Journal of Applied Mathematics and Physics (ZAMP)
Vol. 35, January 1984

0044-2275/84/001013-11 $ 3.70/0

Elastostatic bounds for the stiffness of an elliptical disc inclusion embedded at a transversely isotropic bi-material interface

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1. Introduction

The stress analysis of elastic media which are reinforced with either elastic or rigid inclusions is of considerable interest to the study of composite materials. Solutions developed for spheroidal and ellipsoidal inhomogeneities embedded in homogeneous isotropic elastic media are given by Eshelby [1], Edwards [2] and Lur'e [3]. Comprehensive accounts of inclusion problems in classical elasticity are given by Mura [4], Willis [5] and Walpole [6].

Disc inclusions are a particular limiting case of the general class of three-dimensional inhomogeneities. The articles by Collins [7], Keer [8], Kassir and Sih [9], Kanwal and Sharma [10] and Selvadurai [11–16] are primarily concerned with the study of disc shaped inclusions embedded in bonded contact with isotropic and transversely isotropic elastic solids of infinite extent. Several authors have extended these studies to include a variety of other features including flexural behaviour of the disc inclusion, delamination at the inclusion-elastic medium interface and the influence of externally applied loads. References to these studies are given by Selvadurai [17] and Selvadurai and Singh [18]. A study of the literature on inclusion problems indicates that a majority of the investigations concentrate on the response of disc shaped inclusions which are embedded in homogeneous elastic media. Analogous problems related to inclusions embedded in non-homogeneous elastic media have received little attention. The response of inclusions which are embedded in bi-material elastic solids is of interest to the study of precipitation hardening effects in multiphase composites. This paper examines the problem of the load-displacement response of a rigid elliptical disc inclusion which is embedded in bonded contact at a transversely isotropic bi-material interface. An exact formulation of the associated elasticity problem yields a system of simultaneous singular integral equations with complicated kernel functions. These equations are, however, not amenable to exact solution. As such, recourse must be made to numerical methods of analysis in order to develop results for the load-displacement response of the embedded elliptical inclusion. For this reason it is desirable to explore alternative techniques for estimating the load-displacement characteristics of the embedded inclusion. The method of analysis of the elliptical inclusion problem discussed here focusses on the development of a set of bounds for the stiffness of the inclusion which is located in bonded contact at the bi-material interface. The upper bound is obtained by imposing an inextensibility constraint at the bi-material interface. The lower bound assumes the presence of a frictionless bi-material interface with continuity of normal stresses and axial displacements. These bounds are developed in exact closed form.

2. Fundamental formulae

Comprehensive accounts of the methods employed for the analysis of three dimensional problems in transversely isotropic elastic media are given by Elliott [19, 20], Shield [21], Green and Zerna [22] and Kassir and Sih [23]. In the absence of body forces, the displacements and stress fields in a transversely isotropic elastic solid can be expressed in terms of two "harmonic functions" $\varphi_1^i(x, y, z_1)$ and $\varphi_2^i(x, y, z_2)$ which are solutions of

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_\alpha^2}\right\} \varphi_\alpha^i(x, y, z_\alpha) = 0; \qquad (\alpha = 1, 2) \tag{1}$$

where $z_\alpha = z/\sqrt{v_\alpha^i}$ and v_α^i are the roots of the equation

$$c_{11}^i\, c_{44}^i\, (v^i)^2 + [c_{13}^i\,(2\,c_{44}^i + c_{13}^i) - c_{11}^i\, c_{33}^i]\, v^i + c_{33}^i\, c_{44}^i = 0$$

and the *superscript* i can be assigned the values $i = A, B$ corresponding to the halfspace regions $A\,(z \geq 0)$ and $B\,(z \leq 0)$ respectively. We note that c_{kl}^i are the elastic constants of the transversely isotropic elastic material and the z-axis is normal to the plane of transverse isotropy. The relevant displacement and stress components in the transversely isotropic elastic medium can be expressed in the following forms

$$\left\{u_x^i;\, u_y^i;\, u_z^i\right\} = \left\{\frac{\partial}{\partial x}(\varphi_1^i + \varphi_2^i);\quad \frac{\partial}{\partial y}(\varphi_1^i + \varphi_2^i);\quad \frac{\partial}{\partial z}(k_1^i\, \varphi_1^i + k_2^i\, \varphi_2^i)\right\} \tag{2}$$

$$\sigma_{xz}^i = c_{44}^i \left\{(1 + k_1^i)\frac{\partial^2 \varphi_1^i}{\partial x\, \partial z} + (1 + k_2^i)\frac{\partial^2 \varphi_2^i}{\partial x\, \partial z}\right\} \tag{3}$$

$$\sigma_{yz}^i = c_{44}^i \left\{(1 + k_1^i)\frac{\partial^2 \varphi_1^i}{\partial y\, \partial z} + (1 + k_2^i)\frac{\partial^2 \varphi_2^i}{\partial y\, \partial z}\right\} \tag{4}$$

$$\sigma_{zz}^i = (k_1^i\, c_{33}^i - v_1^i\, c_{13}^i)\frac{\partial^2 \varphi_1^i}{\partial z^2} + (k_2^i\, c_{33}^i - v_2^i\, c_{13}^i)\frac{\partial^2 \varphi_2^i}{\partial z} \tag{5}$$

where

$$k_\alpha^i = \frac{c_{11}^i \nu_\alpha^i - c_{44}^i}{c_{13}^i + c_{44}^i}; \quad (\alpha = 1, 2) \tag{6}$$

and no summation is implied on any repeated superscripts.

3. The upper bound estimate

We examine the problem related to an elliptical disc shaped rigid inclusion which is embedded in bonded contact at the bi-material transversely isotropic elastic interface (Fig. 1). For ease of reference we shall adopt the following nomenclature. Referring to the plane $z = 0$, the region occupied by the rigid elliptical inclusion (i.e. $(x^2/a^2) + (y^2/b^2) \leq 1$ where a and b are the major and minor semiaxes of the ellipse) is denoted by S_i. The region exterior to the inclusion is denoted by S_e; also $S = S_i \cup S_e$. To develop the upper bound estimate for the stiffness of the elliptical inclusion we assume that the bi-material interface region S_e exhibits inextensible behaviour. Since the rigid elliptical inclusion is embedded in bonded contact with the bi-material elastic solid, the inextensibility conditions are also satisfied in the region S_i. The displacement and traction boundary conditions on S, associated with the axial displacement of the elliptical inclusion embedded at the constrained bi-material interface can be stated as follows:

$$u_x^A(x, y, 0^+) = u_x^B(x, y, 0^-) = 0; \quad (x, y) \in S_e \tag{7}$$

$$u_y^A(x, y, 0^+) = u_y^B(x, y, 0^-) = 0; \quad (x, y) \in S_e \tag{8}$$

$$u_z^A(x, y, 0^+) = u_z^B(x, y, 0^-); \quad (x, y) \in S_e \tag{9}$$

$$\sigma_{zz}^A(x, y, 0^+) = \sigma_{zz}^B(x, y, 0^-); \quad (x, y) \in S_e \tag{10}$$

where the signs $(\ )^+$ and $(\ )^-$ refer to the surface of the inclusion in contact with the material regions A and B respectively. In the elliptical inclusion region we have

$$u_x^A(x, y, 0^+) = u_x^B(x, y, 0^-) = 0; \quad (x, y) \in S_i \tag{11}$$

$$u_y^A(x, y, 0^+) = u_y^B(x, y, 0^-) = 0; \quad (x, y) \in S_i \tag{12}$$

$$u_z^A(x, y, 0^+) = u_z^B(x, y, 0^-) = \delta; \quad (x, y) \in S_i . \tag{13}$$

From (7), (8), (10) and (11) it is evident that

$$u_x^A(x, y, 0^+) = u_x^B(x, y, 0^-) = 0; \quad (x, y) \in S \tag{14}$$

$$u_y^A(x, y, 0^+) = u_y^B(x, y, 0^-) = 0; \quad (x, y) \in S . \tag{15}$$

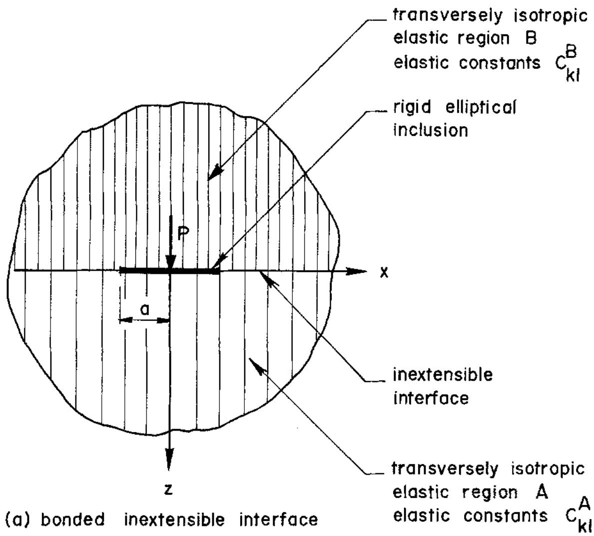

Figure 1
Geometry of the embedded rigid elliptical disc inclusion embedded in bonded contact at a bonded inextensible elastic interface – upper bound estimate.

In order to satisfy the boundary conditions (14) and (15) we select solutions of (1) which take the form

$$\varphi_1^i = \varphi^i(x, y, z_1); \qquad \varphi_2^i = -\varphi^i(x, y, z_2) \tag{16}$$

where $\nabla^2 \varphi^i = 0$ and ∇^2 is Laplace's operator referred to the rectangular Cartesian coordinate system. Considering the result (16) and the expressions for u_z^i and σ_{zz}^i we obtain the following results:

$$u_z^i = \left\{\frac{k_1^i \sqrt{v_2^i} - k_2^i \sqrt{v_1^i}}{\sqrt{v_1^i v_2^i}}\right\} \frac{\partial \varphi^i}{\partial z}; \qquad (i = A, B) \tag{17}$$

$$\sigma_{zz}^i = c_{33}^i \left\{\frac{k_1^i v_2^i - k_2^i v_1^i}{v_1^i v_2^i}\right\} \frac{\partial^2 \varphi^i}{\partial z^2}; \qquad (i = A, B). \tag{18}$$

Considering (17) and the boundary conditions (9) and (13) we obtain

$$\varphi^A = \left\{\frac{k_1^B \sqrt{v_2^B} - k_2^B \sqrt{v_1^B}}{k_1^A \sqrt{v_2^A} - k_2^A \sqrt{v_1^A}}\right\} \sqrt{\frac{v_1^A v_2^A}{v_1^B v_2^B}}\, \varphi^B = m\, \varphi^B = \varphi^* . \tag{19}$$

Similarly, using (19) the boundary condition (10) can be reduced to the form

$$\left[c_{33}^A \left\{\frac{k_1^A v_2^A - k_2^A v_1^A}{v_1^A v_2^A}\right\} - \frac{c_{33}^B}{m} \left\{\frac{k_1^B v_2^B - k_2^B v_1^B}{v_1^B v_2^B}\right\}\right] \frac{\partial^2 \varphi^*}{\partial z^2} = 0; \quad (x, y) \in S_e . \tag{20}$$

The boundary conditions (13) and (10) can now be rewritten as

$$\frac{\partial \varphi^*}{\partial z} = \frac{\delta \sqrt{v_1^A v_2^A}}{\{k_1^A \sqrt{v_2^A} - k_2^A \sqrt{v_1^A}\}}; \qquad (x, y) \in S_i \tag{21}$$

$$\frac{\partial^2 \varphi^*}{\partial z^2} = 0; \quad (x, y) \in S_e. \tag{22}$$

For the analysis of the above mixed boundary value problem we follow Lamb [24] and Green and Sneddon [25] and observe that $\partial\varphi^*/\partial z$ represents the velocity potential of the motion of a perfect fluid flowing through an elliptical aperture in a thin boundary. We thus obtain

$$\frac{\partial \varphi^*}{\partial z} = \frac{\delta\, a \sqrt{v_1^A v_2^A}}{2\{k_1^A \sqrt{v_2^A} - k_2^A \sqrt{v_1^A}\}\, K(e_0)}; \int_{\xi}^{\infty} \frac{ds}{[s(a^2+s)(b^2+s)]^{1/2}} \tag{23}$$

where (ξ, η, ζ) are the ellipsoidal coordinates of the points (x, y, z) and they are roots of the equation

$$\frac{x^2}{(a^2+\theta)} + \frac{y^2}{(b^2+\theta)} + \frac{z^2}{\theta} - 1 = 0. \tag{24}$$

In the ellipsoidal coordinate system S_i corresponds to $\xi = 0$ and S_e corresponds to $\eta = 0$. In (23) $K(e_0)$ is the complete elliptical integral of the first kind defined by

$$K(e_0) = \int_0^1 \frac{dt}{[(1-t^2)(1-e_0^2 t^2)]^{1/2}} \tag{25}$$

where $e_0 = (a^2 - b^2)/a^2$. It is also convenient to express the ellipsoidal coordinates in terms of Jacobian elliptic functions, e.g.

$$\xi = a^2 (sn^{-2} u - 1) \tag{26a}$$

where $sn\, u$ is the Jacobian elliptic function which has real and imaginary roots $4K$ and $2iK'$ respectively corresponding to the moduli e_0 and $e_0'(= b/a)$. The variable u takes all real values between 0 and K. [It may also be noted that from the definition of incomplete elliptic integrals (see e.g. Byrd and Friedman [26])

$$u(x, e_0) = u = \int_0^x \frac{dt}{[(1-t^2)(1-e_0^2 t^2)]^{1/2}}. \tag{26b}$$

For a given value of x, (26b) defines u. The elliptic function $sn\, u$ gives the inverse of this equation; $sn\, u$ is the value of x for which the integral yields u.]

In order to complete the analysis of the problem it is necessary to determine the explicit form of φ^* such that (23) is satisfied. For the purposes of this paper it is possible to obtain u_z^i and σ_{zz}^i directly from (23) and the equations (17)–(19). In particular, the normal stresses at the bonded faces of the inclusion are given by

$$\sigma_{zz}^i(x, y, 0^{\pm}) = \pm \frac{c_{33}^i (k_1^i v_2^i - k_2^i v_1^i)\,\delta}{\sqrt{v_1^i v_2^i}\,\{k_1^i \sqrt{v_2^i} - k_2^i \sqrt{v_1^i}\}\, b\, K(e_0) \left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]^{1/2}} \tag{27}$$

where $i = A, B$; $(x, y) \in S_i$ and $(0)^+$ and $(0)^-$ refer to the faces of the inclusion in contact with the regions A and B respectively. The force acting on the rigid elliptical inclusion located at the constrained bi-material interface is given by

$$P = \iint_{S_i} [\sigma_{zz}^A(x, y, 0^+) - \sigma_{zz}^B(x, y, 0^-)]\, \mathrm{d}x\, \mathrm{d}y. \tag{28}$$

Evaluating (28) we obtain the force-displacement response for the inclusion. Assuming that δ occurs in the direction of the applied force we obtain the upper bound estimate as follows

$$P = \frac{2\pi a \delta}{K(e_0)}\left[\sum_{i=A,B} \frac{c_{33}^i \{k_1^i v_2^i - k_2^i v_1^i\}}{\sqrt{v_1^i v_2^i}\,\{k_1^i \sqrt{v_2^i} - k_2^i \sqrt{v_1^i}\}}\right]. \tag{29}$$

4. The lower bound estimate

In order to develop the lower bound estimate for the elastic stiffness of a rigid elliptical inclusion embedded at a bi-material interface, we impose a frictionless constraint at the interface. According to this assumption, the elliptical inclusion is embedded at an interface which exhibits zero shear tractions in the entire region S. The smoothly embedded elliptical inclusion is subjected to a central force P which causes a rigid displacement δ in the z direction. It is assumed that for all values of δ complete contact is maintained at the smooth interface S such that the normal stresses σ_{zz}^i and axial displacements u_z^i $(i = A, B)$ exhibit continuity across the interface S_e. In order to physically realize this continuity requirement the smooth interface can be subjected to a sufficiently large uniform precompression σ_0 (Fig. 2). When separation at the smooth interface is suppressed the magnitude of σ_0 has no effect on the elastic stiffness of the embedded inclusion. The interface conditions associated with the lower bound estimate are as follows: in the entire interface region we require

$$\sigma_{xz}^A(x, y, 0^+) = \sigma_{xz}^B(x, y, 0^-) = 0; \quad (x, y) \in S \tag{30}$$

$$\sigma_{yz}^A(x, y, 0^+) = \sigma_{yz}^B(x, y, 0^-) = 0; \quad (x, y) \in S \tag{31}$$

$$u_z^A(x, y, 0^+) = u_z^B(x, y, 0^-); \quad (x, y) \in S \tag{32}$$

in the inclusion and exterior regions we require

$$\sigma_{zz}^A(x, y, 0^+) = \sigma_{zz}^B(x, y, 0^-); \quad (x, y) \in S_e \tag{33}$$

$$u_z^A(x, y, 0^+) = u_z^B(x, y, 0^-) = \delta; \quad (x, y) \in S_i. \tag{34}$$

In order to satisfy the interface conditions (30) and (31) we select solutions of (1) which take the form

$$\varphi_1^i = \frac{\sqrt{v_1^i}}{(1 + k_1^i)} \varphi^i(x, y, z_1); \quad \varphi_2^i = \frac{\sqrt{v_2^i}}{(1 + k_2^i)} \varphi^i(x, y, z_1) \tag{35}$$

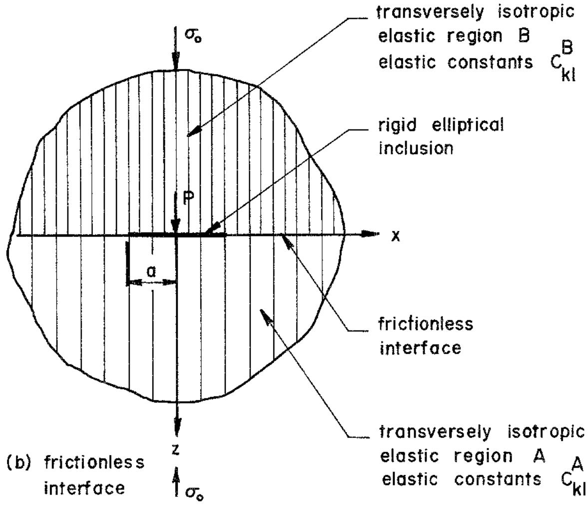

Figure 2
Rigid elliptical disc inclusion embedded in frictionless contact at a frictionless bi-material elastic interface – lower bound estimate.

where $\nabla^2 \varphi^i = 0$. Considering the result (35) and the expressions for u^i_z and σ^i_{zz} we obtain the following results

$$u^i_z = \frac{(k^i_1 - k^i_2)}{(1 + k^i_1)(1 + k^i_2)} \frac{\partial \varphi^i}{\partial z}; \quad (i = A, B) \tag{36}$$

$$\sigma^i_{zz} = \left\{ \frac{(k^i_1 c^i_{33} - v^i_1 c^i_{13})}{\sqrt{v^i_1}(1 + k^i_1)} - \frac{(k^i_2 c^i_{33} - v^i_2 c^i_{13})}{\sqrt{v^i_2}(1 + k^i_2)} \right\} \frac{\partial^2 \varphi^i}{\partial z^2}; \quad (i = A, B). \tag{37}$$

Considering (36) and the boundary conditions (32) and (34) it is evident that

$$\varphi^A = \left\{ \frac{(1 + k^A_1)(1 + k^A_2)(k^B_1 - k^B_2)}{(1 + k^B_1)(1 + k^B_2)(k^A_1 - k^A_2)} \right\} \varphi^B = n\,\varphi^B = \tilde{\varphi}. \tag{38}$$

Using the above result the boundary condition pertaining to the continuity of normal traction (33) can be reduced to the form

$$\left[\frac{(k^A_1 c^A_{33} - v^A_1 c^A_{13})}{\sqrt{v^A_1}(1 + k^A_1)} - \frac{(k^A_2 c^A_{33} - v^A_2 c^A_{13})}{\sqrt{v^A_2}(1 + k^A_2)} \right.$$
$$\left. - \frac{1}{n} \left\{ \frac{(k^B_1 c^B_{33} - v^B_1 c^B_{13})}{\sqrt{v^B_1}(1 + k^B_1)} - \frac{(k^B_2 c^B_{33} - v^B_2 c^B_{13})}{\sqrt{v^B_2}(1 + k^B_2)} \right\} \right] \frac{\partial^2 \tilde{\varphi}}{\partial z^2} = 0; \quad (x, y) \in S_e. \tag{39}$$

The boundary conditions (34) and (33) can now be rewritten as

$$\frac{\partial \tilde{\varphi}}{\partial z} = \frac{\delta (1 + k^A_1)(1 + k^A_2)}{(k^A_1 - k^A_2)}; \quad (x, y) \in S_i \tag{40}$$

$$\frac{\partial^2 \tilde{\varphi}}{\partial z^2} = 0; \quad (x, y) \in S_e. \tag{41}$$

The solution of the mixed boundary value problem defined by (40) and (41) closely follows the procedures outlined in the previous section. In this section we shall restrict our attention to the evaluation of the load-displacement behaviour of the rigid elliptical disc inclusion which is embedded in smooth contact at the bi-material elastic interface. The normal stresses at the faces of the inclusion are given by

$$\sigma_{zz}^{i}(x, y, 0^{\pm}) = \pm\left[\frac{\sqrt{v_2^i}\,(1+k_2^i)\,(k_1^i c_{33}^i - v_1^i c_{13}^i) - \sqrt{v_1^i}\,(1+k_1^i)\,(k_2^i c_{33}^i - v_2^i c_{13}^i)}{\sqrt{v_1^i v_2^i}\,(k_1^i - k_2^i)\left[1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}\right]^{1/2}}\right]\frac{\delta}{K(e_0)\,b}\,. \tag{42}$$

Using (28) and (42) it is possible to evaluate the load-displacement behaviour of the rigid elliptical inclusion which is embedded at the smooth bi-material interface. We have

$$P = \frac{2\pi a\delta}{K(e_0)}\left[\sum_{i=A,B}\left\{\frac{\sqrt{v_2^i}\,(1+k_2^i)\,(k_1^i c_{33}^i - v_1^i c_{13}^i) - \sqrt{v_1^i}\,(1+k_1^i)\,(k_2^i c_{33}^i - v_2^i c_{13}^i)}{\sqrt{v_1^i v_2^i}\,(k_1^i - k_2^i)}\right\}\right] \tag{43}$$

5. Bounds for the elastic stiffness

Considering the results derived in the preceding sections it is proposed that the elastic stiffness for the rigid elliptical disc shaped inclusion embedded in bonded contact at a bonded bi-material interface can be presented in the following set of bounds:

$$\sum_{i=A,B}\left[\frac{c_{33}^i}{c_{44}^*}\right]\frac{\{k_1^i v_2^i - k_2^i v_1^i\}}{\sqrt{v_1^i v_2^i}\,\{k_1^i\sqrt{v_2^i} - k_2^i\sqrt{v_1^i}\}} \geq \frac{P}{\{2\pi a\delta c_{44}^*/K(e_0)\}} \geq \sum_{i=A,B}\left\{\frac{\sqrt{v_2^i}\,(1+k_2^i)\,(k_1^i c_{33}^i - v_1^i c_{13}^i) - \sqrt{v_1^i}\,(1+k_1^i)\,(k_2^i c_{33}^i - v_2^i c_{13}^i)}{c_{44}^*\sqrt{v_1^i v_2^i}\,(k_1^i - k_2^i)}\right\} \tag{44}$$

where

$$c_{44}^* = c_{44}^A + c_{44}^B\,.$$

It is of interest to note that the upper and lower bounds presented in equation (44) depend solely on the elastic properties of the two transversely isotropic regions. The accuracy of these bounds can be established by assigning suitable properties for the transversely isotropic elastic media. As examples, we consider the bounds for the elastic stiffnesses of rigid elliptical inclusion embedded at bi-material interfaces composed of magnesium, cadmium and β-quartz. The transversely isotropic elastic constants for these materials are given by Kassir and Sih [9] and Chen [27] and they are presented in Table 1. The Table 2 summarizes the upper and lower bound estimates for the elastic stiffness derived from (44). These results indicate that the bounding procedures outlined in this paper form efficient techniques for the estimation of the elastic stiffness of rigid elliptical inclusions embedded at the bi-material interface. In the limiting case of material isotropy we note that v_α^i ($\alpha = 1,2$; $i = A, B$) $\to 1$ and

$$c_{11}^i = c_{33}^i = (\lambda^i + 2\mu^i); \quad c_{13}^i = c_{12}^i = \lambda^i; \quad c_{44}^i = \mu^i \tag{45}$$

where λ^i and μ^i are the classical Lamé constants. Also $\lambda^i = E_i v_i/(1 + v_i)(1 - 2v_i)$.

When the bi-material regions exhibit an isotropic elastic response the result (44) reduces to the following

$$\frac{4\{(1 - v_A)(3 - 4v_B) + \Gamma(1 - v_B)(3 - 4v_A)\}}{(3 - 4v_A)(3 - 4v_B)(1 + \Gamma)} \geq \frac{P}{2\pi a \delta \mu^*/K(e_0)} \geq \frac{\{(1 - v_B) + \Gamma(1 - v_A)\}}{(1 - v_A)(1 - v_B)(1 + \Gamma)} \tag{46}$$

where $\mu^* = \mu_A + \mu_B$ and $\Gamma = \mu_B/\mu_A$. When $v_i(i = A, B) \to 1/2$, the bounds (46) converge to the single result

$$P = 4\pi\delta a \mu^*/K(e_0). \tag{47}$$

Table 1
Values of elastic constants for some transversely isotropic materials (in psi).

Material	c_{11}	c_{12}	c_{13}	c_{33}	c_{44}
Magnesium	5.97	2.62	2.17	6.17	1.64
Cadmium	11.00	4.04	3.83	4.69	1.56
β-Quartz	11.66	1.67	3.28	11.04	3.61

Table 2
Upper and lower bound estimates for the elastic stiffness of the rigid elliptical inclusion embedded at the bi-material interface.

Bi-material interface	$\frac{P}{\{2\pi a \delta c_{44}^*/K(e_0)\}}$	
	Upper bound	Lower bound
Magnesium-Cadmium	1.5927	1.4164
Magnesium-β Quartz	1.5940	1.4204
Cadmium-β Quartz	1.4957	1.3471

The result (47) indicates that in the limit of material incompressibility (of both regions) the bi-material interface essentially behaves as an inextensible interface which transmits only normal tractions. When $\nu_A = \nu_B = \nu$ and $\mu_A = \mu_B = \mu$ the asymmetry of the deformation ensures that $u_x = u_y = 0$ on $x_i \in S$; consequently the upper bound result yields the exact solution for the elastic stiffness of a rigid elliptical disc inclusion embedded in bonded contact with an isotropic elastic solid. [9, 28] i.e.

$$P = \frac{16\pi\mu(1-\nu)a\delta}{(3-4\nu)K(e_0)}. \tag{48}$$

Similarly, when $c_{ij}^B \to 0$ we obtain from (44) a set of bounds which may be used to estimate the elastic stiffness of an elliptical rigid punch which is in adhesive contact with a transversely isotropic elastic halfspace.

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Abstract

The present paper examines the problem related to the axial translation of a rigid elliptical disc inclusion which is embedded in bonded contact at the interface of a transversely isotropic bimaterial elastic medium. The result for the stiffness of the elliptical disc inclusion is presented in the form of a set of bounds. These bounds are developed, in exact closed form, by imposing displacement and traction constraints at the bi-material interface.

Resumé

On examine le problème d'une inclusion rigide de form elliptique soumise à une translation axiale. Cette inclusion est située à l'interface d'un milieu élastique bi-matériel à isotropie transversale et elle est en contact lié avec se dernier. La rigidité de l'inclusion elliptique est exprimée par un ensemble de limites et la forme exacte de ces limites à été obtenue en imposant des contraintes de déplacement et traction à l'interface du bi-matériau.

(Received: June 17, 1983)