar{u}}_0|$ shown in Figs. 1 and 2 have been obtained by making use of the

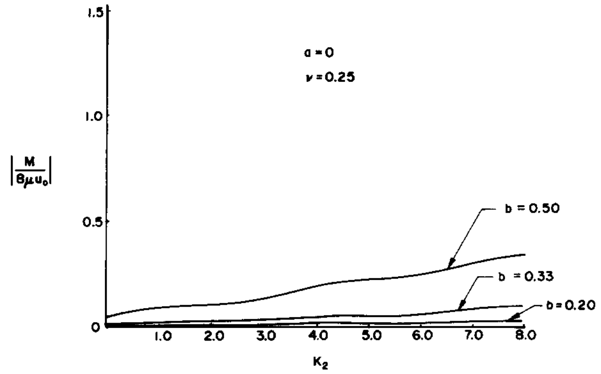

Figure 6. Variation of K_2 with $|M/8\mu u_0|$ for the solid inclusion

result (39). The numerical values of the function $H_1(\phi)$ required for the numerical evaluation of (39) have been obtained from a numerical solution of the integral equation (36). The results shown in Figs. 1 and 2 have been evaluated for the cases when (i) $\rho_2/\rho_1 = 1$ and for various values of μ_1/μ_2 and (ii) $\mu_1/\mu_2 = 1$ and for various values of ρ_1/ρ_2. The results given in Figs. 3 and 4 refer to the limiting case of a solid penny-shaped inclusion. Similarly the numerical values for $|M/8\mu u_0|$ shown in Figs. 5 and 6 have been obtained by making use of the result (74). The numerical values of the function R (s) required for the numerical evaluation of (74) have been obtained from a numerical solution of the integral equation (70). The results given in Fig. 6 refer to the limiting case of a solid penny-shaped inclusion. It may be noted that k_1 and k_2 have dimensions of $(\text{length})^{-1}$. By taking $K_1 = bk_1$ and $K_2 = bk_2$ we find that K_1 and K_2 are dimensionless. Since b in Figs. 1-6 have been set equal to unity the values of K_1 and K_2 are dimensionless.

REFERENCES

1 Loeber, J. F. and Sih, G. C. Transmission of anti-plane shear waves past an interface crack in dissimilar media, *Eng. Fracture Mech.* 1973, 5, 699

2 Loeber, J. F. and Sih, G. C. Torsional wave scattering about a penny-shaped crack lying on a bimaterial interface, *Dynamic Crack Propagation*, Sih, G. C., ed., Noordhoff International Publishers, Leyden, pp. 513-528, 1973

3 Shindo, Y. Diffraction of torsional waves by a flat annular crack, in an infinite elastic medium, *J. Appl. Mech.* 1979, 46, 827

4 Shindo, Y. Normal compression waves scattering at a flat annular crack in an infinite elastic medium, *Quart. Appl. Math.* 1981, 39, 305

5 Singh, B. M., Dhaliwal, R. S. and Vrbik, J. Diffraction of torsional waves by a flat annular crack at the interface of two bonded dissimilar elastic solids, *Eng. Fracture Mech.* (to appear)

6 Singh, B. M., Rokne, J. and Dhaliwal, R. S. Diffraction of torsional wave by a circular rigid disc at the interface of two bonded dissimilar elastic solids, *Acta Mechanica* 1983, 49, 139

7 Cooke, J. C. Triple integral equations, *Quart. J. Mech. Appl. Math.* 1963, 16, 193

8 Mal, A. K., Ang, D. D. and Knopff, L. Diffraction of elastic waves by a rigid circular disc, *Proc. Camb. Phil. Soc.* 1968, 64, 237

9 Mal, A. K. Motion of a rigid disc in an elastic solid, *Bull. Seism. Soc. America* 1971, 61, 1717