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THE ROTATION OF A RIGID ELLIPTICAL DISC INCLUSION EMBEDDED IN A TRANSVERSELY ISOTROPIC ELASTIC SOLID

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Abstract

This paper examines the problem of asymmetric rotation of a rigid elliptical disc inclusion embedded in bonded contact with a transversely isotropic elastic solid of infinite extent. The moment-rotation relationship for the embedded inclusion is evaluated in explicit closed form.

Introduction

Problems related to inclusions embedded in elastic media are of interest to the study of composite and multiphase elastic solids. A comprehensive account of developments pertaining to inclusion problems is given by Mura [1]. Specific application of these micromechanical theories to composite materials are documented by Christensen [2], Willis [3] and Walpole [4]. In a recent paper Selvadurai [5] examined the problem of the axial displacement of a rigid elliptical disc inclusion which is embedded in bonded contact with a transversely isotropic elastic solid. The generalized treatment of a three-dimensional ellipsoidal inclusion embedded in a transversely isotropic elastic solid is quite complicated. The elliptical disc inclusion problem therefore serves as a useful limiting case of the three-dimensional inclusion problem. This paper examines the problem in which the elliptical inclusion is subjected to a rotation about one of its principal axes. The plane of the rigid elliptical disc inclusion is assume to coincide with the plane of transverse isotropy (Figure 1). The solution to the elliptical inclusion problem is obtained by making use of an ellipsoidal coordinate formulation of the associated mixed boundary value problem. The results developed in this paper can be used in conjunction with Betti's reciprocal relationship to examine the interaction between elliptical defects and nuclei of strain.

Fundamental Formulae

Following Elliott [6,7], Green and Zerna [8] and Kassir and Sih [9] it can be shown that, in the absence of body forces, the displacement and stress fields in a transversely isotropic elastic medium can be expressed in terms of two 'harmonic' functions ϕ_i (i = 1,2) which are solutions of

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_i^2}\right\} \phi_i(x,y,z_i) = 0; \qquad (i = 1,2) \tag{1}$$

where $z_i = z/\sqrt{\nu_i}$ and ν_i are the roots of the equation

$$c_{11}c_{44}\nu^2 + [c_{13}(2c_{44}+c_{13}) - c_{11}c_{13}]\nu + c_{33}c_{44} = 0. \tag{2}$$

We note that c_{ij} are the elastic constants of the transversely isotropic elastic material and the z-axis is normal to the plane of isotropy. The displacement and stress components relevant to the present problem can be expressed in terms the derivatives of ϕ_i. We have

$$u_x = \frac{\partial}{\partial x}(\phi_1 + \phi_2); \qquad u_y = \frac{\partial}{\partial y}(\phi_1 + \phi_2) \tag{3}$$

$$u_z = \frac{\partial}{\partial z}(k_1\phi_1 + k_2\phi_2) \tag{4}$$

$$\sigma_{zz} = (k_1c_{33}-\nu_1c_{13}) \frac{\partial^2\phi_1}{\partial z^2} + (k_2c_{33}-\nu_2c_{13}) \frac{\partial^2\phi_2}{\partial z^2} \tag{5}$$

where k_1 and k_2 are given by

$$k_i = \frac{c_{11}\nu_i - c_{44}}{c_{13}+c_{44}}; \qquad (i = 1,2). \tag{6}$$

The Rigid Elliptical Disc Inclusion Problem

We examine the problem of a rigid elliptical disc inclusion which is embedded in bonded contact with the transversely isotropic elastic medium of infinite extent. For purposes of reference, it is convenient to adopt the following nomenclature. Referring to the plane $z = 0$ which contains the rigid elliptical inclusion, the inclusion region (i.e. $(x^2/a^2) + (y^2/b^2) \leq 1$; where a and b are respectively the major and minor semi-axes of the elliptical disc inclusion) is denoted by S_I; the region exterior to the inclusion region is denoted by S_E; also $S = S_I \cup S_E$. The disc inclusion experiences a rigid rotation Ω about the y-axis. The state of deformation induced in the elastic medium due to the rotation of the rigid inclusion is such that the displacements u_x and u_y and the stress σ_{zz} exhibit a state of asymmetry about the plane $z = 0$. The analysis of the inclusion problem can thus be restricted to the analysis of a single half-space region in which the plane $z = 0^+$ (the positive superscript refers to the halfspace region occupying $z \geq 0$) is subjected to appropriate mixed boundary conditions. Owing to the fully bonded conditions at the inclusion-elastic medium interface it is evident that u_x and u_y are zero in the region S_I.

From the preceding discussion we note that

$$u_x(x,y,0^+) = u_y(x,y,0^+) = 0; \quad (x,y) \in S_E \tag{7}$$

$$\sigma_{zz}(x,y,0^+) = 0 \quad ; \quad (x,y) \in S_E. \tag{8}$$

In the inclusion region

$$u_z(x,y,0^+) = \Omega x \quad ; \quad (x,y) \in S_I \tag{9}$$

$$u_x(x,y,0^+) = u_y(x,y,0^+) = 0; \quad (x,y) \in S_I \tag{10}$$

From (7) and (10) it is evident that

$$u_x(x,y,0^+) = u_y(x,y,0^+) = 0; \quad (x,y) \in S. \tag{11}$$

To satisfy the displacement boundary conditions (11) we select solutions of (1) which are of the form

$$\phi_1 = \Phi(x,y,z_1) \; ; \quad \phi_2 = \Phi(x,y,z_2) \tag{12}$$

where $\nabla^2\Phi = 0$ and ∇^2 is Laplace's operator referred to the rectangular Cartesian coordinate system. Using (4), (5) and (12) the mixed boundary conditions (8) and (9) can be expressed in the form

$$\frac{\partial\Phi}{\partial z} = \frac{\Omega x\sqrt{\nu_1\nu_2}}{\{k_1\sqrt{\nu_2}-k_2\sqrt{\nu_1}\}} \; ; \qquad (x,y) \in S_I \tag{13}$$

$$\frac{\partial^2\Phi}{\partial z^2} = 0 \; ; \qquad (x,y) \in S_E. \tag{14}$$

For the analysis of the mixed boundary value problem defined by (13) and (14) it is convenient to express the potential Φ in relation to a system of ellipsoidal coordinates (ξ,η,ζ) which are defined in such a way that ξ, η and ζ are the roots of the equation

$$\frac{x^2}{(a^2+s)} + \frac{y^2}{(b^2+s)} + \frac{z^2}{s} = 1. \tag{15}$$

The coordinates ξ , η and ζ have values subject to the restrictions

$$-a^2 < \zeta < -b^2 < \eta < 0 < \xi. \tag{16}$$

In the ellipsoidal coordinate system S_I corresponds to $\xi = 0$ and S_E corresponds to $\eta = 0$. When $\xi = 0$ the remaining coordinates η and ζ are obtained from the roots of the equation

$$\frac{x^2}{(a^2+s)} + \frac{y^2}{(b^2+s)} = 1 \tag{17}$$

and the product of the roots is found to be

$$\eta\zeta = a^2b^2 \{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\}. \tag{18}$$

Using the techniques outlined by Segedin [10], Kassir and Sih [11] and Stallybrass [12] it can be shown that the appropriate solution for $(\partial\Phi/\partial z)$ is

$$\frac{\partial\Phi}{\partial z} = Ax \int_{\xi}^{\infty} \frac{ds}{(a^2 + s)[s(a^2 + s)(b^2 + s)]^{1/2}} = \frac{2Ax[u - E(u)]}{a^3 e_0^2} \tag{19}$$

where A is an arbitrary constant and $e_0^2 = (a^2 - b^2)/a^2$. The independent variable u is related to the ellipsoidal coordinate ξ by

$$\xi^2 = a^2(\mathrm{sn}^{-2}u - 1) \tag{20}$$

and

$$E(u) = \int_0^u (\mathrm{dn}^2 t)dt. \tag{21}$$

The quantities sn u, dn u, etc., represent the Jacobian elliptic functions (see e.g. [13]) which have real and imaginary roots 4K and 2iK* respectively corresponding to the moduli e_0 and $e_0^* = b/a$. It may also be noted that

$$K(e_0) = \int_0^1 \frac{dt}{[(1-t^2)(1-e_0^2t^2)]^{1/2}} \; ; \; E(e_0) = \int_0^1 \left\{\frac{1 - e_0^2t^2}{1 - t^2}\right\}^{1/2} dt. \tag{22}$$

Considering (19) and the boundary condition (13), it is possible to determine the arbitrary constant A. This completes the formal analysis of the problem. The normal stress σ_{zz} acting on the plane of asymmetry z = 0 can be evaluated in the following form:

$$\sigma_{zz}(x,y,0^+) = -\frac{2x\Omega a^2 e_0^2[\eta\zeta(a^2+\xi)(b^2+\xi)]^{1/2} c_{33}\{k_1\nu_2-k_2\nu_1\}}{b(a^2+\xi)[K(e_0)-E(e_0)](\xi-\eta)(\xi-\zeta)\sqrt{\nu_1\nu_2}\{k_1\sqrt{\nu_2}-k_2\sqrt{\nu_1}\}} \tag{23}$$

The normal stress σ_{zz} in the inclusion region S_I is given by

$$\sigma_{zz}(x,y,0^+) = -\frac{2c_{33}\{k_1\nu_2-k_2\nu_1\}e_0^2\Omega x}{\sqrt{\nu_1\nu_2}\{k_1\sqrt{\nu_2}-k_2\sqrt{\nu_1}\}[K(e_0)-E(e_0)]b[1-\frac{x^2}{a^2}-\frac{y^2}{b^2}]^{1/2}} \tag{24}$$

From the asymmetry of the state of stress, $\sigma_{zz}(x,y,0^+) = -\sigma_{zz}(x,y,0^-)$. The normal stresses in the inclusion region can be used to evaluate the moment-rotation relationship for the embedded inclusion. The moment (M) acting on the rigid elliptical inclusion is given by

$$M = \iint_{S_I} [\sigma_{zz}(x,y,0^+) - \sigma_{zz}(x,y,0^-)]x\, dx\, dy. \tag{25}$$

Evaluating (25), we obtain the moment-rotation relationship for the embedded inclusion. Assuming that the rotation Ω occurs in the direction of application of M we have

$$M = \frac{4\pi c_{33} a^3 e_0^2 \Omega\{k_1\nu_2-k_2\nu_1\}}{3\sqrt{\nu_1\nu_2}\{K(e_0)-E(e_0)\}\{k_1\sqrt{\nu_2}-k_2\sqrt{\nu_1}\}} . \tag{26}$$

The analysis and results presented considers only the case where the inclusion rotates about the y-axis. The analytical techniques can be extended to cover the situation in which the inclusion is subjected to a moment M* which induces a rotation Ω^* about the x axis. In this case the moment-rotation relationship takes the form

$$M^* = \frac{4\pi c_{33} a^3 e_0^2 (1-e_0^2)\Omega^*\{k_1\nu_2-k_2\nu_1\}}{3\sqrt{\nu_1\nu_2}\{E(e_0)-(1-e_0^2)K(e_0)\}\{k_1\sqrt{\nu_2}-k_2\sqrt{\nu_1}\}} . \tag{27}$$

Limiting Cases

In the limiting case when $\nu_i \to 1$, we recover from (26) the moment-rotation relationship for a rigid elliptical disc inclusion which is embedded in bonded contact with an isotropic elastic medium. We note that as $\nu_i \to 1$

$$c_{11} = c_{33} = \lambda + 2\mu; \quad c_{44} = \mu$$
$$\frac{k_1\nu_2 - k_2\nu_1}{k_1\sqrt{\nu}_2 - k_2\sqrt{\nu}_1} = \frac{2\mu}{\lambda + 3\mu} \tag{28}$$

where λ and μ are Lame's constants. Using these results (26) can be reduced to the form

$$M = \frac{16\pi a^3 \mu e_0^2 \Omega(1-\nu)}{3(3-4\nu)\{K(e_0)-E(e_0)\}} . \tag{29}$$

In the limit as $a \to b$, (26) gives the moment-rotation relationship for a rigid circular disc inclusion embedded in a transversely isotropic elastic solid: i.e.

$$M = \frac{16c_{33}a^3\Omega\{k_1\nu_2 - k_2\nu_1\}}{3\sqrt{\nu_1\nu_2}\{k_1\sqrt{\nu}_1 - k_2\sqrt{\nu}\ \}} . \tag{30}$$

The expression (30) is in agreement with the result for the rotational stiffness of a rigid circular disc inclusion embedded in a transversely isotropic elastic solid determined, separately, by making use of a Hankel transform development of the governing mixed boundary value problem [14].

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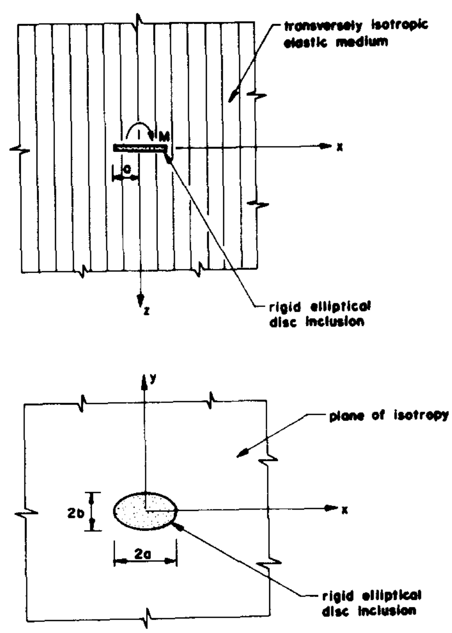

Figure 1: Geometry of the embedded rigid elliptical disc inclusion