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# A contact problem for a Reissner plate and an isotropic elastic halfspace

*Problème de contact d'une plaque de Reissner sur un demi-espace élastique isotrope*

by

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ABSTRACT. — The axially symmetric contact problem for a thick circular plate resting in smooth contact with an isotropic elastic halfspace and subjected to a uniform load over a finite area is examined by using a variational method. The thick plate theory utilized in this paper takes into account the effect of shearing deformations in the plate as developed in the thick plate theory of Reissner. Numerical results presented in this paper illustrate the manner in which the plate deflection, contact stresses and flexural moments can be influenced by the thickness effects in the plate.

RÉSUMÉ. — On étudie par une méthode variationnelle le problème de contact axisymétrique pour une plaque circulaire épaisse, soumise à une charge répartie uniforme, et reposant sur un demi espace élastique isotrope dans l'hypothèse d'un contact sans frottement. La théorie de plaque épaisse utilisée dans cet article prend en compte l'effet des déformations de cisaillement dans une plaque selon la théorie de Reissner. Les résultats numériques présentés mettent en évidence de quelle façon la déflection de la plaque, les contraintes au contact et les moments fléchissants sont influencés par les effets d'épaisseur de la plaque.

## 1. Introduction

The analysis of flexural interaction between finite elastic plates and elastic media is of particular interest in geotechnical engineering. Solutions to such problems have application in the structural design of raft foundations resting on soil and rock media. This paper investigates the flexural interaction of a thick circular plate resting on an isotropic elastic halfspace. Conventional treatments of the flexural interaction between a circular plate and an elastic medium usually assume that the flexural behaviour of the plate can be adequately described by the classical Poisson-Kirchhoff thin plate theory. (Poulos and Davis [1]; Selvadurai [2]). When dealing with moderately thick plates (i. e. thickness to diameter ratios of the order of 1/8) subjected to localized loads, it becomes necessary to examine the influence of shearing deformations on the raft settlement and flexural moments. The thick plate theory proposed by Reissner [3] can be adopted for the analysis of this class of interaction problem (*see* e. g. Gladwell and Iyer [4] and Svec [5]).

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The method of analysis employed in this paper utilizes a variational formulation in which the transverse deflection of the thick plate [$w(r)$] and the shear strain [$\gamma(r)$] are represented in the form of power series in the radial coordinate $r$. These distributions are specified to within a set of undetermined constants ordered in such a way that (i) the kinematic constraints of the axisymmetric deformation and (ii) the boundary conditions at the free edge of the raft, are identically satisfied. The variational formulation proposed here requires the development of a total potential energy functional for the thick plate-elastic halfspace system which consists of (i) the strain energy of the halfspace region, (ii) the strain energy of the circular plate region and (iii) the work component of the external loads. The strain energy component of the halfspace region can be developed by computing the work component of the surface tractions which compose the contact stresses at the thick plate-elastic medium interface. These contact stresses associated with the imposed displacement field $w(r)$ can be uniquely determined by making use of integral equation methods developed for mixed boundary value problems in classical elasticity (Sneddon [6]). The strain energy of the thick circular plate is composed of the flexural and shear energies corresponding to the prescribed functions $w(r)$ and $\gamma(r)$. The total potential energy functional thus developed is defined in terms of constants which characterize the two distributions for the thick plate deflection and shear strain. These constants are uniquely determined from the linearly independent algebraic equations generated from the minimization of the total potential energy functional.

The general procedure outlined above is used to analyse the flexure of a thick circular raft resting in smooth contact with an isotropic elastic halfspace and subjected to a uniform circular load over a finite area. The deflections and shear strains in the raft are represented by even and odd order power series to the sixth and third powers in $r$, respectively. Numerical results presented in this paper illustrate the manner in which raft settlements and flexural moments can be influenced by the raft thickness and the localized nature of the external load.

## 2. The thick plate theory

The classical Poisson-Kirchhoff thin plate theory has been widely applied in the examination of several important problems pertaining to soil-foundation interaction (*see* e. g. Borowicka [7], Barden [8], Brown [9], Timoshenko and Woinowsky-Krieger [10]). A comprehensive account of the axisymmetric interaction problem related to a thin plate resting on an isotropic elastic halfspace is given by Selvadurai [2]. There are, however, many engineering situations for which the use of the classical thin plate theory may be open to objection. Such situations include the stress analysis of thick circular rafts with thickness to diameter ratios of the order of 1/8 and rafts subjected to highly localized loads. The various thick plate theories that have been proposed in the literature provide useful refinements to the Poisson-Kirchhoff thin plate theory. The thick plate theories of Reissner [3] and Mindlin [11] incorporate the effects of transverse shear deformations in the analysis of flexure. Further accounts of the various thick plate theories are given by Timoshenko and Woinowsky-Krieger [10], Naghdi [12] and Selvadurai [2].

In the thick plate theory proposed by Reissner [3] the total rotation of a cross-section is expressed as the sum of the following: (i) the rotation corresponding to the normal which remains perpendicular to the middle surface of the plate and (ii) a straight line rotation which represents the shear deformation effect. In the ensuing, attention is restricted to the axisymmetric flexure of a thick circular plate, the deformation of which is characterized by the displacement components $\bar{u}(r, z)$ and $\bar{w}(r, z)$ in the $r$ and $z$ directions respectively. We have:

$$\bar{u}(r, z) = z\,\Phi(r); \qquad \bar{w}(r, z) = w(r), \tag{1}$$

where $\Phi$ is the average rotation of a section $r$=constant, and $w(r)$ is the mid-plane transverse displacement. The rotation $\Phi$ can be written as:

$$\Phi(r) = -\frac{dw}{dr} + \gamma; \qquad \gamma = \frac{c\,Q}{G\,t}, \tag{2}$$

where $\gamma$ is the additional rotation due to transverse shear; Q is the shear force; $G(=E/2(1+\nu))$ is the linear elastic shear modulus; $t$ is the plate thickness and $c$ is a shear coefficient representing the restraint of the cross-section against warping. This constant can vary between 1.0 and 1.5 depending on the degree of restraint and is usually taken as 6/5. The curvature and force equilibrium relationships yield:

$$k_r = -\left\{\frac{d^2w}{dr^2} - \frac{d\gamma}{dr}\right\}; \qquad k_\theta = -\left\{\frac{1}{r}\frac{dw}{dr} - \frac{\gamma}{r}\right\}, \tag{3}$$

and:

$$\frac{dQ}{dr} + \frac{Q}{r} + p(r) - q(r) = 0, \tag{4}$$

respectively, where $q(r)$ is the contact stress at the thick plate-elastic medium interface and $p(r)$ is the external load. Using the above results, the moment-curvature relationships yield the following expressions for the flexural moments $M_r(r)$ and $M_\theta(r)$ :

$$M_r(r) = -D\left[\frac{d^2w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr} - \left\{\frac{d\gamma}{dr} + \frac{\nu}{r}\gamma\right\}\right], \tag{5 a}$$

$$M_\theta(r) = -D\left[\frac{1}{r}\frac{dw}{dr} + \nu\frac{d^2w}{dr^2} - \left\{\frac{\gamma}{r} + \nu\frac{d\gamma}{dr}\right\}\right], \tag{5 b}$$

where:

$$D = \frac{E\,t^3}{12(1-\nu^2)}, \tag{6}$$

is the plate rigidity. The shear force in the deformed thick plate is defined by the second equation of (2). By utilizing the above relationships it can be shown that the total strain energy of the thick plate is given by:

$$(7)\quad U_{T.P}=\frac{1}{2}\iint_S\left[\{\nabla^2 w(r)\}^2-\frac{2(1-\nu)}{r}\frac{dw}{dr}\frac{d^2w}{dr^2}+\left\{\frac{d\gamma}{dr}+\frac{\gamma}{r}\right\}^2-\frac{2(1-\nu)}{r}\gamma\frac{d\gamma}{dr}-2\frac{d\gamma}{dr}\left\{\frac{d^2w}{dr^2}+\frac{\nu}{r}\frac{dw}{dr}\right\}-\frac{2\gamma}{r}\left\{\frac{1}{r}\frac{dw}{dr}+\nu\frac{d^2w}{dr^2}\right\}+\beta\gamma^2\right]rdrd\theta,$$

where $\beta=6(1-\nu)/t^2c$ and S corresponds to the plate region.

## 3. The strain energy of the halfspace region

The second component of the total potential energy functional corresponds to the elastic strain energy of the isotropic elastic halfspace region which is subjected to the displacement field $w(r)$ in the region $r\leq a$, where $a$ is the radius of the thick plate. It is assumed that the external loading configuration on the thick plate is such that no tensile tractions are generated at the smooth thick plate-elastic medium interface. To physically achieve continuous contact in the region $r\leq a$, the contact stresses should be compressive. This fact has to be verified upon completion of the interaction analysis. From a practical point of view, the self weight stresses of a thick foundation raft may be sufficient to prevent seperation at the smooth interface. The normal contact stresses at the smooth interface can be uniquely determined by making use of the integral equation methods developed by Sneddon [6], Green [13] and others for the analysis of mixed boundary value problems in classical elasticity. The normal contact stress associated with the imposed displacement $w(r)$ (in the region $r\leq a$) is given by:

$$(8)\qquad \sigma_{zz}(r,0)=q(r)=\frac{G_s}{(1-\nu_s)r}\frac{d}{dr}\int_a^r\frac{sg(s)\,ds}{\sqrt{s^2-r^2}},$$

where:

$$(9)\qquad g(s)=\frac{2}{\pi}\frac{d}{ds}\int_0^s\frac{rw(r)\,dr}{\sqrt{s^2-r^2}},$$

and $G_s$ and $\nu_s$ are, respectively, the linear elastic shear modulus and Poisson's ratio of the elastic medium. From the above results, the elastic strain energy of the halfspace region is given by:

$$(10)\qquad U_{H.S}=\frac{G_sa^3}{\pi(1-\nu_s)}\iint_S\frac{w(r)}{a}\left[\frac{1}{r}\frac{d}{dr}\int_a^r\frac{s}{\sqrt{s^2-r^2}}\left\{\frac{d}{ds}\int_0^s\frac{rw(r)\,dr}{a^2\sqrt{s^2-r^2}}\right\}ds\right]rdrd\theta.$$

The potential energy of the external loads is given by:

$$U_L = - \iint_{S_L} p(r)\, w(r)\, r dr d\theta, \tag{11}$$

where $S_L$ is the region occupied by the external load $p(r)$.

## 4. The total potential energy functional

The total potential energy functional U for the thick plate-elastic medium system is obtained by combining (7), (10) and (11) i. e.:

$$U = U_{T.P} + U_{H.S} + U_L. \tag{12}$$

For the total potential energy functional to satisfy the principle of stationary potential energy:

$$\delta U = 0, \tag{13}$$

where $\delta U$ is the variation in the functional. In order to apply the principle of total potential energy to the interaction problem it may be assumed that the deflection of the thick plate $w(r)$ and the shear strain $\gamma(r)$ can be represented in the form:

$$w(r) = a \sum_{i=0}^{n} C_{2i}\, \varphi_{2i}(r); \qquad \gamma(r) = \sum_{i=1}^{m} C_i\, \chi_i(r), \tag{14}$$

where $C_i$ and $C_{2i}$ are arbitrary constants and $\varphi_i(r)$ and $\chi_i(r)$ are arbitrary functions which satisfy the kinematic constraints of the axisymmetric deformation associated with a distributed loading. In addition, the arbitrary constants $C_i$ and $C_{2i}$ are ordered in such a way that the flexural moments and shear forces derived from (14) satisfy the boundary conditions applicable to the free edge of a thick plate. For axial symmetry these reduce to:

$$\begin{cases} M_r(a) = -D\left[\dfrac{d^2 w}{dr^2} + \dfrac{\nu}{r}\dfrac{dw}{dr} + \dfrac{d\gamma}{dr} + \dfrac{\nu\gamma}{r}\right]_{r=a} = 0, \\ Q(a) = -\dfrac{Gt}{c}\,[\gamma]_{r=a} = 0. \end{cases} \tag{15}$$

Using the representations (14) the total potential energy functional for the thick plate-elastic medium system can be represented in terms of $(m+n+1)$ independent constants $C_j\,(j=i,\, 2i)$. The principle of total potential energy requires that U be an extremum with respect to the kinematically admissible deformation fields characterized by (14). Hence:

$$\frac{\partial U}{\partial C_j} = 0; \qquad (j = 0, 2, 4, \ldots, n;\; j = 1, 3, \ldots, m). \tag{16}$$

The above minimization procedure yields $(m+n+1)$ linearly independent equations for the undetermined constants $C_j$.

## 5. The interaction problem

In order to apply the formal theory developed in the preceding sections to the analysis of axisymmetric interaction between the thick circular plate and the isotropic elastic halfspace, explicit representations of (14) are considered. Since complete contact is assumed, the functions $w(r)$ and $\gamma(r)$ are approximated by a power series of the form:

$$(17) \qquad w(r) = a \sum_{i=0}^{3} C_{2i} \left( \frac{r}{a} \right)^{2i}; \qquad \gamma(r) = \sum_{i=1}^{2} C_{2i-1} \left( \frac{r}{a} \right)^{2i-1}.$$

The particular choice of functions corresponding to $\varphi_i(r)$ and $\chi_i(r)$ give kinematically admissible deflection fields and finite flexural moments and shearing forces in the plate region $0 \leqq r \leqq a$. The distributions (17), when combined with the constraints:

$$(18) \qquad C_1 = -C_3;\ C_1 = -\{(1+\nu)C_2 + 2C_4(3+\nu) + 3C_6(5+\nu)\},$$

identically satisfy the boundary conditions applicable to the free edge $r=a$. The contact stress distribution corresponding to the imposed displacement field $w(r)$ of (17) can be determined by making use of the relationships (8) and (9); thus:

$$(19) \qquad \sigma_{zz}(r,0) = q(r) = \frac{2G_s a}{\pi(1-\nu_s)\sqrt{a^2-r^2}} \left[ C_0 + C_2 \left\{ -2 + 4\frac{r^2}{a^2} \right\} + \frac{8}{9} C_4 \left\{ -1 - 4\frac{r^2}{a^2} + 8\frac{r^4}{a^4} \right\} + \frac{16}{25} C_6 \left\{ -1 - 2\frac{r^2}{a^2} - 8\frac{r^4}{a^4} + 16\frac{r^6}{a^6} \right\} \right].$$

By using the expressions for $w(r)$ and $\gamma(r)$ in the generalized expression for U given by (12) it can be shown that:

$$(20) \qquad U = \frac{2G_s a^3}{(1-\nu_s)} \left[ \left\{ \beta_{00} C_0^2 + \beta_{22} C_2^2 + \beta_{44} C_4^2 + \beta_{66} C_6^2 + \beta_{02} C_0 C_2 + \beta_{04} C_0 C_4 + \beta_{06} C_0 C_6 + \beta_{24} C_2 C_4 + \beta_{26} C_2 C_6 + \beta_{46} C_4 C_6 \right\} - \frac{\pi p_0 (1-\nu_s)\alpha^2}{2G_s} \left\{ k_0 C_0 + k_2 C_2 + k_4 C_4 + k_6 C_6 \right\} \right].$$

In (20), the constants $\beta_{ij}$ take the form:

$$(21) \qquad \beta_{ij} = \rho_{ij} + K\,\eta_{ij},$$

where:

(22) $$K=\frac{\pi(1-\nu_s^2)}{12(1-\nu^2)}\,\frac{E}{E_s}\left(\frac{t}{a}\right)^3,$$

is a relative rigidity parameter for the plate-elastic medium system and the constants $\rho_{ij}$ and $\eta_{ij}$ are defined in Appendix A. The constants $\eta_{ij}$ depend on a parameter $\Omega$ where:

(23) $$\Omega=\frac{8\,c\,(t/a)^2+3\,(1-\nu)}{12\,c\,(t/a)^2},$$

is a parameter which governs the effect of shear deformations in the thick plate. The constants $k_n\,(n=0,\,2,\,4,\,6)$ are given by:

(24) $$k_0=1;\qquad k_2=\frac{\alpha^2}{2};\qquad k_4=\frac{\alpha^4}{3};\qquad k_6=\frac{\alpha^6}{4},$$

and $\alpha a$ is the radius of the loaded region (*Fig.* 1). The constants $C_0$, $C_2$, $C_4$ and $C_6$ can be determined from the set of linear equations which are generated from the minimization conditions:

(25) $$\frac{\partial U}{\partial C_i}=0;\qquad (i=0,\,2,\,4,\,6).$$

The solution of this set of equations can be written in the compact form:

(26) $$[\mathbf{C}]=\frac{P(1-\nu_s^2)}{a^2\,E_s}\,[\boldsymbol{\beta}]^{-1}\,[\mathbf{k}],$$

where:

(27 *a*) $$[\mathbf{C}]^T=[C_0\,C_2\,C_4\,C_6],$$

(27 *b*) $$[\mathbf{k}]^T=[k_0\,k_2\,k_4\,k_6],$$

(27 *c*) $$[\boldsymbol{\beta}]=\begin{bmatrix}2\,\beta_{00} & \beta_{02} & \beta_{04} & \beta_{06}\\ \beta_{02} & 2\,\beta_{22} & \beta_{24} & \beta_{26}\\ \beta_{04} & \beta_{24} & 2\,\beta_{44} & \beta_{46}\\ \beta_{06} & \beta_{26} & \beta_{46} & 2\,\beta_{66}\end{bmatrix},$$

and $P=p_0\,\pi\alpha^2\,a^2$ is the total external load on the thick plate.

## 6. Flexural deflections, contact stresses and flexural moments

The accuracy of the solution for the deflection of the thick plate $w(r)$ developed by employing an energy approach can be examined by assigning suitable limits to the relative rigidity parameter K (the numerical results presented in this section are valid for $\nu=0.3$).

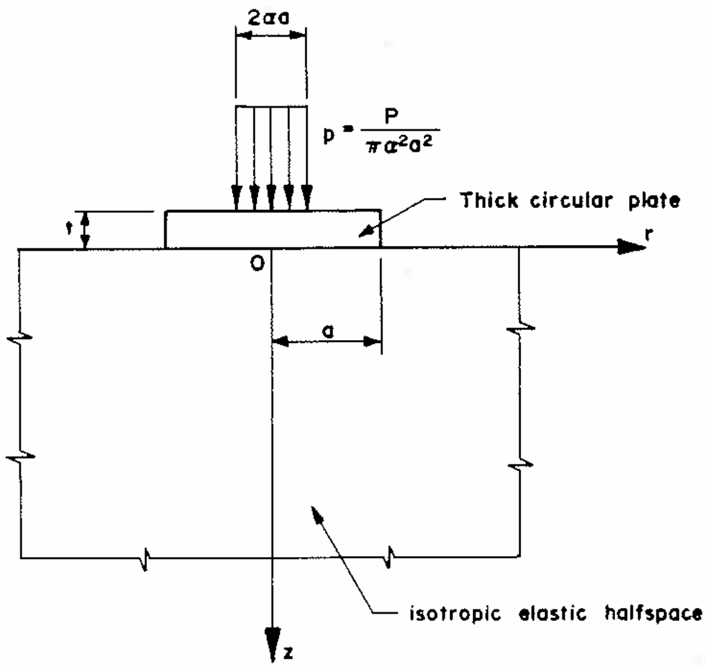


Fig. 1. — The geometry of the thick circular plate and the external loading.

*Fig. 1. — Géométrie de la plaque circulaire épaisse et du chargement extérieur.*

(i) *Infinitely rigid circular plate*

It is evident that as $K \to \infty$, the thick circular plate resembles a rigid foundation. This limiting value may be achieved by increasing either the modular ratio $(E/E_s)$ or the thickness ratio $(t/a)$. In the particular limit as $K \to \infty$, however, it is assumed that $w(r) = w_0 = \text{constant}$, in the region $r \leqq a$. From the results of the energy solution we obtain:

$$w_0 = \frac{P(1-\nu_s^2)}{2\alpha^2 a E_s}. \tag{28}$$

This result is in agreement with the classical solution obtained by Boussinesq [14] and Harding and Sneddon [15] for the displacement of a rigid circular punch resting in smooth contact with an isotropic elastic halfspace. These exact solutions were derived by considering results of potential theory and integral equation methods respectively.

(ii) *Flexible circular loading*

As $K \to 0$ and as $\mu(=t/a) \to 0$, the interaction problem reduces to that of the axisymmetric loading of an isotropic elastic halfspace by a uniform circular load of radius $\alpha a$ and stress intensity $p_0(=P/\pi\alpha^2 a^2)$. For convenience, $\alpha$ is set equal to unity. The two results of particular engineering interest are the maximum deflection $\{w(0)\}$ and differential deflection $\{w(0)-w(a)\}$ within the loaded area. The exact solutions corresponding to these can be readily obtained by an integration of Boussinesq's solution for the normal loading of a halfspace by a concentrated force (Timoshenko and Goodier, [16]). A comparison of the energy estimates with the exact solutions yields the following:

$$[\{w(0)\}_{\text{Energy}}: \{w(0)\}_{\text{Exact}}] = \frac{P(1-\nu_s^2)}{\pi a E_s}[2.039: 2.000]. \tag{29}$$

Similarly, a comparison of the differential displacement $w_d = \{w(0) - w(a)\}$ gives:

$$[\{w_d\}_{\text{Energy}}: \{w_d\}_{\text{Exact}}] = \frac{P(1-\nu_s^2)}{\pi a E_s}[0.730: 0.727]. \tag{30}$$

From (29) and (30) it is evident that the energy estimates for the displacement of the infinitely rigid and perfectly flexible plates compare favourably with known exact solutions.

(iii) *Flexural deflections in the thick plate*

In order to examine the effect of the plate thickness on the flexural deflections of the thick plate it is appropriate to replace the relative rigidity parameter K by:

$$K = K_M \mu^3, \tag{31 a}$$

where:

$$K_M = \frac{\pi(1-\nu_s^2)E}{12(1-\nu^2)E_s}, \tag{31 b}$$

is a reduced modular ratio. The Figures 2-3 illustrate the variation in the central deflection of the thick circular raft ($w_0$) with $\mu$ for a range of values of $K_M$. Similarly Figures 4-5 illustrate the distribution of the thick plate deflection along a diametral section. It is evident that the flexural deflections of the thick circular raft are considerably influenced by the thickness ratio $\mu$ and the loading configuration $\alpha$.

(iv) *The contact stress*

The contact stress distribution at the thick plate-elastic halfspace interface can be evaluated by making use of the equation (19) and the results of (28). The formal expression for the contact stress can be written as:

$$\bar{q}(r_0) = \frac{\sigma_{zz}(r, 0)}{P/\pi a^2} = \frac{1}{\sqrt{1-r_0^2}}\Big[C_0 - 2C_2\{1-2r_0^2\} - \frac{8}{9}C_4\{1+4r_0^2-8r_0^4\} - \frac{16}{25}C_6\{1+2r_0^2+8r_0^4-16r_0^6\}\Big], \tag{32}$$

where $r_0 = r/a$.

The expression for the contact stress (32), can be evaluated to establish the limits of applicability of the interaction problem examined in this paper. The assumption of tensionless contact at the frictionless interface is fundamental to the developments presented in the preceding sections.

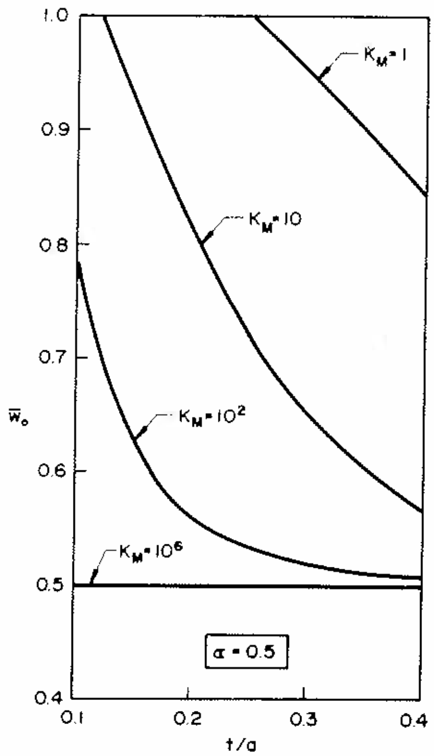


Fig. 2

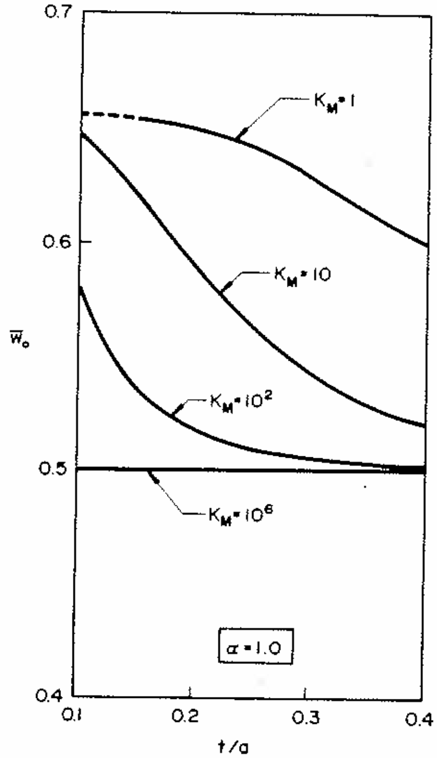


Fig. 3

Figs. 2-3. — The central deflection of the thick circular plate:

*Fig. 2-3. — Déflection centrale de la plaque circulaire épaisse :*

$$w(0) = \frac{P(1-\nu_s^2)}{\alpha^2 a E_s} \overline{w}_0; \qquad P = p_0 \pi \alpha^2 a^2.$$

For the energy solution to be physically admissible it is essential that the contact stresses developed at the interface remain compressive for various combinations of the loading configuration $\alpha$, the thickness ratio, $\mu$, and the reduced modular ratio, $K_M$. Should the contact stresses become tensile in any region of the interface then the interaction problem becomes one of unilateral contact between the plate and the elastic halfspace. A detailed account of investigations pertaining to such tensionless contact problems is given by Selvadurai [2] and Gladwell [17]. It is found that frictionless contact between thin plates and elastic media induced by highly localized or concentrated loads are susceptible to such separation effects. The variation of the contact stress at the thick plate-elastic halfspace interface computed for various combinations of $\alpha$, $\mu$ and $K_M$ is illustrated in Figures 6-7. It is evident that tensile contact stresses tend to develop in the instance where relatively thin plates are subjected to localized external loads.

(v) *Flexural moments in the thick plate*

The flexural moments induced in the thick plate due to its interaction with the isotropic elastic halfspace can be obtained by one of two methods. In the first method, the

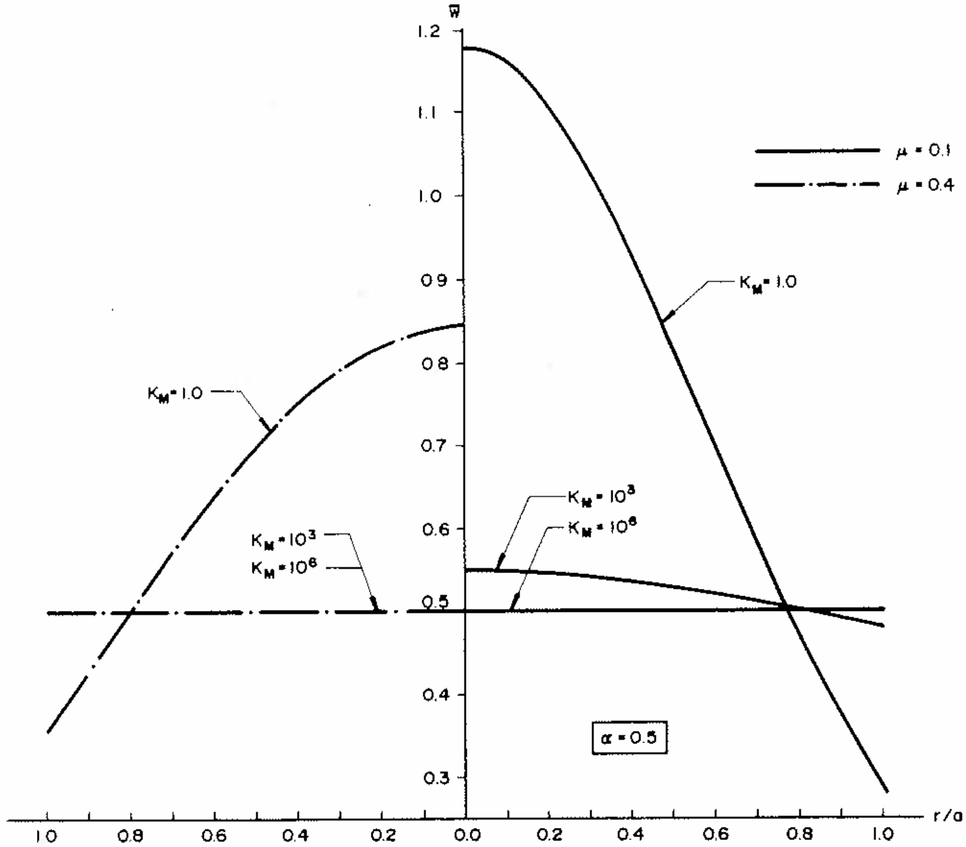


Fig. 4

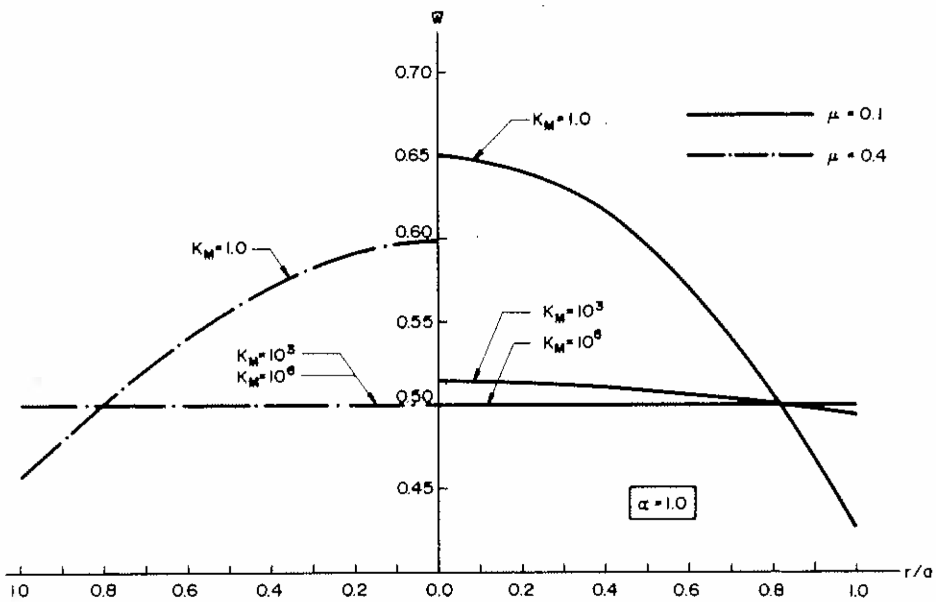


Fig. 5

Figs. 4-5. — The flexural deflections of the thick circular plate:

*Fig. 4-5. — Les déflections de flexion de la plaque circulaire épaisse:*

$$w(r) = \frac{P(1-\nu_s^2)}{\alpha^2 a E_s}\,\bar{w}; \qquad P = p_0 \pi \alpha^2 a^2.$$

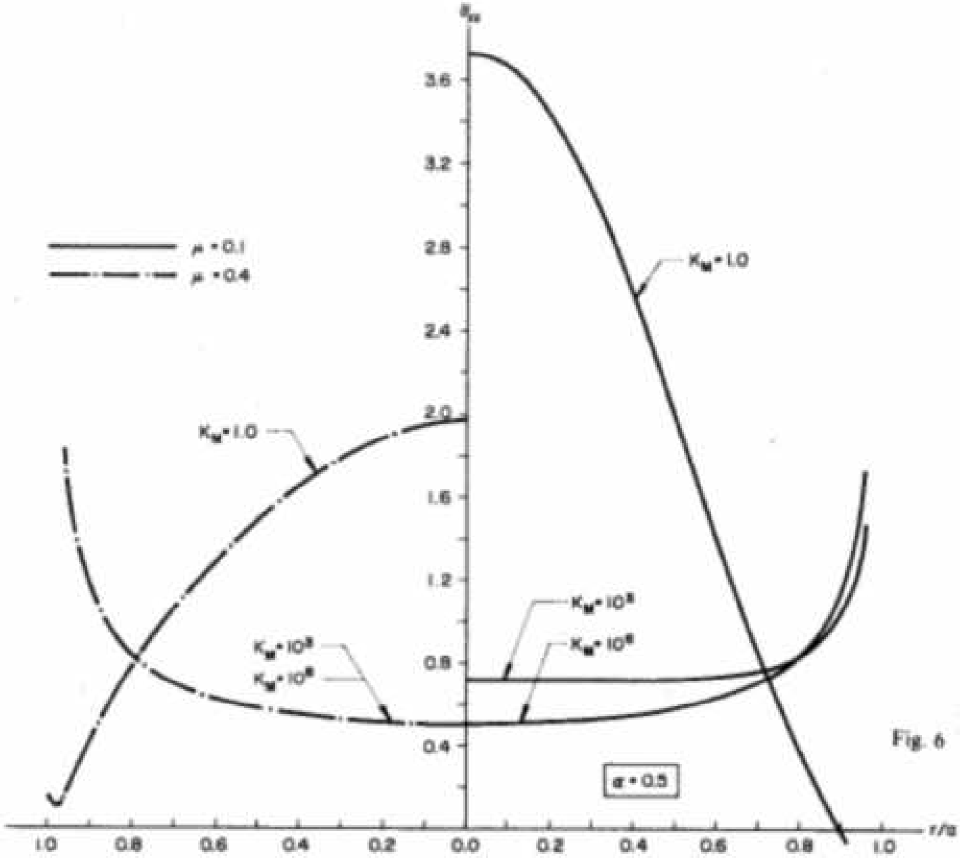


Fig. 6

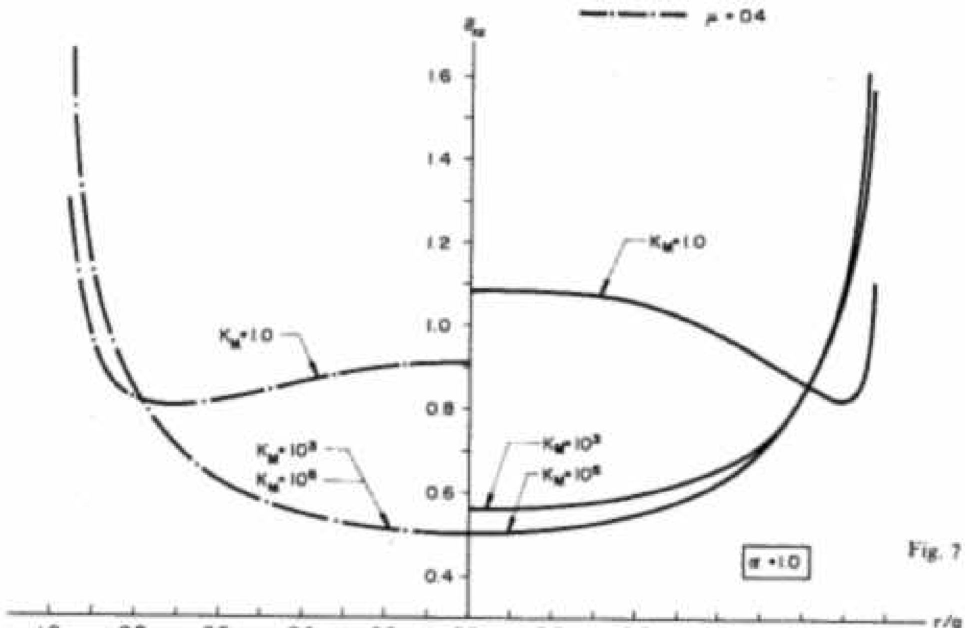


Fig. 7

Figs. 6-7. – The contact stress distribution at the thick plate-elastic medium interface:

*Fig. 6-7. – Distribution des contraintes de contact à l'interface plaque épaisse-milieu élastique :*

$$\sigma_{zz} = \frac{P}{\pi \alpha^2 a^2} \bar{\sigma}_{zz}$$

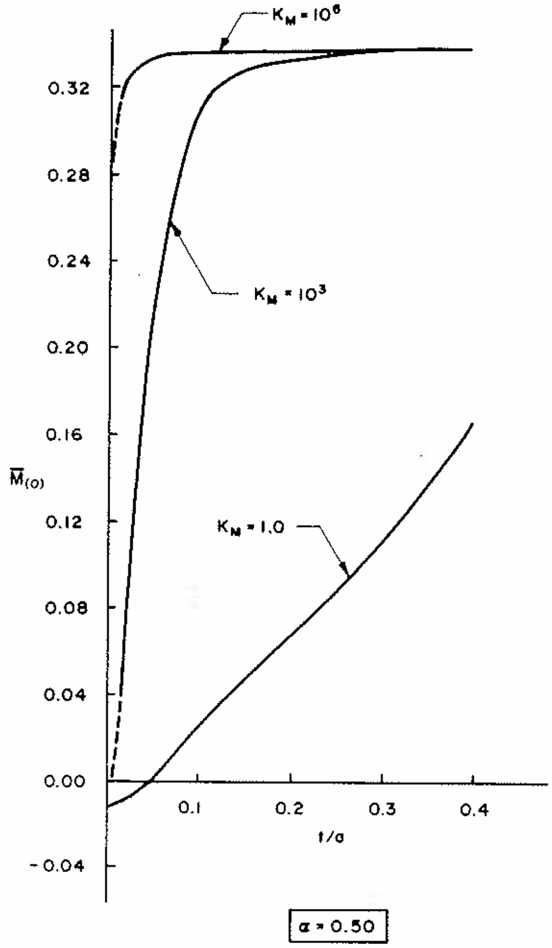


Fig. 8

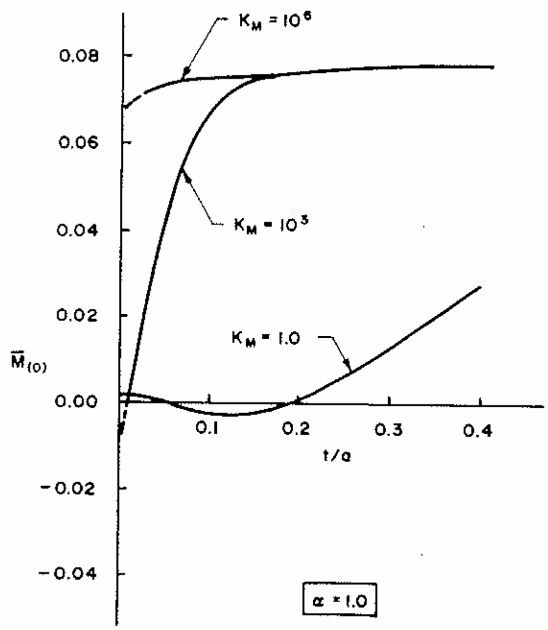


Fig. 9

Fig. 8-9. — The central flexural moment in the thick plate:

*Fig. 8-9. — Moment fléchissant central dans la plaque épaisse :*

$$M_r(0) = M_\theta(0) = \frac{P}{\pi}\overline{M}_{(0)}.$$

distributions developed for $w(r)$ and $\gamma(r)$ are directly utilized in the relationships for $M_r$ and $M_\theta$ given by (5). It is clear that the energy method provides accurate estimates of the plate deflection; however, the accuracy with which $w(r)$ and $\gamma(r)$ are able to predict flexural moments in the thick plate is, in general, considerably less. As observed by Dym and Shames [18] and others, any inaccuracies that may be present in the energy estimates for $w(r)$ and $\gamma(r)$ are greatly magnified in the computation of $M_r$ and $M_\theta$, owing to the presence of derivatives up to the second-order. In the second method, the flexural moments in the thick plate are computed by considering the action of the external load distribution and the contact stress distribution. For this purpose, it is convenient to employ basic solutions developed for a thick plate simply supported along its boundary. Avoiding details of calculation, it can be shown that the flexural moment at boundary. Avoiding details of calculation, it can be shown that the flexural moment at the centre of the thick circular plate ($M_r(0) = M_\theta(0) = M_0$) is given by:

$$\frac{M_0}{P/\pi} = \frac{4-(1-\nu)\alpha^2}{16} - \frac{(1+\nu)\ln\alpha}{4} + (1-\nu)\left\{-\frac{C_0}{12} + \frac{C_2}{30} + \frac{2C_4}{35} + \frac{4C_6}{63}\right\}$$
$$+(1+\nu)\left\{\left[\frac{C_0}{2} + \frac{C_2}{3} + \frac{4C_4}{15} + \frac{8C_6}{35}\right]\ln 2 - \left[\frac{C_0}{2} + \frac{C_2}{9} + \frac{4C_4}{75} + \frac{8C_6}{245}\right]\right\}. \tag{33}$$

Figures 8-9 illustrate the variation in the central moment. These results indicate that the central flexural moment is signicantly influenced by the loading configuration, the reduced modular ratio and the thickness ratio. These solutions agree with known results [2] for the central flexural moment for a 'rigid' plate resting on an isotropic elastic halfspace.

## Conclusions

It is shown that the analysis of axisymmetric interaction between a thick circular plate resting in smooth contact with an istropic elastic halfspace can be examined by employing a variational approach. The plate deflections and the shear deformation are prescribed functions. The results derived for the variational method agree closely with known exact solutions for limiting values of the relative rigidity between the thick plate and the elastic halfspace. The various numerical results presented in this paper indicate the manner in which the deflections, flexural moments and contact stresses in the thick plate can be influenced by (i) the thickness to diameter ratio of the plate, (ii) the modular ratio of the plate material and the supporting medium, and (iii) the extent of localization of the applied loads.

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## Appendix A

The constants $\rho_{ij}$ and $\eta_{ij}$ ($i=0, 2, 4, 6$; $j=0, 2, 4, 6$) are defined as follows:

$$\rho_{00} = 1; \qquad \rho_{22} = \frac{4}{5}; \qquad \rho_{44} = \frac{64}{81}; \qquad \rho_{66} = \frac{256}{325};$$

$$\rho_{02}=\frac{4}{3};\qquad \rho_{04}=\frac{16}{15};\qquad \rho_{06}=\frac{800}{875};\qquad \rho_{24}=\frac{32}{21};$$

$$\rho_{26}=\frac{64}{45};\qquad \rho_{46}=\frac{256}{165}.$$

$$\eta_{22}=4+4\nu+\Omega\,\zeta_2^2-\zeta_2\,\tau_2,$$

$$\eta_{44}=\frac{80}{3}+16\nu+\Omega\,\zeta_4^2-\zeta_4\,\tau_2,$$

$$\eta_{66}=\frac{1008}{5}+36\nu+\Omega\,\zeta_6^2-\zeta_6\,\tau_6,$$

$$\eta_{24}=16+16\nu+2\Omega\,\zeta_2\,\zeta_4-\zeta_2\,\tau_4-\zeta_4\,\tau_2,$$

$$\eta_{26}=24+24\nu+2\Omega\,\zeta_2\,\zeta_6-\zeta_2\,\tau_6-\zeta_6\,\tau_2,$$

$$\eta_{46}=96+48\nu+2\Omega\,\zeta_4\,\zeta_6-\zeta_4\,\tau_6-\zeta_6\,\tau_4,$$

and all other $\eta_{0j}(j=0,\ 2,\ 4,\ 6)=0$. Also,

$$\tau_2=0;\qquad \tau_4=-\frac{16}{3};\qquad \tau_6=-12.$$

and :

$$\zeta_2=(1+\nu);\qquad \zeta_4=2(3+\nu);\qquad \zeta_6=3(5+\nu).$$

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