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# On the expansion of a penny-shaped crack by a rigid circular disc inclusion

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## Abstract

This paper examines the problem of the symmetric indentation of a penny-shaped crack by a smoothly embedded rigid circular thin disc inclusion. The analysis of the problem yields a system of triple integral equations which are solved in an approximate manner. An expression for the stress intensity factor at the boundary of the penny-shaped crack is evaluated in the form of a series which involves the ratio of the radius of the rigid circular inclusion to the radius of the penny-shaped crack.

## 1. Introduction

The class of problems which examine the loading of a penny-shaped crack by tractions applied at the surfaces of the crack have been examined in detail by several investigators including Sneddon [1] and Barenblatt [2]. Detailed accounts of the stress analysis of these problems and an assessment of the associated stress intensity factors are given by Sneddon and Lowengrub [3], Kassir and Sih [4] and Cherepanov [5]. The problem related to the internal loading of a penny-shaped crack is of some importance to the modelling of hydraulically induced fracture of resource bearing geological formations. In this paper we examine the problem related to the indentation of a penny-shaped crack by a smooth disc

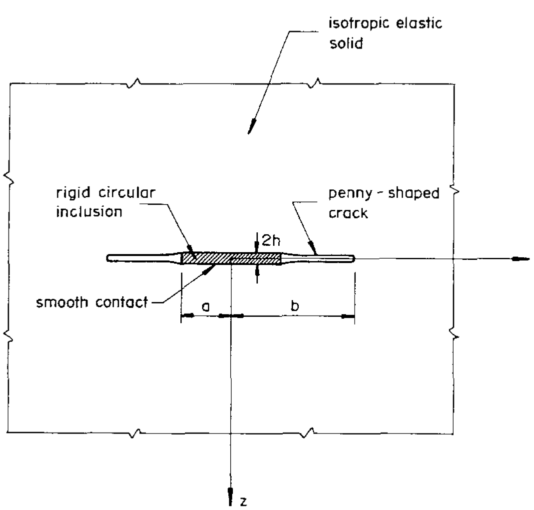


Figure 1. Indentation of a penny-shaped crack by a smooth circular disc inclusion.

shaped rigid inclusion (Fig. 1). To the authors' knowledge a solution to this problem is not available in literature on fracture mechanics. When displacement boundary conditions are prescribed at the surface of the penny-shaped crack the character of the problem changes essentially from a two-part to a three-part mixed boundary value problem. A Hankel transform development of the problem yields a system of triple integral equations which can be solved in an approximate fashion. The analysis of the problem focusses on the estimation of the stress intensity factor for the indented penny-shaped crack. This stress intensity factor is evaluated in a series form in terms of a non-dimensional parameter which involves the ratio of the radius of the rigid circular inclusion to the radius of the penny-shaped crack. The penny-shaped crack problem discussed here is of some interest to the modelling of fracture processes in composite elastic materials which are reinforced with dilute concentrations of rigid circular inclusions. In this category of problem the indentation of the penny-shaped crack can be caused by thermally induced dilatation of the inclusion.

## 2. Basic formulae

For the solution of the axisymmetric elastostatic problem discussed previously it is convenient to adopt a formulation which is based on Love's strain potential approach [6,7]. For a medium free of body forces, the solution of the displacement equations of equilibrium can be represented in terms of a biharmonic function $\Phi(r, z)$; i.e.

$$\nabla^2\nabla^2\Phi(r, z) = 0 \tag{1}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \tag{2}$$

is Laplace's operator referred to the cylindrical polar coordinate system $(r, \theta, z)$. The components of the displacement vector $\boldsymbol{u}$ and the Cauchy stress tensor $\boldsymbol{\sigma}$ referred to the cylindrical polar coordinate system can be expressed in terms of the derivatives of $\Phi(r, z)$. We have

$$2Gu_r = -\frac{\partial^2\Phi}{\partial r\partial z} \tag{3a}$$

$$2Gu_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \tag{3b}$$

where $G$ and $\nu$ are the linear elastic shear modulus and Poisson's ratio of the material respectively. Similarly, the components of the stress tensor are given by

$$\sigma_{rr} = \frac{\partial}{\partial z}\left(\nu\nabla^2\Phi - \frac{\partial^2\Phi}{\partial r^2}\right) \tag{4a}$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left(\nu\nabla^2\Phi - \frac{1}{r}\frac{\partial\Phi}{\partial r}\right) \tag{4b}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left([2-\nu]\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right) \tag{4c}$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left([1-\nu]\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right). \tag{4d}$$

## 3. The indentation problem

We consider the isotropic elastic region $(r \in (0, \infty);\ z \in (\infty, -\infty))$, which is bounded internally by a penny-shaped crack of radius "$b$". The traction free plane surfaces of the penny-shaped crack are indented by a smooth, rigid circular disc inclusion of radius "$a$" and thickness "$2h$" (Fig. 1). The indentation process is assumed to be such that complete contact is maintained between the elastic medium and the plane ends of the rigid circular disc inclusion. Since the problem exhibits a state of symmetry about the plane $z = 0$ we can restrict our attention to a single halfspace region occupying $z > 0$ and denote by $z = 0^+$ the plane of symmetry associated with that region. The mixed boundary conditions associated with the problem are

$$\sigma_{rz}(r, 0^+) = 0; \quad r \geq 0 \tag{5}$$

$$u_z(r, 0^+) = h; \quad 0 \leq r \leq a \tag{6}$$

$$\sigma_{zz}(r, 0^+) = 0; \quad a < r < b \tag{7}$$

$$u_z(r, 0^+) = 0; \quad b \leq r < \infty. \tag{8}$$

In order to examine the mixed boundary value problem defined by (5)–(8), it is convenient to employ a solution of Love's strain potential which is based on a Hankel transform development of the governing differential equation (1). The integral representation for $\Phi(r, z)$ can be chosen such that the stresses and displacements derived from $\Phi(r, z)$ reduce to zero as $(r^2 + z^2)^{1/2} \to \infty$. The relevant solution is (Sneddon [8])

$$\Phi(r, z) = \int_0^\infty [A(\xi) + zB(\xi)]\, e^{-\xi z} J_0(\xi r) \mathrm{d}\xi \tag{9}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions which need to be determined by satisfying the boundary conditions (5)–(8).

The stresses and displacements in the elastic medium can be determined by making use of Love's strain potential (9) and the relevant expressions (3a–b) and (4a–d). Consequently the mixed boundary conditions (5) to (8) yield the following set of triple integral equations for a single unknown function $R(\xi)$ (the functions $A(\xi)$ and $B(\xi)$ can be expressed in terms of $R(\xi)$). We have

$$H_0[\xi^{-2}R(\xi); r] = h^*; \quad 0 \leq r \leq a \tag{10}$$

$$H_0[\xi^{-1}R(\xi); r] = 0; \quad a < r < b \tag{11}$$

$$H_0[\xi^{-2}R(\xi); r] = 0; \quad b \leq r < \infty \tag{12}$$

where $H_0$ is the Hankel operator defined by

$$H_0[f(\xi); r] = \int_0^\infty \xi f(\xi) J_0(\xi r) \mathrm{d}\xi \tag{13}$$

and

$$h^* = -\frac{Gh}{(1-\nu)}. \tag{14}$$

The set of triple integral equations defined by (10) to (12) can be solved by employing the approximate procedures developed by Cooke [9]. General accounts of the techniques that may be employed for the solution of systems of triple integral equations are given by Williams [10], Tranter [11], Sneddon [12] and Kanwal [13]. For the solution of the triple system (10) to (12) we assume that (11) admits a representation of the form

$$\int_0^\infty R(\xi) J_0(\xi r) \mathrm{d}\xi = \begin{cases} f_1(r); & 0 < r < a \quad (15) \\ f_2(r); & b < r < \infty \quad (16) \end{cases}$$

Making use of the Hankel inversion theorem we obtain

$$R(\xi) = \xi \left[ \int_0^a \eta f_1(\eta) J_0(\eta\xi) \mathrm{d}\eta + \int_b^\infty \eta f_2(\eta) J_0(\eta\xi) \mathrm{d}\eta \right]. \tag{17}$$

Substituting the value of R($\xi$) defined above in (10) and (12) and changing the order of the integration we obtain, respectively,

$$\int_0^a \eta f_1(\eta) L_1(\eta, r) \mathrm{d}\eta + \int_b^\infty \eta f_2(\eta) L_1(\eta, r) \mathrm{d}\eta = h^*; \quad 0 \le r \le a \tag{18}$$

$$\int_0^a \eta f_1(\eta) L_1(\eta, r) \mathrm{d}\eta + \int_b^\infty \eta f_2(\eta) L_1(\eta, r) \mathrm{d}\eta = 0; \quad b < r < \infty \tag{19}$$

where

$$L_1(\eta, r) = \int_0^\infty J_0(\xi\eta) J_0(\xi r) \mathrm{d}\xi. \tag{20}$$

Following Noble [14] and Cooke [9] it can be shown that

$$L_1(\eta, r) = \begin{cases} \dfrac{2}{\pi} \displaystyle\int_0^{\min(\eta, r)} \dfrac{\mathrm{d}s}{[(\eta^2 - s^2)(r^2 - s^2)]^{1/2}} & (21) \\ \dfrac{2}{\pi} \displaystyle\int_{\max(\eta, r)}^\infty \dfrac{\mathrm{d}s}{[(s^2 - \eta^2)(s^2 - r^2)]^{1/2}} & (22) \end{cases}$$

where $\min(\eta, r)$ and $\max(\eta, r)$ denote the minimum and maximum values of $\eta$ and $r$ respectively. Substituting (21) in (18) and changing the order of the integration we obtain

$$\int_0^r \frac{\mathrm{d}s}{(r^2 - s^2)^{1/2}} \int_s^a \frac{\eta f_1(\eta) \mathrm{d}\eta}{(\eta^2 - s^2)^{1/2}} + \frac{\pi}{2} \int_b^\infty \eta f_2(\eta) L_1(\eta, r) \mathrm{d}\eta = \frac{\pi}{2} h^*; \quad 0 \le r \le a. \tag{23}$$

Denote

$$\int_s^a \frac{\eta f_1(\eta) \mathrm{d}\eta}{(\eta^2 - s^2)^{1/2}} = N(s); \quad 0 < s < a. \tag{24}$$

Then by using the solution for Abel's integral equation (23) can be reduced to the form

$$N(s) = h^* - \frac{\mathrm{d}}{\mathrm{d}s} \int_0^s \frac{r \mathrm{d}r}{(s^2 - r^2)^{1/2}} \int_b^\infty \eta f_2(\eta) L_1(\eta, r) \mathrm{d}\eta; \; 0 < s < a. \tag{25}$$

It can be shown that (see e.g. the result A3.2 given by Cooke [9])

$$\frac{\mathrm{d}}{\mathrm{d}s} \int_0^s \frac{r \mathrm{d}r}{(s^2 - r^2)^{1/2}} \int_b^\infty \eta f_2(\eta) L_1(\eta, r) \mathrm{d}\eta = \begin{cases} \displaystyle\int_b^\infty \dfrac{\eta f_2(\eta) \mathrm{d}\eta}{(\eta^2 - s^2)^{1/2}} & ; \eta > s \quad (26) \\ 0 & ; \eta < s \quad (27) \end{cases}$$

and from (24) we have

$$\eta f_1(\eta) = -\frac{2}{\pi} \frac{\mathrm{d}}{\mathrm{d}\eta} \int_\eta^a \frac{s N(s) \mathrm{d}s}{(s^2 - \eta^2)^{1/2}} \tag{28}$$

Now substituting the value of $N(s)$ defined by (24) into (28) and using the results (26)–(27) and the relationship

$$\frac{\mathrm{d}}{\mathrm{d}\eta} \int_\eta^a \frac{\mathrm{d}s}{[(s^2 - \eta^2)(\xi^2 - s^2)]^{1/2}} = \frac{-\eta(\xi^2 - a^2)^{1/2}}{(a^2 - \eta^2)^{1/2}(\xi^2 - \eta^2)} \tag{29}$$

we obtain

$$\frac{\pi}{2}(a^2 - \eta^2)^{1/2} f_1(\eta) = \eta h^* - \int_b^\infty \frac{t(t^2 - a^2)^{1/2} f_2(t) \mathrm{d}t}{(t^2 - \eta^2)}; \quad 0 \le \eta \le a. \tag{30}$$

Similarly it can be shown that by using (19) and (22) we obtain

$$\frac{\pi}{2}(\eta^2-b^2)^{1/2}f_2(\eta) = -\int_0^a \frac{t(b^2-t^2)^{1/2}f_1(t)\mathrm{d}t}{(\eta^2-t^2)}; \quad b<\eta<\infty. \tag{31}$$

By substituting the value of $f_2(\eta)$ defined by (31) into (30) and making use of the substitutions

$$\psi(\eta_1) = \frac{\pi a}{2h^*}(1-\eta_1^2)^{1/2}f_1(a\eta_1); \tag{32}$$

$$\xi=\xi_1 a; \quad \eta=\eta_1 a \tag{33}$$

we obtain the following Fredholm integral equation of the second kind for $\psi(\eta_1)$;

$$\psi(\eta_1) = \eta_1 + \int_0^1 \psi(\xi_1)K(\xi_1,\eta_1)\mathrm{d}\xi_1; \quad 0\le\eta_1\le 1 \tag{34}$$

where

$$K(\xi_1,\eta_1) = \frac{4c\xi_1\eta_1(1-c^2\xi_1^2)^{1/2}}{\pi^2(1-\xi_1^2)^{1/2}}\int_1^\infty F(\xi_1,\eta_1,t_1)\mathrm{d}t_1 \tag{35}$$

$$F(\xi_1,\eta_1,t_1) = \frac{\left[1-\dfrac{c^2}{t_1^2}\right]^{1/2}}{t_1^3\left[1-\dfrac{1}{t_1^2}\right]^{1/2}\left[1-\dfrac{c^2\xi_1^2}{t_1^2}\right]\left[1-\dfrac{c^2\eta_1^2}{t_1^2}\right]} \tag{36}$$

and $c=a/b$. In order to develop an approximate solution for the Fredholm integral equation of the second kind defined by (34) we assume that the kernel function $K(\xi_1,\eta_1)$ can be expressed as a series in terms of a small parameter. A number of such parameters can be assigned; however, in the present discussion we shall assume that the radii ratio $c$ is such that $c<1$. Accordingly, $K$ can be expressed in the form

$$\begin{aligned}K(\xi_1,\eta_1) = \frac{4\xi_1\eta_1}{\pi^2(1-\xi_1^2)^{1/2}}\Bigg[&c + c^3\left\{\frac{\xi_1^2}{6}+\tfrac{2}{3}(\eta_1^2-\tfrac{1}{2})\right\}\\ &+c^5\left\{\frac{8}{15}\left(\xi_1^4+\eta_1^4+\xi_1^2\eta_1^2-\frac{1}{2}\xi_1^2-\frac{1}{2}\eta_1^2-\frac{1}{8}\right)-\frac{\xi_1^4}{8}-\frac{\xi_1^2}{3}\left(\xi_1^2+\eta_1^2-\frac{1}{2}\right)\right\}\\ &+c^7\left\{\frac{16}{35}\left(\xi_1^6+\eta_1^6-\frac{3}{16}-\frac{1}{8}[\xi_1^2+\eta_1^2]-\frac{1}{2}[\xi_1^4+\eta_1^4+\xi_1^2\eta_1^2]\right.\right.\\ &\left.+\xi_1^4\eta_1^2+\eta_1^4\xi_1^2\right)+\frac{4}{15}\xi_1^2\left(\xi_1^4+\eta_1^4+\xi_1^2\eta_1^2-\frac{1}{2}[\xi_1^2+\eta_1^2]\right)\\ &\left.\left.-\frac{3}{16}\xi_1^6-\frac{\xi_1^4}{12}\left[\xi_1^2+\eta_1^2-\frac{1}{2}\right]\right\}\right]+\mathrm{O}(c^9)\end{aligned} \tag{37}$$

where $\mathrm{O}(c^n)$ is the Landau symbol. We assume that the function $\psi(\eta_1)$ can be expressed in the form

$$\psi(\eta_1) = \sum_{i=0}^{n} c^n\psi_n(\eta_i). \tag{38}$$

Substituting the above result for $\psi(\eta_1)$ in (34) and comparing terms of order $c^n (n = 0, 1, \dots 6)$ we can obtain a solution for $\psi_n(\eta_1)$. It can be shown that

$$\psi(\eta_1) = \eta_1 + \frac{c\eta_1}{\pi} + \frac{c^2\eta_1}{\pi^2} + c^3\left\{\frac{\eta_1}{\pi^2}\left(\frac{1}{\pi} + 4\pi\left[\frac{\eta_1^2}{6} - \frac{5}{96}\right]\right)\right\}$$

$$+ c^4\left\{\frac{4\eta_1}{\pi^3}\left(\frac{5\pi}{48} + \frac{1}{4\pi} + \frac{\pi}{6}\left[\eta_1^2 - \frac{1}{2}\right]\right)\right\} + c^5\left\{\frac{4\eta_1}{\pi^4}\left(\frac{7\pi}{48} + \frac{1}{4\pi}\right)\right.$$

$$\left. + \frac{2\eta_1}{\pi^3}\left(\frac{\eta_1^2}{3} - \frac{5}{48}\right) + \frac{4\eta_1}{\pi}\left(\frac{2\eta_1^4}{15} - \frac{7\eta_1^2}{240} - \frac{91}{3840}\right)\right\} + \mathrm{O}(c^6) \tag{39}$$

and the function $f_1(a\eta_1)$ can be obtained by making use of (32) and (39). The result (31) can now be employed to derive a solution for the function $f_2(\eta)$. Introduce the change of variables according to

$$\chi(\eta_2) = \frac{\pi b}{2}(\eta_2^2 - 1)^{1/2} f_2(b\eta_2)$$

$$\eta = \eta_2 b. \tag{40}$$

Then (31) can be written in the following form:

$$\chi(\eta_2) = -\frac{2ch^*}{\pi\eta_2^2}\left[\int_0^1 \frac{\xi_1\psi(\xi_1)}{(1-\xi_1^2)^{1/2}}\left\{1 + \frac{c^2\xi_1^2}{2\eta_2^2}(2 - \eta_2^2)\right.\right.$$

$$\left.\left. + c^4\xi_1^4\left(\frac{8 - \eta_2^4 - 4\eta_2^2}{8\eta_2^4}\right) + \mathrm{O}(c^6)\right\}\mathrm{d}\xi_1\right]; \quad 1 < \eta_2 < \infty. \tag{41}$$

Substituting (39) into (41) and performing the necessary integrations it can be shown that

$$\chi(\eta_2) = -h^*\left[\frac{c}{2\eta_2^2} + \frac{c^2}{2\pi\eta_2^2} - c^3\left\{\frac{1}{2\pi^2\eta_2^2} + \frac{3(2-\eta_2^2)}{16\eta_2^4}\right\}\right.$$

$$+ c^4\left\{\frac{1}{\pi\eta_2^2}\left(\frac{7}{48} + \frac{1}{2\pi^2} + \frac{3(2-\eta_2^2)}{16\eta_2^2}\right)\right\}$$

$$\left. + c^5\left\{\frac{2}{\pi\eta_2^2}\left(\frac{5\pi[8 - \eta_2^4 - 4\eta_2^2]}{256\eta_2^4} + \frac{3[2-\eta_2^2]}{32\pi\eta_2^2} + \frac{1}{4\pi^3} + \frac{7}{48\pi}\right)\right\} + \mathrm{O}(c^6)\right]. \tag{42}$$

In summary, we note that the relevant series approximations for the functions $f_1(r)$ and $f_2(r)$ are given by

$$f_1(r) = \frac{2h^*}{\pi(a^2 - r^2)^{1/2}}\left\{\rho + \frac{c\rho}{\pi} + \frac{c^2\rho}{\pi^2} + c^3\left(\frac{\rho}{\pi}\left[\frac{1}{\pi} + 4\pi\left(\frac{\rho^2}{6} - \frac{5}{96}\right)\right]\right)\right.$$

$$+ c^4\left[\frac{4\rho}{\pi^3}\left(\frac{5\pi}{48} + \frac{1}{4\pi} + \frac{\pi}{6}\left[\rho^2 - \tfrac{1}{2}\right]\right)\right] + c^5\left[\frac{4\rho}{\pi^4}\left(\frac{7\pi}{48} + \frac{1}{4\pi}\right) + \frac{2\rho}{\pi^3}\left(\frac{\rho^3}{3} - \frac{5}{48}\right)\right.$$

$$\left.\left. + \frac{4\rho}{\pi}\left(\frac{2}{15}\rho^4 - \frac{7}{240}\rho^2 - \frac{91}{3840}\right)\right] + \mathrm{O}(c^6)\right\} \tag{43}$$

and

$$f_2(r) = -\frac{2h^*}{\pi(r^2-b^2)^{1/2}}\left\{\frac{c}{2\gamma^2}+\frac{c^2}{2\pi\gamma^2}+c^3\left(\frac{1}{2\pi\gamma^2}+\frac{3[2-\gamma^2]}{16\gamma^4}\right)\right.$$

$$+c^4\left(\frac{1}{\pi\gamma^2}\left\{\frac{7}{48}+\frac{1}{2\pi^2}+\frac{3(2-\gamma^2)}{16\gamma^2}\right\}\right)$$

$$\left.+c^5\left(\frac{2}{\pi\gamma^2}\left\{\frac{5\pi}{256\gamma^4}[8-\gamma^4-4\gamma^2]+\frac{3(2-\gamma^2)}{32\pi\gamma^2}+\frac{1}{4\pi^3}+\frac{7}{48\pi}\right\}\right)+O(c^6)\right\} \tag{44}$$

where $\rho = r/a$ and $\gamma = r/b$.

This formally completes the approximate solution of the system of triple integral equations (10) to (12). The stresses and displacements in the elastic medium can be expressed in terms of the function $R(\xi)$ or the functions $f_i(r)$ $(i = 1, 2)$.

## 4. The stress intensity factor

In this section we shall examine the influence of the wedging action of the rigid circular disc inclusion on the stress concentration at the boundary of the penny-shaped crack.

From (11) and (16) it is evident that

$$\sigma_{zz}(r, 0) = -\int_0^\infty R(\xi)J_0(\xi r)\mathrm{d}\xi = -f_2(r); \quad b < r < \infty. \tag{45}$$

The stress intensity factor at the crack boundary $r = b$ is defined by

$$K_\mathrm{I}^\mathrm{T} = \lim_{r\to b^+}[2(r-b)]^{1/2}\sigma_{zz}(r, 0). \tag{46}$$

The superscript "T" is intended to signify the result derived from a solution of the system of triple integral equations. Using the result (44) and (45) in (46) and performing the necessary simplifications we obtain

$$K_\mathrm{I}^\mathrm{T} = \frac{hG}{\pi(1-\nu)\sqrt{b}}\left[c+c^2\left\{\frac{1}{\pi}\right\}+c^3\left\{\frac{1}{\pi^2}+\frac{3}{8}\right\}+c^4\left\{\frac{2}{3\pi}+\frac{1}{\pi^3}\right\}\right.$$

$$\left.+c^5\left\{\frac{23}{24\pi^2}+\frac{15}{64}+\frac{1}{\pi^4}\right\}+O(c^6)\right] \tag{47}$$

and $c = a/b$. The variation of the normalized stress intensity factor $\bar{K}_\mathrm{I}^\mathrm{T} = K_\mathrm{I}^\mathrm{T}\pi(1-\nu)\sqrt{b}/hG$ with $a/b$ is shown in Fig. 2.

It is of interest to note that when the radii ratio $c$ is such that terms of order $c^2$ and higher can be neglected then the stress intensity factor for the crack indentation problem can be estimated separately by making use of classical results given by Sneddon [15] and Kassir and Sih [4]. To enable such a derivation we shall assume that for small values of $a/b$, the normal contact stress at the disc inclusion—crack surface interface can be approximated by a Boussinesq type stress distribution derived for a loaded rigid circular punch resting in frictionless contact with an elastic halfspace.

Considering (15) and (43), the stress distribution in the disc inclusion region is given by (to $O(c^2)$)

$$\sigma_{zz}(r, 0^+) = -\frac{2h^*}{\pi(a^2-r^2)^{1/2}}\left\{\frac{r}{a}\left(1+\frac{c}{\pi}\right)\right\}. \tag{48}$$

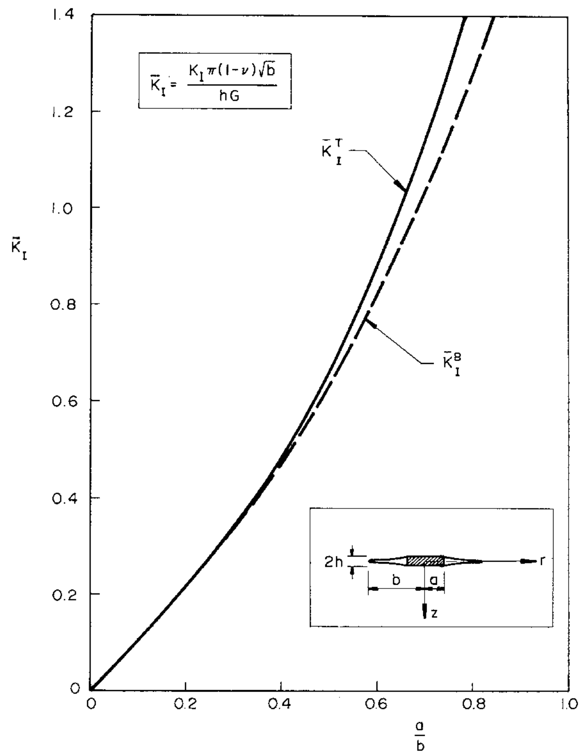


Figure 2. Stress intensity factors for a penny-shaped crack indented by a smooth rigid circular inclusion.

Therefore to O($c$), the total load exerted on the inclusion by the wedging action is given by

$$P^* = 2\pi\int_0^a \sigma_{zz}(r, 0^+) r \mathrm{d}r = -\frac{\pi hGa}{(1-\nu)}\left(1+\frac{c}{\pi}\right). \tag{49}$$

The associated "Boussinesq" stress distribution is given by

$$\sigma_{zz}^{\mathrm{B}}(r, 0^+) = \frac{P^*}{2\pi a(a^2-r^2)^{1/2}} = -\frac{hG}{2(1-\nu)(a^2-r^2)^{1/2}}\left(1+\frac{c}{\pi}\right). \tag{50}$$

When this distribution of surface tractions act on the faces of the penny-shaped crack within the region $r \leq a$, the stress intensity factor $K_{\mathrm{I}}$ can be obtained from the relationship

$$K_{\mathrm{I}}^{\mathrm{B}} = -\frac{1}{\pi\sqrt{b}}\int_0^a \frac{\sigma_{zz}^{\mathrm{B}}(r, 0^+) r \mathrm{d}r}{\sqrt{b^2-r^2}}. \tag{51}$$

The superscript "B" is intended to signify the stress intensity factor associated with the simplified "Boussinesq" distribution of normal tractions. Evaluating (51) we obtain

$$K_{\mathrm{I}}^{\mathrm{B}} = \frac{hG}{2\pi\sqrt{b}(1-\nu)}\left(1+\frac{c}{\pi}\right)\ln\left\{\frac{1+c}{1-c}\right\}. \tag{52}$$

Expanding the logarithmic term in powers of $c$ and neglecting terms of order higher than $c^3$, (51) yields

$$K_{\mathrm{I}}^{\mathrm{B}} = \frac{hG}{\pi\sqrt{b}(1-\nu)}\left[c + c^2\left\{\frac{1}{\pi}\right\} + \dots\right]. \tag{53}$$

The result (53) is in agreement with the stress intensity factor derived earlier. As a matter of interest, if order terms in $c^5$ are retained in (53) we obtain

$$K_{\rm I}^{\rm B} \simeq \frac{hG}{\pi\sqrt{b}\,(1-\nu)}\left[c + c^2\left\{\frac{1}{\pi}\right\} + c^3\left\{\frac{1}{3}\right\} + c^4\left\{\frac{1}{3\pi}\right\} + c^5\left\{\frac{1}{5}\right\} + \mathrm{O}(c^6)\right]. \tag{54}$$

The approximate nature of the result (54) stems from the specified order of $c$ used in the estimation of $P^*$. For purposes of comparison, numerical results for the normalized stress intensity factor $\bar{K}_{\rm I}^{\rm B}$, derived from (54), are shown in Fig. 2. The results indicate a favourable comparison between the two estimates for values of $c \in (0, 0.60)$.

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## Résumé

Dans le mémoire, on examine le problème du marquage symétrique d'une fissure en angle noyée par une inclusion mince et lisse, en forme de disque circulaire rigide. L'analyse du problème conduit à un système d'équations intégrales triples, que l'on résoud par approximations. On obtient une expression du facteur d'intensité de contrainte aux frontières de la fissure sous la forme d'une série comportant le rapport du rayon de l'inclusion rigide circulaire au rayon de la fissure.