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THE FLEXURE OF AN INFINITE STRIP OF FINITE WIDTH EMBEDDED IN AN ISOTROPIC ELASTIC MEDIUM OF INFINITE EXTENT

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SUMMARY

This paper examines the problem of the flexure of a strip of finite width which is embedded in bonded contact with an isotropic elastic medium of infinite extent. The analysis of the problem is applicable to flexible beams of narrow width which experience flexure only in the longitudinal direction. Numerical results are developed for the maximum flexural moment induced in the embedded beam due to the action of a concentrated force.

INTRODUCTION

The group of elastostatic problems which examines the interaction between structural elements such as beams, plates, shells etc., and elastic media is of interest to several branches of engineering. Solutions to such problems are of particular relevance to geomechanical applications which include the analysis of interaction between structural foundations and soil and rock media. Comprehensive accounts of the subject of interaction between beams and plates resting on elastic media are given by Hetenyi,[1] de Pater and Kalker,[2] Desai and Christian,[3] Selvadurai[4] and Gladwell.[5]

A majority of the interaction studies examines the flexural behaviour of structural elements which are located on the surface of deformable elastic media. The flexural interaction of structural foundations fully embedded in elastic media has received only lmited attention.[6-8] Investigations which relate to interaction of structural foundations partially embedded in elastic media have received considerable attention owing to their potential use in the study of soil-pile interaction.[9] The flexural interaction of beams completely embedded in elastic media is of interest to the analysis and design of anchor plates and conduits such as pipelines or duct banks embedded in soil and rock media. This paper investigates one such problem which concerns the flexure of an infinitely long beam of finite width which is embedded in bonded contact with an isotropic elastic medium of infinite extent (Figure 1). The analysis is restricted to the category of beams of narrow width which experience flexure only in the longitudinal direction. The condition related to no flexure in the transverse direction is satisfied in an approximate manner by employing the techniques outlined by Biot,[10] Rvachev[11] and Lekkerkerker,[12] in connection with their analysis of the flexure of a beam of finite width resting in smooth contact with an isotropic elastic halfspace. Specific numerical results are developed for the flexural moment induced in the embedded beam due to the action of a concentrated force.

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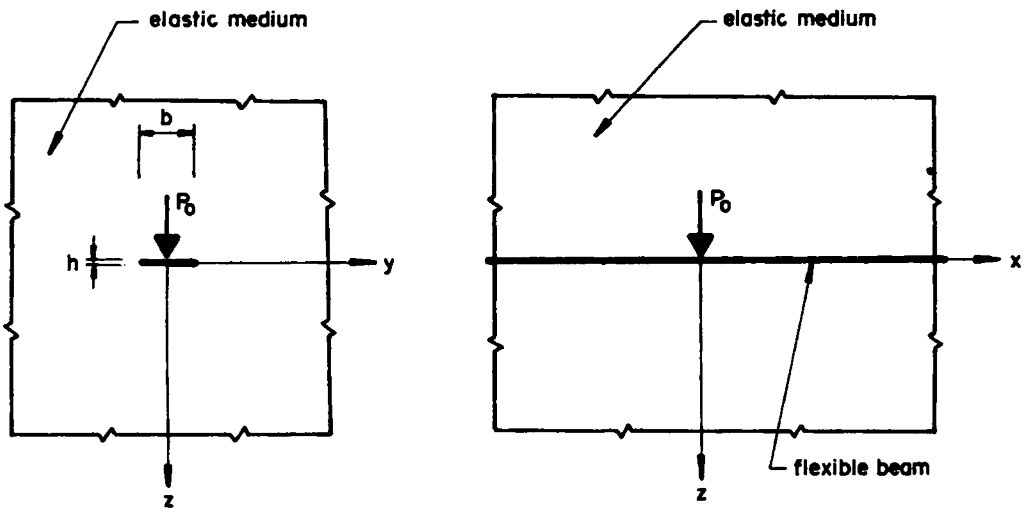

Figure 1. The geometry and loading of the embedded beam

THE FLEXURAL INTERACTION PROBLEM

We consider the problem of an infinitely long flexible beam of finite width b and flexural rigidity E_bI which is embedded in bonded contact with the isotropic elastic medium of infinite extent. We assume that the thickness to width ratio of the beam h/b is sufficiently small to ensure that the embedded strip can be idealized as a ribbon shaped inclusion, which exhibits flexure only in the longitudinal direction. The flexural behaviour of the beam is characterized by the classicial Bernoulli–Euler beam theory. According to this assumption the beam exhibits inextensibility in the longitudinal direction. The embedded beam is subjected to a distributed load $\tilde{p}(x, y)$ which causes symmetric flexure of the beam in the $x-z$ plane. The contact stresses at the beam-elastic medium interfaces are denoted by $\tilde{q}^{+}(x, y)$ and $\tilde{q}^{-}(x, y)$. The positive and negative superscripts signify the components of the contact stresses at the faces of the beam in contact with the halfspace regions $z>0$ and $z<0$, respectively. In the absence of flexure in the transverse plane (y, z), the general differential equation governing flexure of the beam takes the form

$$E_bI\frac{d^4w(x)}{dx^4}+\int_{-b/2}^{b/2}\{\tilde{q}^{+}(x, y)-\tilde{q}^{-}(x, y)\}\,dy=\int_{-b/2}^{b/2}\tilde{p}(x, y)\,dy \tag{1}$$

Owing to the 'one-dimensional' nature of the flexure problem it is convenient to consider a Fourier integral development of the interaction problem. The beam deflection, the contact stresses and the applied loads are expressed as averages taken over the width of the beam, i.e.

$$w(x)=\int_0^{\infty}\bar{w}(m)\cos\left(\frac{mx}{a}\right)dm=\int_0^{\infty}\left\{\frac{1}{b}\int_{-b/2}^{b/2}w(m, y)\,dy\right\}\cos\left(\frac{mx}{a}\right)dm \tag{2}$$

$$bq(x)=\int_0^{\infty}\bar{q}(m)\cos\left(\frac{mx}{a}\right)dm=\int_0^{\infty}\left\{\int_{-b/2}^{b/2}\bar{q}(m, y)\,dy\right\}\cos\left(\frac{mx}{a}\right)dm \tag{3}$$

$$bp(x)=\int_0^{\infty}\bar{p}(m)\cos\left(\frac{mx}{a}\right)dm=\int_0^{\infty}\left\{\int_{-b/2}^{b/2}\bar{p}(m, y)\,dy\right\}\cos\left(\frac{mx}{a}\right)dm \tag{4}$$

Also, $\bar{p}(m)$ is given by

$$\bar{p}(m)=\int_0^\infty p(\zeta)\cos\left(\frac{m\zeta}{a}\right)\mathrm{d}\zeta \tag{5}$$

Considering the Fourier representations of $w(x)$, $q(x)$ and $p(x)$, the differential equation governing flexure of the beam can be written in the form

$$E_bI\frac{\mathrm{d}^4w(x)}{\mathrm{d}x^4}+b\{q^+(x)-q^-(x)\}=bp(x) \tag{6}$$

In order to complete the description of the problem it is necessary to define the 'kernel function' which relates the contact stresses $q^+(x)$ and $q^-(x)$ to the beam deflection $w(x)$ (note that from the spatial symmetry of the problem $q^+(x)=-q^-(x)$). We shall assume that the kernel function can be defined as a relationship between the transformed variables of $w(x)$ and $q(x)$; namely $\bar{w}(m)$ and $\bar{q}(m)$, respectively. Accordingly, we establish a relationship between $\bar{w}(m)$ and $\bar{q}(m)$ in the following form;

$$\bar{w}(m)=\frac{\bar{q}(m)}{K(m)} \tag{7}$$

A complete discussion of the evaluation of $K(m)$ will be presented in the subsequent section. Substituting (2), (3), (4) and (7) in (6) we obtain

$$E_bI\frac{\mathrm{d}^4}{\mathrm{d}x^4}\int_0^\infty \bar{w}(m)\cos\left(\frac{mx}{a}\right)\mathrm{d}m+2\int_0^\infty \bar{w}(m)K(m)\cos\left(\frac{mx}{a}\right)\mathrm{d}m=b\int_0^\infty \bar{p}(m)\cos\left(\frac{mx}{a}\right)\mathrm{d}m \tag{8}$$

A solution of (8) gives

$$w(x)=\frac{ba^4}{E_bI}\int_0^\infty\frac{\bar{p}(m)\cos(mx/a)\,\mathrm{d}m}{[m^4+\Omega(m)]} \tag{9}$$

where

$$\Omega(m)=\frac{2a^4K(m)}{E_bI} \tag{10}$$

The result (9) can be used to generate flexural moments, shear forces etc., induced in the beam due to the loading $p(x)$. In the particular case where the loading corresponds to a concentrated force of magnitude P_0, it can be shown that the integral expressions for the deflection and flexural moment in the beam reduce to the forms

$$w(x)=\frac{P_0a^3}{\pi E_bI}\int_0^\infty\frac{\cos(mx/a)\,\mathrm{d}m}{[m^4+\Omega(m)]} \tag{11}$$

and

$$M(x)=\frac{P_0a}{\pi}\int_0^\infty\frac{m^2\cos(mx/a)\,\mathrm{d}m}{[m^4+\Omega(m)]} \tag{12}$$

respectively, where $M(x)$ represents the numerical value of the bending moment.

EVALUATION OF THE KERNEL FUNCTION $K(m)$

The kernel function $K(m)$ is a mathematical relationship between the displacements and interface tractions at the beam–elastic medium interface. The interfaces occupy the regions $-\infty < x < +\infty$; $-b/2 < y \leq +b/2$ and $z = 0^+$ and $z = 0^-$. The flexible beam experiences a constant displacement across its width (i.e. $-b/2 \leq y \leq +b/2$) at any cross section $x =$ const. Also, owing to the assumed bonded conditions at the beam–elastic medium interface and the spatial symmetry of the loading $\bar{p}(x, y)$, the displacements $u(x, y, z)$ and $v(x, y, z)$ and the stress component $\sigma_{zz}(x, y, z)$ referred to the rectangular cartesian co-ordinate system exhibit a state of asymmetry about the plane $z = 0$ (Figure 1).

As a prelude to the development of the kernel function we consider an auxiliary problem in which the infinite space region is subjected to a load

$$p(x, y) = 2\sigma_0 \cos\left(\frac{mx}{a}\right) \cos\left(\frac{ny}{a}\right) \tag{13}$$

which acts on the plane $z = 0$ along the positive z direction. This loading is transmitted as compressive normal tractions $\sigma_{zz}(x, y, 0^+) = -\sigma_0 \cos(mx/a) \cos(ny/a)$ on the boundary of the halfspace region $z > 0$ and tensile normal tractions on the boundary of the halfspace region $z < 0$. In the examination of the above problem we restrict the analysis to a single halfspace region $z \geq 0$ in which the plane $z = 0$ is subjected to the boundary conditions

$$u(x, y, 0^+) = 0 \tag{14}$$

$$v(x, y, 0^+) = 0 \tag{15}$$

$$\sigma_{zz}(x, y, 0^+) = -\sigma_0 \cos\left(\frac{mx}{a}\right) \cos\left(\frac{ny}{a}\right) \tag{16}$$

where the negative sign in (16) indicates a compressive stress and a is a length parameter in the problem. Also the + superscript indicates reference to the halfspace region occupying $z \geq 0$. For the analysis of the problem we select a Galerkin stress function approach.[13,14] The displacement and stress fields in the elastic solid can be uniquely represented in terms of the Galerkin stress function $Z(x, y, z)$ which satisfies the biharmonic equation

$$\nabla^2\nabla^2 Z(x, y, z) = 0 \tag{17}$$

where ∇^2 is Laplace's operator. The displacement and stress components of particular interest to the present problem can be expressed in terms of $Z(x, y, z)$ as follows:

$$2Gu(x, y, z) = -\frac{\partial^2 Z}{\partial x\, \partial z} \tag{18}$$

$$2Gv(x, y, z) = -\frac{\partial^2 Z}{\partial y\, \partial z} \tag{19}$$

$$2Gw(x, y, z) = 2(1-\nu)\nabla^2 Z - \frac{\partial^2 Z}{\partial z^2} \tag{20}$$

and

$$\sigma_{zz}(x, y, z) = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2 Z - \frac{\partial^2 Z}{\partial z^2}\right] \tag{21}$$

respectively. (G = linear elastic shear modulus; ν = Poisson's ratio). It can be shown that the stress function

$$Z(x, y, z) = -\frac{\sigma_0}{2(1-\nu)}[2(1-\nu)+cz]\,\mathrm{e}^{-cz}\cos\left(\frac{mx}{a}\right)\cos\left(\frac{ny}{a}\right) \tag{22}$$

where $c^2 = (m^2+n^2)/a^2$ satisfies (17) and gives displacement and stress fields which reduce to zero as $z \to \infty$. The expression for the displacement $w(x, y, z)$ takes the form

$$w(x, y, z) = \frac{\sigma_0}{4Gc(1-\nu)}\{3-4\nu+cz\}\,\mathrm{e}^{-cz}\cos\left(\frac{mx}{a}\right)\cos\left(\frac{ny}{a}\right) \tag{23}$$

From (16) and (22) it is evident that

$$w(x, y, 0^+) = \frac{(3-4\nu)a}{4G(1-\nu)(m^2+n^2)^{1/2}}\sigma_{zz}(x, y, 0^+) \tag{24}$$

The displacement $w(x, y, 0^+)$ and the stress $\sigma_{zz}(x, y, 0^+)$ can be identified with the deflection of the beam and the contact stress at the beam-elastic medium interface, respectively. For beams of narrow width (which are embedded in bonded contact with the elastic medium) it is assumed that no flexure takes place in the transverse direction (i.e. in the $y-z$ plane), consequently the displacements of the elastic medium satisfy the constraint

$$w(x, y, 0^+) = w(x, y, 0^-) = w(x) \tag{25}$$

Also the symmetry in the normal stress σ_{zz} gives

$$\sigma_{zz}(x, y, 0^+) = -\sigma_{zz}(x, y, 0^-) = q(x, y) \tag{26}$$

In order to develop the kernel function $K(m)$ we further investigate the problem where the internal loading is applied only within the beam region $|x| \leq \infty$; $|y| \leq b/2$. In particular, the loading applied ($q(x, y)$) should be such that the displacement induced in the beam region is constant across the width of the beam. To develop such a solution for this problem it is necessary to solve the integral equation (see e.g. Reference 5)

$$w(x, y) = \frac{(3-4\nu)}{2\pi G(1-\nu)}\int_{-\infty}^{\infty}\int_{-b/2}^{b/2}\frac{q(\xi, \eta)\,\mathrm{d}\xi\,\mathrm{d}\eta}{[(x-\xi)^2+(y-\eta)^2]^{1/2}} \tag{27}$$

Since (27) cannot be solved in an exact fashion we can attempt to develop certain approximate solutions in which the condition $w(x, y) = w(x)$ is satisfied approximately. Such techniques were used by Biot,[10] Rvachev[11] and Lekkerkerker[12] in connection with the analysis of beams of finite width resting in smooth contact with an isotropic elastic halfspace. The results derived earlier can be used to generate solutions for the displacement field induced in the beam region due to specific forms of internal loading. For example, when the loading in the beam region of the infinite space corresponds to a harmonic strip loading of the type

$$q(x, y) = \begin{cases} 2q^*\cos\left(\dfrac{mx}{a}\right); & |y| \leq \dfrac{b}{2} \\ 0; & ;\; y \geq \dfrac{b}{2} \end{cases} \tag{28}$$

the displacement field induced in the beam region takes the form

$$w(x,y)=\frac{aq^*(3-4\nu)}{4\pi G(1-\nu)}\left\{\int_0^\infty \frac{R(n,y)\,dn}{n(m^2+n^2)^{1/2}}\right\}\cos\left(\frac{mx}{a}\right) \tag{29a}$$

where

$$R(n,y)=\sin\left\{\frac{n}{2a}(2y+b)\right\}-\sin\left\{\frac{n}{2a}(2y-b)\right\} \tag{29b}$$

Successive superposition of solutions of the uniform loading type (28) can be used to achieve a nearly constant displacement across the width of the beam. Following the techniques outlined by Biot[15] it can be shown that for a near uniform displacement across the width of the beam, the relationship between $\bar{w}$ and $\bar{q}$ takes the form

$$\bar{w}(m)=\frac{Ca(3-4\nu)\bar{q}(m)}{4Gmb(1-\nu)\Psi(\beta)} \tag{30}$$

where $1\leq C\leq 1{\cdot}13$; $\beta=bm/2a$ and $\Psi(\beta)$ is a function which can be expressed as follows:

$\beta=$	0·1	0·5	1	3	8	∞
$\Psi(\beta)=$	4·8	1·9	1·42	1·13	1·04	1·00

and for $\beta<0{\cdot}1$, $\Psi(\beta)\simeq(2/\pi\beta)\{\ln(1/\beta)+0{\cdot}923\}^{-1}$.

Improved estimates for the result (2) can also be obtained by adopting the approximate techniques proposed by Rvachev[11] and Lekkerkerker[12] for the analogous halfspace problem. The details of these methods are also given by Selvadurai.[4] The salient results are summarized here for completeness. It can be shown that the contact stress distribution $q(x,y)$ can be expressed in the form

$$q(x,y)=\tilde{q}(m,y)\cos\left(\frac{mx}{a}\right) \tag{31}$$

where $\tilde{q}(m,y)$ is a solution of the itegral equation

$$\bar{w}(m)=\frac{(3-4\nu)}{4\pi G(1-\nu)}\int_{-b/2}^{b/2}\tilde{q}(m,t)K_0\left[m\frac{|y-t|}{a}\right]dt \tag{32}$$

and K_0 is the modified Bessel function of the second kind. From the definition of the finite Fourier transform we have

$$b\bar{q}(m)=\int_{-b/2}^{b/2}\tilde{q}(m,y)\,dy \tag{33}$$

A solution of (32) therefore formally completes the analysis of the problem and establishes a relationship between $\bar{w}(m)$ and $\bar{q}(m)$. An approximate solution of the integral equation (32) can be obtained by employing the following methods. In the first method an infinite series representation of $\tilde{q}(m,y)$ in terms of Mathieu functions is used to solve (32). In the second method an asymptotic series solution is obtained by employing a series expansion of the kernel function in terms of the small parameter β. The associated relationships between $\bar{w}(m)$ and $\bar{q}(m)$ reduce to the forms

$$\bar{w}(m)=\frac{\bar{q}(m)(3-4\nu)a}{4G(1-\nu)(a+bm)} \tag{34}$$

and

$$\bar{w}(m)=\frac{\bar{q}(m)(3-4\nu)}{4G(1-\nu)}\left\{(2\beta+\tfrac{1}{2})\,\mathrm{erf}\,(\sqrt{2\beta})+\left(\frac{2\beta}{\pi}\right)^{1/2}\mathrm{e}^{-2\beta}-\beta\right\}^{-1} \tag{35}$$

respectively. The former result is accurate for small values of β and the latter is applicable for large values of β. In (35) erf (x) corresponds to the error function

$$\mathrm{erf}\,(x)=\frac{2}{\sqrt{\pi}}\int_0^x \mathrm{e}^{-t^2}\,\mathrm{d}t \tag{36}$$

The results given by (30), (34), and (35) provide three estimates for $K(m)$. Accordingly, three separate expressions can be developed for the function $\Omega(m)$.

The function $\Omega(m)$ $(i=1,2,3)$ takes the following definitions consistent with the relationships between $\bar{w}$ and $\bar{q}$ proposed in equations (30), (34) and (35). Taking these equations in order we have

$$\begin{bmatrix}\Omega_1(m)\\ \Omega_2(m)\\ \Omega_3(m)\end{bmatrix}=\frac{16a^4G(1-\nu)}{(3-4\nu)E_bI}\begin{bmatrix}\beta\Psi(\beta)/C\\ \tfrac{1}{2}(2\beta+1)\\ \{(2\beta+\tfrac{1}{2})\,\mathrm{erf}\,(\sqrt{2\beta})+(2\beta/\pi)^{1/2}\,\mathrm{e}^{-2\beta}-\beta\}^{-1}\end{bmatrix} \tag{37a–c}$$

The result (37) can be used to generate the flexural moments, shear forces etc., induced in the embedded beam due to the loading $p(x)$.

NUMERICAL RESULTS

A result of some importance to engineering applications concerns the maximum flexural moment induced in the embedded beam due to the application of the concentrated force. The length parameter a can be set equal to the width b of the finite beam. For purposes of illustration we shall consider the flexural moment induced in the embedded beam at the location $x=0$, when the function $\Omega_i(m)$ conforms to the expressions given by (37a–c).

We introduce a relative rigidity parameter (R^*) for the beam-elastic medium system which is defined by

$$R^*=\frac{E_bh^3}{Eb^3} \tag{38}$$

where h is the thickness of the embedded beam. The flexural moment corresponding to $\Omega_1(m)$ can be expressed in the integral form

$$\left[\frac{M(0)}{P_0b/\pi}\right]_{(1)}=\int_0^\infty \frac{CR^*\phi m\,\mathrm{d}m}{[R^*\phi Cm^3+\Psi(m)]} \tag{39}$$

where

$$\phi=\frac{(1+\nu)(3-4\nu)}{48(1-\nu)} \tag{40}$$

The result corresponding to $\Omega_2(m)$ can be expressed in the integral form

$$\left[\frac{M(0)}{P_0b/\pi}\right]_{(2)}=\int_0^\infty \frac{R^*\phi m^2\,\mathrm{d}m}{[R^*\phi m^4+m+1]} \tag{41}$$

The result corresponding to $\Omega_3(m)$ can be expressed in the integral form

$$\left[\frac{M(0)}{P_0 b/\pi}\right]_{(3)} = \int_0^\infty \frac{R^* \phi \Delta m^2 \, dm}{[R^* \phi \Delta m^4 + 4]} \tag{42}$$

where

$$\Delta = \left\{(2m+1)\operatorname{erf}\sqrt{m} + 2\left(\frac{m}{\pi}\right)^{1/2} e^{-m} - m\right\} \tag{43}$$

The infinite integrals encountered in (41) and (42) can be evaluated by employing a numerical scheme based on Gauss–Legendre quadrature. The numerical results corresponding to (42), shown in Figures 2 and 3, have been calculated by using a six-point Gauss–Legendre quadrature.

CONCLUSIONS

In this paper we have examined the problem of flexure of an infinite beam of finite width which is embedded in bonded contact with an elastic medium. The condition related to the rigidity of the beam in the transverse direction is satisfied in an approximate manner. The three approximate formulations developed can be used to examine the flexural response of the infinite beam in the longitudinal direction. The numerical results presented in the paper, for the maximum flexural moment, indicate that the approximations given by $\Omega_i (i = 1, 2, 3)$ correlate well for a range of relative rigidities (R^*) of practical interest. The maximum flexural moment developed in an embedded beam which is subjected to a concentrated force can be

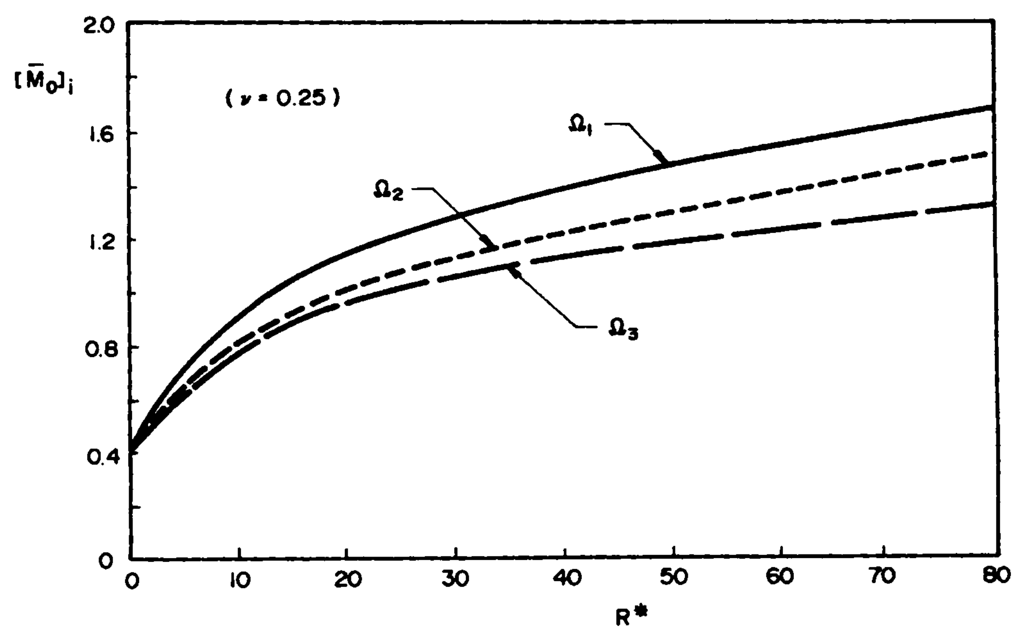

Figure 2. Variation in the maximum flexural moment in the embedded beam with the relative stiffness R^*: $[\bar{M}_0]_i = [M(0)/P_0 b/\pi]_i$; $i = 1, 2, 3$

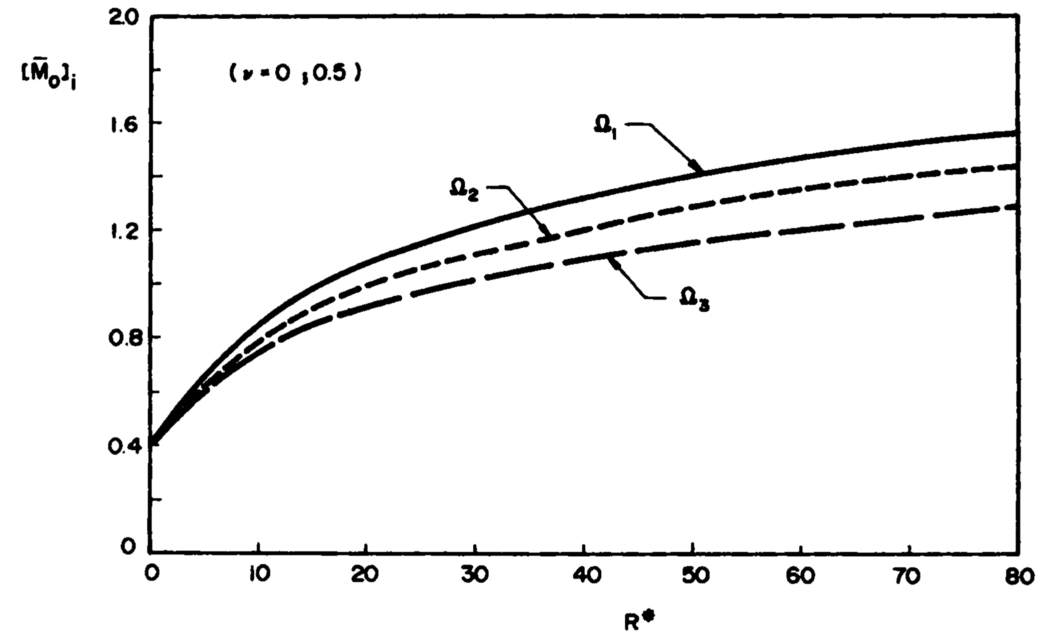

Figure 3. Variation in the maximum flexural moment in the embedded beam with the relative stiffness R^*: $[M_0]_i = [M(0)/P_0b/\pi]_i$; $i = 1, 2, 3$

conveniently estimated from the relationship

$$\frac{M(0)}{P_0 b/\pi} = \{R^*\phi\}^{0 \cdot 277}; \qquad (R^* > 0)$$

The procedure and results given in this paper can be further used to develop influence functions for flexural deflections, bending moments, etc. developed in an embedded beam. Such solutions are of interest to the structural design of flexible pipeline groups which are embedded in soil media.

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