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SECOND-ORDER ELASTIC EFFECTS IN THE TORSION OF A SPHERICAL ANNULAR REGION

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Abstract—This paper presents the application of a displacement function method to certain problems of second-order incompressible elastic torsion associated with a spherical annular region. Owing to the particular torsional symmetry of the problems considered, the second-order contribution is equivalent to a state of stress which is symmetric about the axis of torsion, the solution of which is facilitated by the use of a displacement function technique.

1. INTRODUCTION

THE THEORY of successive approximations has been widely applied, to obtain approximate solutions to problems in finite elasticity. Comprehensive accounts of the method are given by Truesdell and Noll [1], Green and Adkins [2] and Spencer [3]. The theory of second-order elasticity, which is obtained by terminating the successive approximation procedure after the second approximation, adequately describes the mechanical behaviour of rubber-like materials at moderately large strains. In the theory of successive approximations the classical elasticity solution is usually regarded as the first approximation.

The present paper is concerned with some problems of second-order incompressible elastic torsion of a spherical annular region, in which the deformations are independent of the longitude (φ). Owing to the particular torsional symmetry of these problems, the second-order contribution constitutes a states of stress which is symmetric about the axis of torsion. In a recent paper Selvadurai and Spencer [4] have proposed a method of analysis of purely axially symmetric problems in incompressible second-order elasticity which is based on the use of a 'displacement function'. The close formal resemblance between the 'displacement function' method for incompressible isotropic elastic materials and the formulation of slow flows of a Newtonian fluid in terms of Stokes' 'stream function' is fully discussed in [4]. It is shown that the same technique can be effectively employed to analyse the second-order contribution to this special class of rotationally symmetric torsion problems.

Other noteworthy methods of analysis of torsion problems in second-order elasticity have been proposed by Green and Spratt [5] and Chan and Carlson [6] (see also [1–3]). The displacement function technique however, has a particular appeal to problems where displacement boundary conditions are prescribed but it can also be readily applied to traction boundary value problems.

A summary of the general theory of incompressible finite elasticity expressed in spherical polar coordinates is given in section 2 and the second-order theory for the special class of rotationally symmetric problems is developed in section 3. The formulation of the second-order problem in terms of the displacement function is given in section 4. In section 5 we consider the torsional shear of a bonded spherical annulus. In this problem an incompressible elastic region is in welded contact between two spherical rigid concentric boundaries. The outer sphere is held stationary and the inner sphere is subjected to a pure rotation. Formal solutions to torsion problems associated with a spherical composite and a thick spherical shell are presented in section 6.

The torsion problems thus described are relevant to the study of stress concentrations which occur in bonded rubber mountings and in rubber-like multiphase materials. The purely axially symmetric modes of deformation associated with these problems will be treated elsewhere [7].

2. FINITE ELASTICITY—SPHERICAL POLAR COORDINATES

The rectangular cartesian coordinates of a generic particle in the undeformed configuration and in the deformed configuration are denoted by $X_k (k = 1, 2, 3)$ and $x_i(X_k)$ $(i = 1, 2, 3)$ respectively. The spherical polar coordinates (S, Φ, Θ) and (s, φ, θ) of particles in the reference and deformed configurations are given by

$$X_1 = S \sin\Theta \cos\Phi, \quad X_2 = S \sin\Theta \sin\Phi, \quad X_3 = S\cos\Theta, \tag{2.1}$$

and

$$x_1 = s\sin\theta\cos\varphi, \quad x_2 = s\sin\theta\sin\varphi, \quad x_3 = s\cos\theta, \tag{2.2}$$

respectively.

In general

$$s = s(S, \Phi, \Theta), \quad \varphi = \varphi(S, \Phi, \Theta), \quad \theta = \theta(S, \Phi, \Theta). \tag{2.3}$$

The matrix of deformation gradients in the S, Φ, Θ direction is

$$\mathbf{G}_s = \begin{bmatrix} \dfrac{\partial s}{\partial S} & \dfrac{1}{S\sin\Theta}\dfrac{\partial s}{\partial \Phi} & \dfrac{1}{S}\dfrac{\partial s}{\partial \Theta} \\ s\sin\theta\dfrac{\partial \varphi}{\partial S} & \dfrac{s\sin\theta}{S\sin\Theta}\dfrac{\partial \varphi}{\partial \Phi} & \dfrac{s\sin\theta}{S}\dfrac{\partial \varphi}{\partial \Theta} \\ s\dfrac{\partial \theta}{\partial S} & \dfrac{s}{S\sin\Theta}\dfrac{\partial \theta}{\partial \Phi} & \dfrac{s}{S}\dfrac{\partial \theta}{\partial \Theta} \end{bmatrix}. \tag{2.4}$$

We consider incompressible elastic materials for which

$$\det \mathbf{G}_s = 1. \tag{2.5}$$

The Cauchy–Green strain matrix $\mathbf{C}_s$ referred to (s, φ, θ) coordinates is given by

$$\mathbf{C}_s = \mathbf{G}_s\mathbf{G}_s^T. \tag{2.6}$$

The matrix of the physical components of the symmetrical Cauchy stress tensor is

$$\mathbf{T} = \begin{bmatrix} t_{ss} & t_{s\varphi} & t_{s\theta} \\ t_{s\varphi} & t_{\varphi\varphi} & t_{\theta\varphi} \\ t_{s\theta} & t_{\theta\varphi} & t_{\theta\theta} \end{bmatrix}. \tag{2.7}$$

The general constitutive equation for an incompressible isotropic elastic material can be reduced to the form[3]

$$\mathbf{T} = -p\mathbf{I} + 2C_1\mathbf{C}_s - 2C_2\mathbf{C}_s^{-1}, \tag{2.8}$$

where **I** is the unit matrix, p is an arbitrary scalar hydrostatic pressure and C_1 and C_2 are constants such that

$$2(C_1 + C_2) = \mu, \tag{2.9}$$

where μ is the elastic shear modulus.

In the absence of body forces the equations of equilibrium are

$$\begin{aligned}
&\frac{\partial t_{ss}}{\partial s} + \frac{1}{s}\frac{\partial t_{s\theta}}{\partial \theta} + \frac{1}{s\sin\theta}\frac{\partial t_{s\varphi}}{\partial \varphi} + \frac{1}{s}(2t_{ss} - t_{\theta\theta} - t_{\varphi\varphi} + t_{s\theta}\cot\theta) = 0,\\
&\frac{\partial t_{s\theta}}{\partial s} + \frac{1}{s}\frac{\partial t_{\theta\theta}}{\partial \theta} + \frac{1}{s\sin\theta}\frac{\partial t_{\theta\varphi}}{\partial \varphi} + \frac{1}{s}\{3t_{s\theta} + (t_{\theta\theta} - t_{\varphi\varphi})\cot\theta\} = 0,\\
&\frac{\partial t_{s\varphi}}{\partial s} + \frac{1}{s}\frac{\partial t_{\theta\varphi}}{\partial \theta} + \frac{1}{s\sin\theta}\frac{\partial t_{\varphi\varphi}}{\partial \varphi} + \frac{1}{s}(3t_{s\varphi} + 2t_{\theta\varphi}\cot\theta) = 0.
\end{aligned} \tag{2.10}$$

The components of surface traction $(F_s, F_\varphi, F_\theta)$ in the s, φ, θ directions on a surface $\Gamma(s, \varphi, \theta) = 0$, in the deformed body are

$$\begin{aligned}
t_{ss}n_s + t_{s\varphi}n_\varphi + t_{s\theta}n_\theta &= F_s,\\
t_{s\varphi}n_s + t_{\varphi\varphi}n_\varphi + t_{\theta\varphi}n_\theta &= F_\varphi,\\
t_{s\theta}n_s + t_{\theta\varphi}n_\varphi + t_{\theta\theta}n_\theta &= F_\theta,
\end{aligned} \tag{2.11}$$

where

$$n_s^2 + n_\varphi^2 + n_\theta^2 = 1, \quad \text{and} \quad n_s : n_\varphi : n_\theta = \frac{\partial \Gamma}{\partial s} : \frac{1}{s\sin\theta}\frac{\partial \Gamma}{\partial \varphi} : \frac{1}{s}\frac{\partial \Gamma}{\partial \theta}. \tag{2.12}$$

3. SECOND-ORDER ELASTICITY

In this section we develop the equations of second-order elasticity for rotationally symmetric deformations which are independent of the longitude φ (Fig. 1).

We assume that the spherical polar coordinates s, φ, θ of a particle in the deformed body and the displacement components u, v, w in the S, Φ, Θ directions can be expressed as asymptotic series in terms of a small non-dimensional parameter ϵ, in the form

$$\begin{aligned}
s &= S + \sum_{n=1}^{\infty} \epsilon^n s_n(S, \Theta),\\
\varphi &= \Phi + \sum_{n=1}^{\infty} \epsilon^n \varphi_n(S, \Theta),\\
\theta &= \Theta + \sum_{n=1}^{\infty} \epsilon^n \theta_n(S, \Theta),
\end{aligned} \tag{3.1}$$

and

$$u = \sum_{n=1}^{\infty} \epsilon^n u_n(S, \Theta), \quad v = \sum_{n=1}^{\infty} \epsilon^n v_n(S, \Theta), \quad w = \sum_{n=1}^{\infty} \epsilon^n w_n(S, \Theta), \tag{3.2}$$

respectively.

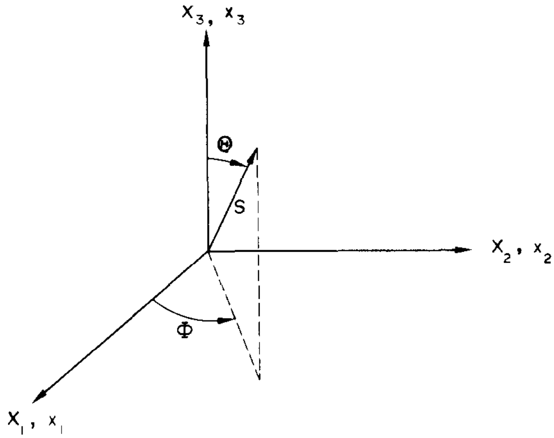

Fig. 1.

By considering the geometry of the deformation it can be shown that

$$u = s\cos(\theta - \Theta) - S, \quad v = s\sin\theta\sin(\varphi - \Phi), \quad w = s\sin(\theta - \Theta). \tag{3.3}$$

By making use of (3.2) and (3.3), (3.1) can now be expressed, to the second-order in ϵ, in the form

$$\begin{aligned} s &= S + \epsilon u_1 + \epsilon^2\left\{u_2 - \frac{w_1^2}{2S}\right\}, \\ \varphi &= \Phi + \frac{\epsilon v_1}{S\sin\Theta} + \epsilon^2\left\{\frac{v_2}{S\sin\Theta} - \frac{v_1}{S\sin\Theta}\left(\frac{u_1}{S} + \frac{w_1}{S}\cot\Theta\right)\right\}, \\ \theta &= \Theta + \epsilon\frac{w_1}{S} + \epsilon^2\left\{\frac{w_2}{S} - \frac{u_1 w_1}{S^2}\right\}. \end{aligned} \tag{3.4}$$

We further restrict our attention to the particular class of rotationally symmetric problems defined in the first-order by displacement components

$$u_1 = 0, \quad v_1 = v_1(S, \Theta), \quad w_1 = 0. \tag{3.5}$$

For the second-order theory we assume [5] that $v_2 = 0$ and u_2, w_2 are independent of Φ. The second-order problem therefore constitutes a state of stress which is symmetric about the axis $\Theta = 0$. By making use of (3.4) and (3.5), (3.1) then reduces to

$$s = S + \epsilon^2 u_2(S, \Theta), \quad \varphi = \Phi + \epsilon\frac{v_1(S, \Theta}{S\sin\Theta}, \quad \theta = \Theta + \epsilon^2\frac{w_2(S, \Theta)}{S}. \tag{3.6}$$

From (2.5) and (3.6) it can be shown that the first-order incompressibility condition is trivially satisfied and the second-order incompressibility condition is

$$\frac{\partial u_2}{\partial S} + 2\frac{u_2}{S} + \frac{1}{S}\frac{\partial w_2}{\partial\Theta} + \frac{w_2}{S}\cot\Theta = 0. \tag{3.7}$$

Also, by making use of (3.6) the Cauchy–Green strain matrix (2.6) can be written in the form

$$\mathbf{C}_s = \mathbf{I} + \epsilon \mathbf{C}_1 + \epsilon^2\{\mathbf{C}_2 + \mathbf{D}_1\}, \tag{3.8}$$

where

$$\mathbf{C}_1 = \begin{bmatrix} 0 & \beta_1 & 0 \\ \beta_1 & 0 & \beta_2 \\ 0 & \beta_2 & 0 \end{bmatrix}, \tag{3.9a}$$

$$\mathbf{C}_2 = \begin{bmatrix} \dfrac{2\partial u_2}{\partial S} & 0 & \dfrac{1}{S}\dfrac{\partial u_2}{\partial \Theta} + \dfrac{\partial w_2}{\partial S} - \dfrac{w_2}{S} \\ 0 & \dfrac{2u_2}{S} + \dfrac{2w_2}{S}\cot\Theta & 0 \\ \dfrac{1}{S}\dfrac{\partial u_2}{\partial \Theta} + \dfrac{\partial w_2}{\partial S} - \dfrac{w_2}{S} & 0 & \dfrac{2u_2}{S} + \dfrac{2}{S}\dfrac{\partial w_2}{\partial \Theta} \end{bmatrix}, \tag{3.9b}$$

$$\mathbf{D}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \beta_1^2 + \beta_2^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{3.9c}$$

and

$$\beta_1 = \frac{\partial v_1}{\partial S} - \frac{v_1}{S}, \quad \beta_2 = \frac{1}{S}\frac{\partial v_1}{\partial \Theta} - \frac{v_1}{S}\cot\Theta. \tag{3.10}$$

Similarly $\mathbf{C}_s^{-1}$ can be written in the form

$$\mathbf{C}_s^{-1} = \mathbf{I} - \epsilon \mathbf{C}_1 - \epsilon^2\{\mathbf{C}_2 + \mathbf{D}_1 - \mathbf{C}_1^2\}. \tag{3.11}$$

It is further assumed that to the second-order in ϵ the stress matrix $\mathbf{T}$ and the arbitrary hydrostatic pressure p can be expressed in the form

$$\mathbf{T} = \mathbf{T}_0 + \epsilon \mathbf{T}_1 + \epsilon^2 \mathbf{T}_2, \quad p = p_0 + \epsilon p_1 + \epsilon^2 p_2. \tag{3.12}$$

We now substitute (3.8), (3.11), (3.12) into (2.8) and equate powers of ϵ. The terms independent of ϵ correspond to a uniform hydrostatic state of stress and the first and second-order constitutive equations take the form

$$\mathbf{T}_1 = -p_1\mathbf{I} + \mu \mathbf{C}_1, \tag{3.13}$$

and

$$\mathbf{T}_2 = -p_2\mathbf{I} + \mu \mathbf{C}_2 + \mathbf{P}_2', \tag{3.14}$$

respectively, where

$$\mathbf{P}_2' = \begin{bmatrix} P'_{ss} & 0 & P'_{s\theta} \\ 0 & P'_{\varphi\varphi} & 0 \\ P'_{s\theta} & 0 & P'_{\theta\theta} \end{bmatrix} = 2C_1\mathbf{D}_1 - 2C_2\{\mathbf{C}_1^2 - \mathbf{D}_1\}. \tag{3.15}$$

Also in the present discussion the differential operators $\partial/\partial s$ and $\partial/\partial\theta$ can be replaced, to order ϵ, by $\partial/\partial S$ and $\partial/\partial\Theta$ respectively; and

$$\frac{\partial}{\partial\varphi}=\frac{\partial}{\partial\Phi}=0. \tag{3.16}$$

By making use of the series expansions for the components of $\mathbf{T}$ and considering that

$$t_{ss}^{(1)}=t_{\varphi\varphi}^{(1)}=t_{\theta\theta}^{(1)}=-p_1,\quad t_{s\theta}^{(1)}=0,\quad t_{s\varphi}^{(2)}=t_{\theta\varphi}^{(2)}=0, \tag{3.17}$$

the equations of equilibrium (2.10) reduce, in the first order, to

$$\frac{\partial t_{ss}^{(1)}}{\partial S}=0,\quad \frac{\partial t_{\theta\theta}^{(1)}}{\partial\Theta}=0, \tag{3.18}$$

$$\frac{\partial t_{s\varphi}^{(1)}}{\partial S}+\frac{3t_{s\varphi}^{(1)}}{S}+\frac{1}{S}\frac{\partial t_{\theta\varphi}^{(1)}}{\partial\Theta}+2t_{\theta\varphi}^{(1)}\cot\Theta=0,$$

and in the second-order to

$$\begin{aligned}&\frac{\partial t_{ss}^{(2)}}{\partial S}+\frac{1}{S}\frac{\partial t_{s\theta}^{(2)}}{\partial\Theta}+\frac{1}{S}\{2t_{ss}^{(2)}-t_{\theta\theta}^{(2)}-t_{\varphi\varphi}^{(2)}+t_{s\theta}^{(2)}\cot\Theta\}=0,\\ &\frac{\partial t_{s\theta}^{(2)}}{\partial S}+\frac{1}{S}\frac{\partial t_{\theta\theta}^{(2)}}{\partial\Theta}+\frac{1}{S}\{3t_{s\theta}^{(2)}+(t_{\theta\theta}^{(2)}-t_{\varphi\varphi}^{(2)})\cot\Theta\}=0.\end{aligned} \tag{3.19}$$

4. DISPLACEMENT FUNCTIONS

By substituting the expressions for $t_{s\varphi}^{(1)}$ and $t_{\theta\varphi}^{(1)}$ in (3.18) we obtain

$$\frac{\partial p_1}{\partial S}=0,\quad \frac{\partial p_1}{\partial\Theta}=0, \tag{4.1}$$

$$\left\{\frac{\partial^2}{\partial S^2}+\frac{2}{S}\frac{\partial}{\partial S}+\frac{1}{S^2}\frac{\partial^2}{\partial\Theta^2}+\frac{\cot\Theta}{S^2}\frac{\partial}{\partial\Theta}-\frac{1}{S^2\sin^2\Theta}\right\}v_1=0. \tag{4.2}$$

We now introduce a 'displacement function' $\Omega(S,\Theta)$ such that

$$v_1=\frac{\Omega}{S\sin\Theta}. \tag{4.3}$$

We note that the function $\Omega(S,\Theta)$ has a formal resemblance to the 'swirl function' which occurs in the equations for slow viscous flow of a Newtonian viscous fluid[8]. Using (4.3), (4.2) can be reduced to the form

$$E^2\Omega=0, \tag{4.4}$$

where E^2 is Stokes' differential operator which has a form

$$E^2 = \frac{\partial^2}{\partial S^2} + \frac{1}{S^2}\frac{\partial^2}{\partial \Theta^2} - \frac{\cot\Theta}{S^2}\frac{\partial}{\partial \Theta}. \tag{4.5}$$

A solution of (4.4) which satisfies the appropriate boundary conditions gives the complete first-order solution. From (4.1) we note that the first-order hydrostatic pressure p_1 is a constant which can be assigned any arbitrary value.

In order to solve the second-order problem we adopt the displacement function method for purely axially symmetric problems proposed by Selvadurai and Spencer [4].

By introducting a 'displacement function' $\Psi(S, \Theta)$ such that

$$u_2 = -\frac{1}{S^2 \sin\Theta}\frac{\partial\Psi}{\partial\Theta}, \quad w_2 = \frac{1}{S\sin\Theta}\frac{\partial\Psi}{\partial S}, \tag{4.6}$$

the second-order incompressibility condition (3.7) is identically satisfied. We now substitute the second-order stress components (3.14), expressed in terms of Ψ, into the corresponding equations of equilibrium (3.19); if from the resulting equations we eliminate p_2 and Ψ in turn we obtain

$$E^4\Psi = \frac{\sin\Theta}{\mu}\left\{\frac{\partial}{\partial S}(SH_2) - \frac{\partial H_1}{\partial\Theta}\right\}, \tag{4.7}$$

and

$$\nabla^2 p_2 = -\left\{\frac{\partial H_1}{\partial S} + \frac{2H_1}{S} + \frac{1}{S}\frac{\partial H_2}{\partial\Theta} + \frac{H_2\cot\Theta}{S}\right\}, \tag{4.8}$$

where

$$H_1 = -\left\{\frac{\partial P'_{ss}}{\partial S} + \frac{1}{S}\frac{\partial P'_{s\theta}}{\partial\Theta} + \frac{1}{S}(2P'_{ss} - P'_{\theta\theta} - P'_{\varphi\varphi} + P'_{s\theta}\cot\Theta)\right\}, \tag{4.9}$$

$$H_2 = -\left\{\frac{\partial P'_{s\theta}}{\partial S} + \frac{1}{S}\frac{\partial P'_{\theta\theta}}{\partial\Theta} + \frac{1}{S}(3P'_{s\theta} + [P'_{\theta\theta} - P'_{\varphi\varphi}]\cot\Theta)\right\}, \tag{4.10}$$

and

$$E^4 = E^2E^2, \quad \nabla^2 = \frac{\partial^2}{\partial S^2} + \frac{2}{S}\frac{\partial}{\partial S} + \frac{1}{S^2}\frac{\partial^2}{\partial\Theta^2} + \frac{\cot\Theta}{S^2}\frac{\partial}{\partial\Theta}. \tag{4.11}$$

The second-order problem for an incompressible isotropic elastic material is therefore reduced to the solution of the two inhomogeneous differential equations (4.7) and (4.8) subject to appropriate boundary conditions.

5. TORSIONAL SHEAR OF A BONDED SPHERICAL ANNULUS

Consider the problem of an incompressible elastic region which is in welded contact between two rigid concentric spheres. The outer sphere is held stationary and the inner sphere is subjected to a rotation ω about the axis $\Theta = 0$. The spherical coordinate systems are chosen such that the origin of coordinates coincides with the common centres of the sphere (Fig. 2).

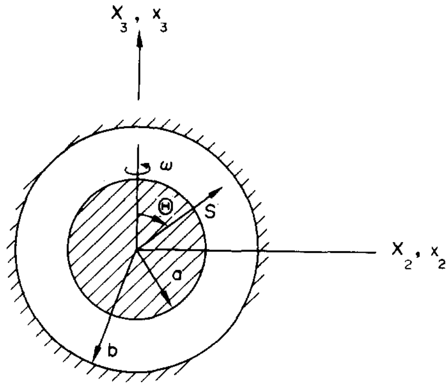

Fig. 2.

First-order solution

The displacement boundary conditions to be satisfied on the surface $S = a$ and $S = b$, where a and b are the inner and outer radii of the spherical shell respectively, are

$$v_1(a, \Theta) = \omega a \sin \Theta, \quad v_1(b, \Theta) = 0. \tag{5.1}$$

We therefore seek solutions of (4.4) of the form

$$\Omega(S, \Theta) = F(S) \sin^2 \Theta. \tag{5.2}$$

Substituting (5.2) in (4.4) we obtain the ordinary differential equation

$$\frac{d^2F}{dS^2} - \frac{2F}{S^2} = 0, \tag{5.3}$$

the general solution of which is

$$F(S) = \rho_1 S^2 + \frac{\rho_2}{S}, \tag{5.4}$$

where ρ_1 and ρ_2 are arbitrary constants.

On satisfying the boundary conditions (5.1) the displacement component v_1 takes the form

$$\epsilon v_1 = \frac{\omega a b^3}{(b^3 - a^3)} \left\{ \frac{a^2}{S^2} - \frac{S a^2}{b^3} \right\} \sin \Theta. \tag{5.5}$$

The non-dimensional parameter ϵ is now chosen to be equal to $\omega/(1 - a^3/b^3)$. As the outer boundary of the spherical annulus, $(S = b)$, recedes to infinity (5.5) gives the first-order displacement field in an infinite elastic solid due to a rotation ω of the rigid sphere.

The first-order stress components corresponding to (5.5) are

$$t^{(1)}_{ss} = t^{(1)}_{\theta\theta} = t^{(1)}_{\varphi\varphi} = p_0, \quad t^{(1)}_{s\theta} = t^{(1)}_{\theta\varphi} = 0, \tag{5.6}$$

$$t^{(1)}_{s\varphi} = -3\mu \frac{a^3}{S^3} \sin\Theta, \tag{5.7}$$

where p_0 is a constant (arbitrary) hydrostatic pressure. The moment acting on the inner sphere is given by

$$M = \int_{-\pi}^{\pi} 2\pi a^3 \sin^2\Theta [t^{(1)}_{s\varphi}]_{S=a}\, d\Theta, \tag{5.8}$$

or

$$M = -\frac{6\pi^2 \mu a^3 \omega}{\left(1 - \frac{a^3}{b^3}\right)}. \tag{5.9}$$

At this point it is convenient to introduce a dimensionless formulation of the problem denoted by ( )* where the change to the physical variables is accomplished according to

$$\begin{aligned} &S = aS^*, && \Psi = a^3\Psi^*, && \Omega = a^2\Omega^*, \\ &v_1 = av_1^*, && u_2 = au_2^*, && w_2 = aw_2^*, \\ &v_1^* = \frac{\Omega^*}{S^* \sin\Theta}, && u_2^* = -\frac{1}{S^{*2}\sin\Theta}\frac{\partial\Psi^*}{\partial\Theta}, && w_2^* = \frac{1}{S^*\sin\Theta}\frac{\partial\Psi^*}{\partial S^*}. \end{aligned} \tag{5.10}$$

We note that

$$v_1^* = \left(\frac{1}{S^{*2}} - \frac{S^*}{\alpha^2}\right)\sin\Theta, \quad \beta_1 = \frac{t^{(1)}_{s\varphi}}{\mu}, \quad \beta_2 = \frac{t^{(1)}_{\theta\varphi}}{\mu} = 0, \tag{5.11}$$

where

$$\alpha = \frac{b}{a}.$$

The second-order problem

From the first-order solution we can determine the elements of $\mathbf{P}_2'$ and the functions $H_1(S^*, \Theta)$ and $H_2(S^*, \Theta)$. The inhomogeneous differential equation (4.7) for the non-dimensional displacement function Ψ^* reduces to

$$E^{*4}\Psi^* = 72\left(4\frac{C_2}{\mu} - 1\right)\frac{1}{S^{*3}}\sin^2\Theta\cos\Theta, \tag{5.12}$$

where

$$E^{*4} = (E^{*2})^2, \quad E^{*2} = \frac{\partial^2}{\partial S^{*2}} + \frac{1}{S^{*2}}\frac{\partial^2}{\partial\Theta^2} - \frac{\cot\Theta}{S^{*2}}\frac{\partial}{\partial\Theta}. \tag{5.13}$$

A particular integral of (5.12) is

$$\Psi_p^* = \frac{\chi}{S^{*3}}\sin^2\Theta\cos\Theta, \tag{5.14}$$

where

$$\chi = \left(\frac{C_2 - C_1}{\mu}\right).$$

The displacement components derived from (5.14) are

$$u^*_{2p} = \frac{\chi}{S^{*5}}(1 - 3\cos^2\Theta), \quad w^*_{2p} = -\frac{3\chi}{S^{*5}}\sin\Theta\cos\Theta. \tag{5.15}$$

Since the elastic medium is in welded contact with the rigid spheres the displacement boundary conditions for the second-order problem are

$$u^*_2(1,\Theta) = u^*_2(\alpha,\Theta) = w^*_2(1,\Theta) = w^*_2(\alpha,\Theta) = 0. \tag{5.16}$$

We seek solutions of the homogeneous equation

$$E^{*4}\Psi^* = 0, \tag{5.17}$$

of the form

$$\Psi^* = S^{*n}\sin^2\Theta\cos\Theta. \tag{5.18}$$

The required homogeneous solution is

$$\Psi^*_h = \chi\left\{\xi_1 + \frac{\xi_2}{S^{*2}} + \xi_3 S^{*3} + \xi_4 S^{*5}\right\}\sin^2\Theta\cos\Theta, \tag{5.19}$$

where ξ_1, ξ_2, ξ_3, ξ_4 are arbitrary constants. By invoking the boundary conditions (5.16) we note that these arbitrary constants are determined by the equations

$$\begin{aligned}
\xi_1 + \xi_2 + \xi_3 + \xi_4 &= -1,\\
\alpha^2\xi_1 + \xi_2 + \alpha^5\xi_3 + \alpha^7\xi_4 &= -\frac{1}{\alpha},\\
-2\xi_2 + 3\xi_3 + 5\xi_4 &= 3,\\
-2\xi_2 + 3\alpha^5\xi_3 + 5\alpha^7\xi_4 &= \frac{3}{\alpha}.
\end{aligned} \tag{5.20}$$

The determinant of the coefficient matrix is

$$\Delta = \alpha^2(\alpha - 1)^4(4\alpha^6 + 16\alpha^5 + 40\alpha^4 + 55\alpha^3 + 40\alpha^2 + 16\alpha + 4). \tag{5.21}$$

From (5.21) we observe that since $\alpha > 1$ the equations (5.20) have a unique finite solution. We have

$$\xi_n = \frac{\Delta_n}{\Delta}, \quad (n = 1, 2, 3, 4) \tag{5.22}$$

where

$$\Delta_1 = \frac{1}{\alpha^2}\{(1-\alpha)(30\alpha^8 - 24\alpha^6 - 6) + (\alpha^6 - 1)(5\alpha^8 + 16\alpha - 21) - (\alpha^8 - 1)(3\alpha^6 + 12\alpha - 15)\},$$

$$\Delta_2 = 15\alpha(\alpha^3 - 1)\{\alpha^3(\alpha^2 - 1) - \alpha^8 + 1\} + 9\alpha(\alpha^5 - 1)(\alpha^6 - 1),$$

$$\Delta_3 = \frac{15}{\alpha}(1-\alpha^2)(\alpha^8 - 1) + \frac{10}{\alpha}(\alpha^7 - 1)(\alpha^3 - 1) + 6\alpha(\alpha - 1)(\alpha^5 - 1),$$

$$\Delta_4 = \frac{9}{\alpha}(1-\alpha^6)(1-\alpha^2) + \frac{6}{\alpha}(\alpha^3 - 1)\{\alpha^2(1-\alpha) - \alpha^5 + 1\}. \tag{5.23}$$

The complete second-order displacement and stress components are

$$u_2 = -\frac{\chi a}{2}\left\{\frac{1}{S^{*5}} + \frac{\xi_1}{S^{*2}} + \frac{\xi_2}{S^{*4}} + \xi_3 S^* + \xi_4 S^{*3}\right\}(1 + 3\cos 2\Theta),$$

$$w_2 = \frac{\chi a}{2}\left\{-\frac{3}{S^{*5}} - 2\frac{\xi_2}{S^{*4}} + 3\xi_3 S^* + 5\xi_4 S^{*3}\right\}\sin 2\Theta, \tag{5.24}$$

and

$$\frac{t_{ss}^{(2)}}{\mu} = \chi\left[\frac{1}{S^{*6}}\left\{\frac{9}{2} + \frac{1}{\chi}\left(-\frac{3C_2}{2\mu} - \frac{3}{4}\right)\right\} + 3\frac{\xi_1}{S^{*3}} + 4\frac{\xi_2}{S^{*5}} - \xi_3 - \frac{1}{2}\xi_4 S^{*2}\right.$$
$$\left. + \left\{\frac{1}{S^{*6}}\left[\frac{27}{2} + \frac{1}{\chi}\left(\frac{3C_2}{2\mu} + \frac{3}{4}\right)\right] + 9\frac{\xi_1}{S^{*3}} + 12\frac{\xi_2}{S^{*3}} - 3\xi_3 + \frac{3}{2}\xi_4 S^{*2}\right\}\cos 2\Theta\right],$$

$$\frac{t_{\varphi\varphi}^{(2)}}{\mu} = \chi\left[\frac{1}{S^{*6}}\left\{-\frac{9}{2} + \frac{1}{\chi}\left(-\frac{3C_2}{2\mu} + \frac{15}{4}\right)\right\} - 3\frac{\xi_2}{S^{*5}} + 2\xi_3 + \frac{15}{2}\xi_4 S^{*2}\right.$$
$$\left. + \left\{\frac{1}{S^{*6}}\left[-\frac{15}{2} + \frac{1}{\chi}\left(\frac{3C_2}{2\mu} - \frac{15}{4}\right)\right] - 5\frac{\xi_2}{S^{*5}} + \frac{25}{2}\xi_4 S^{*2}\right\}\cos 2\Theta\right].$$

$$\frac{t_{\theta\theta}^{(2)}}{\mu} = \chi\left[\frac{1}{S^{*6}}\left\{-\frac{3}{2} + \frac{1}{\chi}\left(\frac{15C_2}{2\mu} - \frac{3}{4}\right)\right\} - \frac{\xi_2}{S^{*5}} - \xi_3 + \frac{5}{2}\xi_4 S^{*2}\right.$$
$$\left. + \left\{\frac{1}{S^{*6}}\left[-\frac{21}{2} + \frac{1}{\chi}\left(-\frac{15C_2}{2\mu} + \frac{3}{4}\right)\right] - 7\frac{\xi_2}{S^{*5}} + 3\xi_3 + \frac{35}{2}\xi_4 S^{*2}\right\}\cos 2\Theta\right],$$

$$\frac{t_{s\theta}^{(2)}}{\mu} = \chi\left[\frac{12}{S^{*6}} + 3\frac{\xi_1}{S^{*3}} + 8\frac{\xi_2}{S^{*5}} + 3\xi_3 + 8\xi_4 S^{*2}\right]\sin 2\Theta. \tag{5.25}$$

The problem is considerably simplified as the outer boundary ($S = b$) recedes to infinity. The first-order displacement field is

$$\epsilon v_1 = \frac{\omega}{S^{*2}}\sin\Theta, \tag{5.26}$$

and the non-dimensional parameter ϵ is chosen to be equal to ω. The boundary conditions at the interface are

$$u_2^*(1,\Theta) = w_2^*(1,\Theta) = 0. \tag{5.27}$$

In order that the second-order displacement and stress components remain finite as $S \to \infty$ the coefficients ξ_3 and ξ_4 in the homogeneous solution (5.19) are set equal to zero. Consequently, the complete second-order displacement and stress components may be obtained by setting $\xi_1 = 1/2$, and $\xi_2 = -3/2$, in (5.24) and (5.25).

The second-order contribution to the couple about the X_3 axis is identically equal to zero. In fact we may observe that from symmetry considerations, for large deformations, the torque (M) must be an odd function of the rotation (ω) and hence ϵ. The expression for the torque given by (5.9) is therefore valid to order ϵ^3. Also, on account of the symmetry of the torsion problem the second-order contribution to the resultant force acting on the outer (or inner) sphere is identically equal to zero.

6. THE SPHERICAL COMPOSITE AND THE THICK SPHERICAL SHELL PROBLEM

In this section we consider the formal solution to some torsion problems associated with an incompressible elastic sphere. The spherical composite consists of a sphere of external radius $b(b < \infty)$ which is bounded internally by a rigid spherical inclusion of radius $a(a < b)$ and the thick spherical shell consists of a sphere bounded internally by a spherical cavity. Tractions are applied on the outer surface of the sphere. The tractions on the hemispheres $X_3 > 0$ and $X_3 < 0$ are statically equivalent to couples about the X_3-axis of equal magnitude but opposite sense. By adopting a method similar to that outlined in section 5 it may be shown that the first-order solutions for the non-dimensional displacement and stress components in the twisted spherical composite (or shell) assume the form

$$\epsilon v_1^* = \omega\left\{c_1 S^{*2} + \frac{c_2}{S^{*3}}\right\} \sin 2\Theta, \tag{6.1}$$

$$\begin{aligned} \frac{t_{s\varphi}^{(1)}}{\mu} &= \left\{c_1 S^* - 4\frac{c_2}{S^{*4}}\right\} \sin 2\Theta, \\ \frac{t_{\theta\varphi}^{(1)}}{\mu} &= -\left\{c_1 S^* + \frac{c_2}{S^{*4}}\right\}(1 - \cos 2\Theta), \end{aligned} \tag{6.2}$$

where c_1 and c_2 are constants. By setting

$$c_1 = 1, \quad c_2 = -1, \quad \epsilon = \omega, \tag{6.3}$$

in (6.1) and (6.2) we obtain the case of the spherical composite where the non-dimensional constant ω can be expressed in terms of the moment (M) acting about the X_3-axis by the relation

$$M = \frac{\pi\mu\omega b^3(\alpha^5 + 4)}{\alpha^4}$$

where $\alpha = b/a$. Similarly by setting

$$c_1 = 1, \quad c_2 = \frac{1}{4}, \quad \epsilon = \omega, \tag{6.4}$$

and

$$M = \frac{\pi\mu\omega b^3(\alpha^5 - 1)}{\alpha^4},$$

we obtain the solution of the thick spherical shell problem. The inhomogeneous differential equation for the non-dimensional second-order displacement function (4.7) reduces to

$$E^{*4}\Psi^* = \left(4\frac{C_2}{\mu}-1\right)\left[\left\{280\frac{c_1c_2}{S^{*4}}-720\frac{c_2^2}{S^{*9}}\right\}\sin^4\Theta\cos\Theta + \left\{-160\frac{c_1c_2}{S^{*4}}+640\frac{c_2^2}{S^{*9}}\right\}\sin^2\Theta\cos\Theta\right]. \tag{6.5}$$

A particular integral of (6.5) is

$$\Psi_p^* = \left(4\frac{C_2}{\mu}-1\right)\left[\left\{c_1c_2-2\frac{c_2^2}{S^{*5}}\right\}\sin^4\Theta\cos\Theta+\frac{4}{3}\frac{c_2^2}{S^{*5}}\sin^2\Theta\cos\Theta\right]. \tag{6.6}$$

The non-dimensional displacement and stress components derived from the particular integral (6.6), (3.18) and (4.6) are

$$u_{2p}^* = 2\chi\left\{\left(5\frac{c_1c_2}{S^{*2}}-10\frac{c_2^2}{S^{*7}}\right)\sin^4\Theta-\left(4\frac{c_1c_2}{S^{*2}}-12\frac{c_2^2}{S^{*7}}\right)\sin^2\Theta-\frac{8}{3}\frac{c_2^2}{S^{*7}}\right\},$$

$$w_{2p}^* = 2\chi\left\{10\frac{c_2^2}{S^{*7}}\sin^3\Theta\cos\Theta-\frac{20}{3}\frac{c_2^2}{S^{*7}}\sin\Theta\cos\Theta\right\}, \tag{6.7}$$

and

$$\frac{t_{ss}^{(2)p}}{\mu} = \left\{2\frac{C_2}{\mu}\left(4c_1^2S^{*2}-92\frac{c_1c_2}{S^{*3}}+274\frac{c_2^2}{S^{*8}}\right)+40\frac{c_1c_2}{S^{*3}}-120\frac{c_2^2}{S^{*8}}\right\}\sin^4\Theta$$
$$+\left\{2\frac{C_2}{\mu}\left(-6c_1^2S^{*2}+96\frac{c_1c_2}{S^{*3}}-320\frac{c_2^2}{S^{*8}}\right)+2c_1^2S^{*2}-40\frac{c_1c_2}{S^{*3}}+144\frac{c_2^2}{S^{*8}}\right\}\sin^2\Theta$$
$$+2\chi\left\{-\frac{16}{3}\frac{c_1c_2}{S^{*3}}+\frac{100}{3}\frac{c_2^2}{S^{*8}}\right\},$$

$$\frac{t_{\varphi\varphi}^{(2)p}}{\mu} = \left\{2\frac{C_2}{\mu}\left(-40\frac{c_1c_2}{S^{*3}}-90\frac{c_2^2}{S^{*8}}\right)+50\frac{c_1c_2}{S^{*3}}\right\}\sin^4\Theta$$
$$+\left\{2\frac{C_2}{\mu}\left(-6c_1^2S^{*2}+48\frac{c_1c_2}{S^{*3}}+\frac{392}{3}\frac{c_2^2}{S^{*8}}\right)+6c_1^2S^{*2}-48\frac{c_1c_2}{S^{*3}}-\frac{52}{3}\frac{c_2^2}{S^{*8}}\right\}\sin^2\Theta$$
$$+2\chi\left\{-\frac{16}{3}\frac{c_1c_2}{S^{*3}}-\frac{68}{3}\frac{c_2^2}{S^{*8}}\right\},$$

$$\frac{t_{\theta\theta}^{(2)p}}{\mu} = \left\{2\frac{C_2}{\mu}\left(-4c_1^2S^{*2}-8\frac{c_1c_2}{S^{*3}}-274\frac{c_2^2}{S^{*8}}\right)+10\frac{c_1c_2}{S^{*3}}+120\frac{c_2^2}{S^{*8}}\right\}\sin^4\Theta$$
$$+\left\{2\frac{C_2}{\mu}\left(-2c_1^2S^{*2}+16\frac{c_1c_2}{S^{*3}}+\frac{904}{3}\frac{c_2^2}{S^{*8}}\right)+2c_1^2S^{*2}-16\frac{c_1c_2}{S^{*3}}-\frac{404}{3}\frac{c_2^2}{S^{*8}}\right\}\sin^2\Theta$$
$$+2\chi\left\{-\frac{16}{3}\frac{c_1c_2}{S^{*3}}-\frac{68}{3}\frac{c_2^2}{S^{*8}}\right\},$$

$$\frac{t_{s\theta}^{(2)p}}{\mu}=\left\{2\frac{C_2}{\mu}\left(4c_1^2S^{*2}+28\frac{c_1c_2}{S^{*3}}-256\frac{c_2^2}{S^{*8}}\right)-20\frac{c_1c_2}{S^{*3}}+120\frac{c_2^2}{S^{*8}}\right\}\sin^3\Theta\cos\Theta$$
$$+2\chi\left\{-8\frac{c_1c_2}{S^{*3}}+\frac{232}{3}\frac{c_2^2}{S^{*8}}\right\}\sin\Theta\cos\Theta \tag{6.8}$$

respectively.

Boundary conditions

By expanding the traction boundary conditions (2.11) in power series of ϵ and taking into consideration the displacement boundary conditions at the rigid inclusion–elastic medium interface it can be shown that for the spherical composite problem the second-order displacement and stress components should satisfy boundary conditions

$$u_2^*(1,\Theta)=w_2^*(1,\Theta)=t_{ss}^{(2)}(\alpha,\Theta)=t_{s\theta}^{(2)}(\alpha,\Theta)=0, \tag{6.9}$$

and for the spherical shell problem the second-order stress components should satisfy boundary conditions

$$t_{ss}^{(2)}(1,\Theta)=t_{s\theta}^{(2)}(1,\Theta)=t_{ss}^{(2)}(\alpha,\Theta)=t_{s\theta}^{(2)}(\alpha,\Theta)=0. \tag{6.10}$$

In addition, the second-order displacement and stress components must be finite and single valued for $0<\Theta<\pi$ and $1<S^*<\alpha$ (the displacement function Ψ^* may be multiple valued). We note that the particular solutions for the second-order displacement and stress components (6.7) and (6.8) respectively, are single valued and finite for $0<\Theta<\pi$ and $1<S^*<\alpha$.

In order to satisfy boundary conditions (6.9) or (6.10) we require ten independent solutions of the homogeneous equation (5.17). We therefore seek solutions of the homogeneous equation (5.17) of the form

$$\Psi_h^*=f_1(S^*)\sin^4\Theta\cos\Theta+f_2(S^*)\sin^2\Theta\cos\Theta+f_3(S^*)\cos\Theta. \tag{6.11}$$

By substituting (6.11) into (5.17) and solving the three resulting simultaneous ordinary differential equations for $f_1(S^*)$, $f_2(S^*)$ and $f_3(S^*)$ we have

$$\Psi_h^*=\left\{\frac{k_1}{S^{*2}}+\frac{k_2}{S^{*4}}+k_3S^{*5}+k_4S^{*7}\right\}\sin^4\Theta\cos\Theta$$
$$+\left\{g_1+\frac{g_2}{S^{*2}}+g_3S^{*3}+g_4S^{*5}-\frac{4}{7}\frac{k_2}{S^{*4}}-\frac{4}{7}k_4S^{*7}\right\}\sin^2\Theta\cos\Theta+l_1\cos\Theta, \tag{6.12}$$

where k_n, g_n $(n=1, 2, 3, 4)$ and l_1 are arbitrary constants. We observe that the nine independent homogeneous solutions (6.12) are chosen such that the displacement and stress components derived from (6.12) are finite for $0<\Theta<\pi$. The remaining solution constitutes a constant hydrostatic state of stress which may be added to the second-order stress components derived from (6.12) without affecting the deformation.

The displacement and stress components derived from (6.12) are

$$u^*_{2h} = 5\left\{\frac{k_1}{S^{*4}} + \frac{k_2}{S^{*6}} + k_3 S^{*3} + k_4 S^{*5}\right\}\sin^4\Theta$$

$$+\left\{-4\frac{k_1}{S^{*4}} - \frac{40}{7}\frac{k_2}{S^{*6}} - 4k_3 S^{*3} - \frac{40}{7}k_4 S^{*5} + 3\left(\frac{g_1}{S^{*2}} + \frac{g_2}{S^{*4}} + g_3 S^* + g_4 S^{*3}\right)\right\}\sin^2\Theta$$

$$+\left\{\frac{8}{7}\frac{k_2}{S^{*6}} + \frac{8}{7}k_4 S^{*5} + \frac{l_1}{S^{*2}} - 2\left(\frac{g_1}{S^{*2}} + \frac{g_2}{S^{*4}} + g_3 S^* + g_4 S^{*3}\right)\right\},$$

$$w^*_{2h} = \left\{-2\frac{k_1}{S^{*4}} - 4\frac{k_2}{S^{*6}} + 5k_3 S^{*3} + 7k_4 S^{*5}\right\}\sin^3\Theta\cos\Theta$$

$$+\left\{\frac{16}{7}\frac{k_2}{S^{*6}} - 4k_4 S^{*5} - 2\frac{g_2}{S^{*4}} + 3g_3 S^* + 5g_4 S^{*3}\right\}\sin\Theta\cos\Theta, \tag{6.13}$$

and

$$\frac{t_{ss}^{(2)h}}{\mu} = \left\{-54\frac{k_1}{S^{*5}} - 60\frac{k_2}{S^{*7}} + 30k_3 S^{*2} + \frac{45}{2}k_4 S^{*4}\right\}\sin^4\Theta$$

$$+\left\{48\frac{k_1}{S^{*5}} + \frac{480}{7}\frac{k_2}{S^{*7}} - 36k_3 S^{*2} - \frac{180}{7}k_4 S^{*4}\right.$$

$$\left.-18\frac{g_1}{S^{*3}} - 24\frac{g_2}{S^{*5}} + 6g_3 - 3g_4 S^{*2}\right\}\sin^2\Theta$$

$$+\left\{-\frac{16}{5}\frac{k_1}{S^{*5}} - \frac{96}{7}\frac{k_2}{S^{*7}} + 8k_3 S^{*2} + \frac{36}{7}k_4 S^{*4}\right.$$

$$\left.+4\frac{g_1}{S^{*3}} + 16\frac{g_2}{S^{*5}} - 4g_3 + 2g_4 S^{*2} - 4\frac{l_1}{S^{*3}} + l_2\right\},$$

$$\frac{t_{\varphi\varphi}^{(2)h}}{\mu} = \left\{18\frac{k_2}{S^{*7}} - \frac{63}{2}k_4 S^{*4}\right\}\sin^4\Theta$$

$$+\left\{4\frac{k_1}{S^{*5}} - \frac{168}{7}\frac{k_2}{S^{*7}} - 10k_3 S^{*2} + 42k_4 S^{*4} + 10\frac{g_2}{S^{*5}} - 25g_4 S^{*2}\right\}\sin^2\Theta$$

$$+\left\{-\frac{16}{5}\frac{k_1}{S^{*5}} + \frac{48}{7}\frac{k_2}{S^{*7}} + 8k_3 S^{*2} - 12k_4 S^{*4} - 8\frac{g_2}{S^{*5}} + 2g_3 + 20g_4 S^{*2} + 2\frac{l_1}{S^{*3}} + l_2\right\},$$

$$\frac{t_{\theta\theta}^{(2)h}}{\mu} = \left\{12\frac{k_1}{S^{*5}} + 42\frac{k_2}{S^{*7}} - 30k_3 S^{*2} - \frac{147}{2}k_4 S^{*4}\right\}\sin^4\Theta$$

$$+\left\{-4\frac{k_1}{S^{*5}} - \frac{312}{7}\frac{k_2}{S^{*7}} + 10k_3 S^{*2} + 78k_4 S^{*4} + 14\frac{g_2}{S^{*5}} - 6g_3 - 35g_4 S^{*2}\right\}\sin^2\Theta$$

$$+\left\{-\frac{16}{5}\frac{k_1}{S^{*5}} + \frac{48}{7}\frac{k_2}{S^{*7}} + 8k_3 S^{*2} - 12k_4 S^{*4} - 8\frac{g_2}{S^{*5}} + 2g_3 + 20g_4 S^{*2} + 2\frac{l_1}{S^{*3}} + l_2\right\},$$

$$\frac{t_{s\theta}^{(2)h}}{\mu} = \left\{30\frac{k_1}{S^{*5}} + 48\frac{k_2}{S^{*7}} + 30k_3 S^{*2} + 48k_4 S^{*4}\right\}\sin^3\Theta\cos\Theta + \left\{-8\frac{k_1}{S^{*5}} - \frac{192}{7}\frac{k_2}{S^{*7}}\right.$$

$$\left.-8k_3 S^{*2} - \frac{192}{7}k_4 S^{*4} + 6\frac{g_1}{S^{*3}} + 16\frac{g_2}{S^{*5}} + 6g_3 + 16g_4 S^{*2}\right\}\sin\Theta\cos\Theta, \tag{6.14}$$

respectively, where l_2 is the constant hydrostatic stress. Explicit values of the constants k_n, g_n, l_1 and l_2 for the spherical composite problem and the thick spherical shell problem can be uniquely determined by making use of boundary conditions (6.9) and (6.10) respectively. The complete second-order solution for the displacement and stress components is obtained by combining the particular solutions (6.7), (6.8) and the homogeneous solutions (6.13) and (6.14).

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Résumé—Cet article présente l'application d'une méthode de fonction de déplacement à certains problèmes de torsion élastique incompréssible du second ordre associée à un domaine annulaire sphérique. A cause de la symétrie de torsion particulière des problèmes considérés, la contribution du second ordre est équivalente à un état de contrainte qui est symétrique par un état de contrainte qui est symétrique par ràpport à l'axe de torsion, et dont la solution est facilitée par l'utilisation d'une technique de fonction de déplacement.

Zusammenfassung—Diese Arbeit legt die Anwendung eines Verdrängungsfunktionsverfahrens auf bestimmte Probleme inkompressibler elastischer Torsion zweiter Ornung vor, die mit einem sphärischen ringfrmigen Bereich verknüpft sind. Infolge der eigentümlichen Torsionssymmetrie der untersuchten Probleme ist der Beitrag zweiter Ordnung einem Spannungszustand gleichwertig, der um die Torsionsachse symmetrisch ist und dessen Lösung durch die Verwendung einer Verdrängungsfunktionsmethode erleichtert wird.

Sommario—Questo articolo presenta l'applicazione di un metodo della funzione dello spostamento a certi problemi di torsione elastica incompressibile del secondo ordine associati ad una regione anulare sferica. A causa della particolare simmetria torsionale dei problemi studiati, la contribuzione del secondo ordine è equivalente ad uno stato di sollecitazione che è simmetrico rispetto all'asse di torsione, la soluzione del quale è facilitata dall'uso di un metodo della funzione dello spostamento.

Абстракт — В работе дано применение метода функции смещения к некоторым проблемам несжимаемого упругого кручения второго порядка, связанным с сферической кольцевой областью. Благодаря определенной симметрии кручения в рассматрываемых проблемах, вклад второго порядка является эквивалентом состоянию напряжения, которое лежит симметрично относительно оси кручения. Применение метода функции смещения облегчает решение этой задачи.