Indentation of Smooth Elastic Interface by Disk Inclusion

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Abstract: This paper examines the problem related to the unilateral interaction between two smoothly precompressed isotropic elastic halfspaces which is caused by the wedging action of a rigid circular inclusion of finite thickness located at the interface. The unilateral nature of the contact leads to the development of a zone of separation between the halfspace regions. This paper discusses a mathematical treatment of the problem and develops an approximate estimate for the radius of the zone of separation. Numerical results presented in the paper illustrate the manner in which the extent of the zone of separation is influenced by the precompression stress and the geometry of the embedded disk inclusion.

Introduction

The category of problem that examines the unilateral contact between deformable elastic solids has received considerable attention. Comprehensive accounts of current development in this area are given by Dundurs and Stippes (9), de Pater and Kalker (8) and Gladwell (13). A particular group of contact problems is concerned with the unilateral interaction between either elastic halfspace regions and elastic layers or elastic layers and rigid boundaries, which can be caused by the action of localized concentrated forces or rigid indentors. For example, Keer and Chantaramungkorn (16), Keer et al. (18) and Tsai et al. (31) have studied a series of problems that examine the unilateral contact between an elastic layer and an elastic halfspace under the action of normal and shear external loadings. The separation at the frictionless interface between an elastic layer and a rigid boundary has been examined by Pu and Hussian (21), Civelek and Erdogan (5) and Gecit and Erdogan (10). Similar problems for elastic layers that are subjected to gravity stresses are considered by Keer and Silva (17), Civelek and Erdogan (4) and Gecit (11). Recent studies by Comninou et al. (6) and Schmueser et al. (25,26) examine the class of unilateral contact problems in which frictional effects are present at the interface. The problem of the body force induced separation at a precompressed transversely isotropic elastic halfspace region is examined by Selvadurai (23). The class of problems that deals with the unilateral contact between beams and plates resting in smooth contact with an elastic halfspace region have been examined by Weitsman (32), Gladwell and Iyer (14) and Gladwell (12). References to further work in this area are also given by Selvadurai (22) and Gladwell (13).

The present paper examines the problem of the frictionless unilateral contact between two precompressed isotropic elastic halfspaces that is perturbed by a smooth penny shaped rigid inclusion of finite thickness

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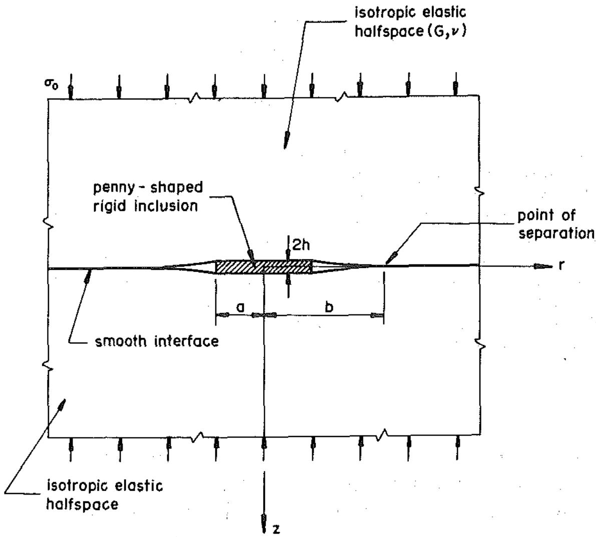

FIG. 1.—Separation of Smooth Precompressed Interface by Smooth Rigid Disk Inclusion

(Fig. 1). The precompression stress σ_0 is assumed to be such that complete contact is maintained between the rigid disk inclusion and the elastic halfspaces. A zone of separation is created beyond the disk inclusion region. Smooth contact is maintained beyond the zone of separation. The radius of the zone of separation (b) is an unknown parameter in the problem which is influenced by the geometric aspect ratio of the disk inclusion (i.e. the thickness to diameter ratio $2h/2a$), the ratio σ_0/G (in which G = the shear modulus of the halfspace region and ν = Poisson's ratio.) In order to evaluate the radius of separation we utilize solutions to two auxiliary problems. These problems consist of a three part mixed boundary value problems in which displacements are prescribed in the interface regions $0 < r \le a$ and $b \le r < \infty$ and normal tractions are prescribed in the region $a < r < b$. Approximate techniques are used to solve these mixed boundary value problems. In particular the auxiliary problems are solved to generate stress intensity factors at the potential boundary of separation $r = b$. The condition of a vanishing stress intensity factor at the point of separation is used to develop a characteristic equation required for the estimation of b. Numerical results presented in the paper illustrate the manner in which the radius of the zone of separation is influenced by the parameters (σ_0/G) and (h/a).

Basic Equations

For the analysis of the axisymmetric problems associated with the unilateral contact problem, it is convenient to employ a formulation which is based on the strain potential approach of Love (19). In the absence of body forces, the solution to the displacement equations of equilibrium can be represented in terms of a biharmonic function $\Phi(r,z)$; i.e.

$$\nabla^2\nabla^2\Phi(r,z) = 0 \qquad (1)$$

in which ∇^2 is Laplace's operator, defined by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \qquad (2)$$

and (r, θ, z) represents the cylindrical polar coordinate system. The components of the displacement vector $\underset{\sim}{u}$ and the Cauchy stress tensor $\underset{\sim}{\sigma}$ referred to the cylindrical polar coordinate system can be expressed in terms of the derivatives of $\Phi(r,z)$. The displacement components are given by

$$2Gu_r = -\frac{\partial^2\Phi}{\partial r\partial z} \qquad (3)$$

$$2Gu_z = 2(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2} \qquad (4)$$

and the stress components are given by

$$\sigma_{rr} = \frac{\partial}{\partial z}\left[\nu\nabla^2\Phi - \frac{\partial^2\Phi}{\partial r^2}\right] \qquad (5)$$

$$\sigma_{\theta\theta} = \frac{\partial}{\partial z}\left[\nu\nabla^2\Phi - \frac{1}{r}\frac{\partial\Phi}{\partial r}\right] \qquad (6)$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left[(2-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right] \qquad (7)$$

$$\sigma_{rz} = \frac{\partial}{\partial r}\left[(1-\nu)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right] \qquad (8)$$

Unilateral Contact Problem

We examine the axisymmetric problem in which two isotropic elastic halfspaces with perfectly smooth boundaries (located at the plane $z = 0$) are precompressed by a uniaxial state of stress σ_0. The smooth contact between the halfspace regions is perturbed by a rigid circular disk inclusion of diameter $2a$ and thickness $2h$. We shall assume that this wedging action creates a zone of separation of radius b at the smooth interface. The magnitude of the precompression σ_0 is such that complete contact is maintained between the halfspace regions beyond $r \geq b$ and that complete contact is maintained between the disc inclusion and the halfspace regions in the region $r \leq a$. The extent of the separation region is derived by evaluating an expression for the stress intensity factor K_I^* at the location $r = b^+$ (the positive superscript denotes a point immediately beyond $r = b$) for the following auxiliary problems. The first problem is concerned with the wedging of a penny shaped crack by a smooth rigid circular inclusion of thickness $2h$. The associated stress intensity factor is denoted by K_I^h. The second problem involves the evaluation of the stress intensity factor $K_I^{\sigma_0}$ at the outer boundary of an annular crack which

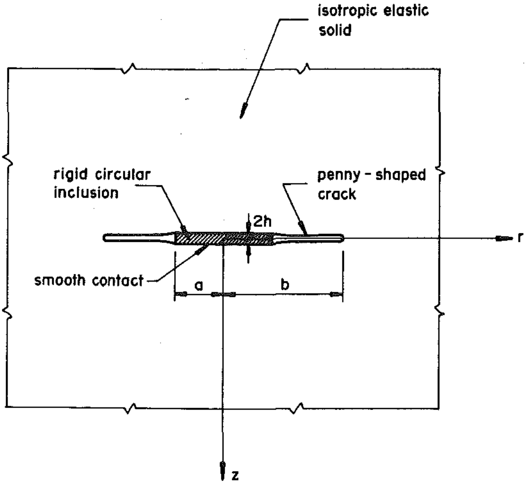

FIG. 2.—Indentation of Penny Shaped Crack by Smooth Rigid Disk Inclusion

is subjected to tensile normal stresses σ_0 in the region $a < r < b$. This stress distribution is required to render the zone of separation traction free. These auxiliary problems are shown in Figs. 2 and 3. Since the precompressed interface is incapable of sustaining tensile tractions the condition $K_I^* = K_I^h + K_I^{\sigma_0} = 0$, yields a characteristic equation from which b can be determined. Since both auxiliary problems exhibit a state of symmetry about $z = 0$, we can restrict our attention to the analysis of a single halfspace region $(z > 0)$. The boundary conditions for the auxiliary problems are as follows:

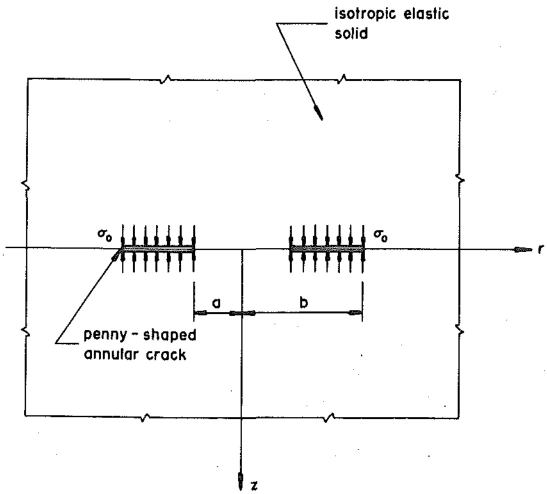

FIG. 3.—Geometry and Loading of Annular Crack

1. For the symmetric indentation of a penny shaped crack by the rigid disk inclusion we require

$\sigma_{rz}(r,0) = 0; \quad r \geq 0$ .... (9)

$u_z(r,0) = h; \quad 0 \leq r \leq a$ .... (10)

$u_z(r,0) = 0; \quad b \leq r < \infty$ .... (11)

$\sigma_{zz}(r,0) = 0; \quad a < r < b$ .... (12)

2. For the tensile loading of the annular crack region we require

$\sigma_{rz}(r,0) = 0; \quad r \geq 0$ .... (13)

$u_z(r,0) = 0; \quad 0 \leq r \leq a$ .... (14)

$u_z(r,0) = 0; \quad b \leq r < \infty$ .... (15)

$\sigma_{zz}(r,0) = \sigma_0; \quad b < r < a$ .... (16)

In order to examine these auxiliary problems it is convenient to employ a Hankel transform development of Eq. 1. The integral representation for $\Phi(r,z)$ can be chosen such that the stress and displacement fields derived from $\Phi(r,z)$ reduce to zero as $(r^2 + z^2)^{1/2} \rightarrow \infty$. The relevant solution is [see e.g. Sneddon (29)]

$$\Phi(r,z) = \int_0^\infty \xi[A(\xi) + zB(\xi)]e^{-\xi z} J_0(\xi r) d\xi \quad \text{.......... (17)}$$

in which $A(\xi)$ and $B(\xi)$ are arbitrary functions which are to be determined by satisfying the mixed boundary conditions applicable to each auxiliary problem.

Indentation of Penny Shaped Crack

The stresses and displacements in the elastic medium can be determined by making use of the strain potential (Eq. 17) and the expressions (Eqs. 3–8). Consequently the mixed boundary conditions (Eqs. 7–12) yield the following set of triple integral equations for a single unknown function $C(\xi)$ (the functions $A(\xi)$ and $B(\xi)$ in Eq. 17 can be uniquely expressed in terms of $C(\xi)$):

$H_0\{\xi^{-2}C(\xi);\ r\} = h^*; \quad 0 \leq r \leq a$ .... (18)

$H_0\{\xi^{-1}C(\xi);\ r\} = 0; \quad a < r < b$ .... (19)

$H_0\{\xi^{-2}C(\xi);\ r\} = 0; \quad b \leq r < \infty$ .... (20)

in which $h^* = -Gh/(1 - \nu)$ and H_0 is the Hankel operator of order zero defined by

$$H_0[f(\xi);\ r] = \int_0^\infty \xi f(\xi) J_0(\xi r) d\xi \quad \text{.......... (21)}$$

The set of triple integral equations defined by Eqs. 18–20 can be solved by employing the approximate procedure suggested by Cooke (7). General accounts of the techniques that may be employed in the solution of triple integral equations are given by Williams (33), Sneddon (30) and Kanwal (15). The details of the method of analysis will not be pursued here. A brief summary of the procedure will be presented for the sake of completeness. We assume that Eq. 19 admits a representation of the form

$$H_0\{\xi^{-1}C(\xi);\ r\} = \begin{cases} f_1(r); & 0<r<a \quad \ldots\ldots (22) \\ f_2(r); & b<r<\infty \quad \ldots\ldots (23) \end{cases}$$

By making use of the Hankel inversion theorem it can be shown that

$$C(\xi) = \xi\left[\int_0^a \eta f_1(\eta)\,J_0(\xi\eta)d\eta + \int_b^\infty \eta f_2(\eta)\,J_0(\eta\xi)d\eta\right] \quad \ldots\ldots (24)$$

Making use of Eq. 24 in Eq. 18 and Eq. 20 we obtain the following simultaneous integral equations for f_1 and f_2:

$$\int_0^a \eta f_1(\eta)L(\eta,r)d\eta + \int_b^\infty \eta f_2(\eta)\,L(\eta,r)d\eta = h^*;\quad 0\le r\le a \quad \ldots\ldots (25)$$

$$\int_0^a \eta f_1(\eta)L(\eta,r)d\eta + \int_b^\infty \eta f_2(\eta)L(\eta,r)d\eta = 0;\quad b<r<\infty \quad \ldots\ldots (26)$$

in which $L(\eta,r) = \int_0^\infty J_0(\xi\eta)J_0(\xi r)d\xi$ ..... (27)

These simultaneous integral equations can be further reduced to Fredholm integral equations of the second kind. For example by introducing the substitutions

$$\psi(\eta_1) = \frac{\pi a}{2h^*}(1-\eta_1^2)^{1/2}f_1(a\eta_1) \quad \ldots\ldots (28)$$

$$\xi = \xi_1 a;\quad \eta = \eta_1 a \quad \ldots\ldots (29)$$

we can obtain the following Fredholm integral equation of the second kind for $\psi(\eta_1)$:

$$\psi(\eta_1) = \eta_1 + \int_0^1 \psi(\xi_1)K(\xi_1,\eta_1)d\xi_1;\quad 0\le\eta_1\le 1 \quad \ldots\ldots (30)$$

in which the kernel function $K(\xi_1,\eta_1)$ is given by

$$K(\xi_1,\eta_1) = \frac{4c\xi_1\eta_1(1-c^2\xi_1^2)^{1/2}}{\pi^2(1-\xi_1^2)^{1/2}}\int_1^\infty F(\xi_1,\eta_1,t_1)dt_1 \quad \ldots\ldots (31)$$

$$F(\xi_1,\eta_1,t_1) = \frac{\left[1-\dfrac{c^2}{t_1^2}\right]^{1/2}}{t_1^3\left[1-\dfrac{1}{t_1^2}\right]^{1/2}\left[1-\dfrac{c^2\xi_1^2}{t_1^2}\right]\left[1-\dfrac{c^2\eta_1^2}{t_1^2}\right]} \quad \ldots\ldots (32)$$

and $c = a/b$. The integral, Eq. 30 can be solved approximately by employing a variety of numerical techniques [see e.g. Atkinson (1) and Baker (2)]. For the purposes of this paper we develop an approximate solution of Eq. 30 in which the kernel function is expressed as a series in terms of the small parameter $c(<1)$. The kernel function $K(\xi_1, \eta_1)$ and $\psi(\eta_1)$ are expressed in the forms

$$K(\xi_1, \eta_1) = \frac{4}{\pi^2(1-\xi_1^2)^{1/2}} \sum_{i=1}^{n} c^i K_i(\xi_1, \eta_1) \qquad (33)$$

and $$\psi(\eta_1) = \sum_{i=1}^{n} c^i \psi_n(\eta_1) \qquad (34)$$

respectively. Eqs. 33 and 34 can be substituted in Eq. 30 and powers of c can be compared to yield a solution for $\psi(\eta_1)$. Finally, the solution for $f_1(r)$ takes the form

$$f_1(r) = \frac{2h^*}{\pi(a^2-r^2)^{1/2}}\left\{\rho + \frac{c}{\pi}\rho + \frac{c^2}{\pi^2}\rho + c^3\left[\frac{\rho}{\pi^2} + 4\pi\rho\left(\frac{\rho^2}{6} - \frac{5}{96}\right)\right]\right.$$
$$+ c^4\left[\frac{4\rho}{\pi^3}\left(\frac{5\pi}{48} + \frac{1}{4\pi} + \frac{\pi}{6}\left(\rho^2 - \frac{1}{2}\right)\right)\right] + c^5\left[\frac{4\rho}{\pi^4}\left(\frac{7\pi}{48} + \frac{1}{4\pi}\right)\right.$$
$$\left.\left. + \frac{2\rho}{\pi^3}\left(\frac{\rho^3}{3} - \frac{5}{48}\right) + \frac{4\rho}{\pi}\left(\frac{2}{15}\rho^4 - \frac{7}{240}\rho^2 - \frac{91}{3{,}840}\right)\right] + 0(c^6)\right\} \qquad (35)$$

in which $\rho = r/a$.

Similarly the series solution of $f_2(r)$ takes the form

$$f_2(r) = \frac{-2h^*}{\pi(r^2-b^2)^{1/2}}\left\{\frac{c}{2\gamma^2} + \frac{c^2}{2\pi\gamma^2} + c^3\left[\frac{1}{2\pi\gamma^2} + \frac{3(2-\gamma^2)}{16\gamma^4}\right]\right.$$
$$+ c^4\left[\frac{1}{\pi\gamma^2}\left(\frac{7}{48} + \frac{1}{2\pi^2} + \frac{3(2-\gamma^2)}{16\gamma^2}\right)\right] + c^5\left[\frac{2}{\pi\gamma^2}\left\{\frac{5\pi}{256\gamma^4}(8-\gamma^4-4\gamma^2)\right.\right.$$
$$\left.\left.\left. + \frac{3(2-\gamma^2)}{32\pi\gamma^2} + \frac{1}{4\pi^3} + \frac{7}{48\pi}\right\}\right] + 0(c^6)\right\} \qquad (36)$$

in which $\gamma = r/b$. The result of primary interest to the analysis of the unilateral contact problem concerns the stress intensity factor at the boundary of the crack $r = b$, which is defined by

$$K_I^h = \lim_{r \to b^+} [2(r-b)]^{1/2} \sigma_{zz}(r, 0) \qquad (37)$$

Noting that $\sigma_{zz}(r,0) = -f_2(r)$ it can be shown that Eq. 37 gives

$$K_I^h = \frac{hG}{\pi(1-\nu)\sqrt{b}}\left[c + \frac{c^2}{\pi} + c^3\left\{\frac{1}{\pi^2} + \frac{3}{8}\right\} + c^4\left\{\frac{2}{3\pi} + \frac{1}{\pi^3}\right\}\right.$$
$$\left. + c^5\left\{\frac{23}{24\pi^2} + \frac{15}{64} + \frac{1}{\pi^4}\right\} + 0(c^6)\right] \qquad (38)$$

During the indentation of the smooth precompressed interface by the rigid circular disk inclusion a zone of continuous separation assumed to occur in the region $a \le r \le b$. Furthermore, the region $0 \le r \le a$ is assumed to maintain full contact with the rigid inclusion. The second auxiliary problem should therefore examine the mixed boundary value problem in which the applied normal stresses (σ_0) are eliminated in the separation zone and the displacements are set equal to zero in the interface regions $0 \le r \le a$ and $b \le r < \infty$. The problem corresponds to the uniform tensile loading of an annular penny shaped crack (Fig. 3). The mixed boundary conditions, Eqs. 13–16, yield the following set of triple integral equations for an unknown function $D(\xi)$ (the functions $A(\xi)$ and $B(\xi)$ can be expressed in terms of $D(\xi)$)

$$H_0[\xi^{-2}D(\xi); r] = 0; \quad 0 \le r \le a \quad \text{(39)}$$

$$H_0[\xi^{-1}D(\xi); r] = \frac{\sigma_0}{2G}; \quad a < r < b \quad \text{(40)}$$

$$H_0[\xi^{-2}D(\xi); r] = 0; \quad b \le r < \infty \quad \text{(41)}$$

Again, the analysis of this system of triple integral equations can be approached by adopting techniques that are similar to those outlined in the section titled Indentation of Penny Shaped Crack, of this paper. The annular crack problem has also been examined by Smetanin (28), Moss and Kobayashi (20) and Shibuya et al. (27) by using different analytical schemes. Choi and Shield (3) have presented a compact analysis of the annular crack located in an elastic solid which is subjected to torsional and axial loads. These authors also provide an estimate of the accuracy of the solutions developed in Refs. 20 and 28. Recently Selvadurai and Singh (24) have re-examined the annular crack problem and presented an approximate solution for the stress concentration factor at the boundary $r = b$. For the purposes of analysis of the unilateral interaction problem we note that the stress intensity factor at $r = b$ is given by (24),

$$K_I^{\sigma_0} = -\frac{2\sigma_0\sqrt{b}}{\pi}\left[1 - \frac{4}{\pi^2}c - \frac{16}{\pi^4}c^2 - c^3\left\{\frac{1}{8} + \frac{64}{\pi^6}\right\} - c^4\left\{\frac{16}{3\pi^4} + \frac{4}{\pi^2}\left(\frac{1}{24} - \frac{8}{9\pi^2} + \frac{64}{\pi^6} + \frac{4}{9\pi^3}\right)\right\} - c^5\left\{\frac{16}{\pi^4}\left(\frac{1}{24} + \frac{64}{\pi^6} - \frac{8}{9\pi^3} + \frac{8}{9\pi^2}\right) + \frac{256}{9\pi^2} - \frac{4}{15\pi^2}\right\}\right] \quad \text{(42)}$$

ZONE OF SEPARATION

The radius of the zone of separation can be estimated from the condition $K_I^h + K_I^{\sigma_0} = 0$. Since both stress intensity factors are evaluated to within the same order of c, we can combine Eqs. 38 and 42 to determine the characteristic equation for b/a. This result can be written in the form

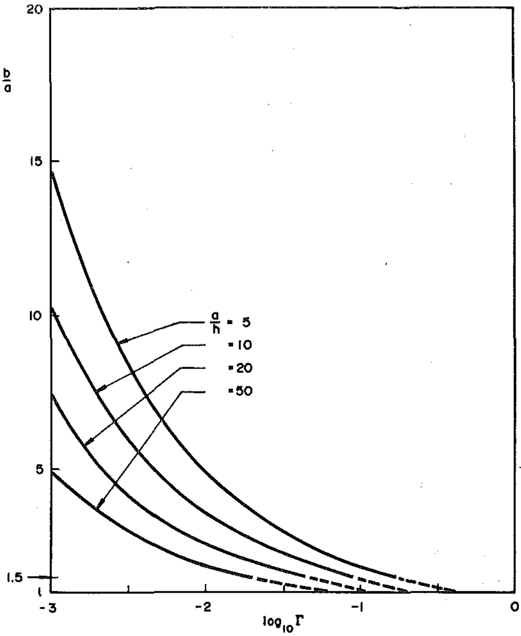

FIG. 4.—Influence of Precompression on Radius of Zone of Separation

$$\Omega_1(\lambda) - \frac{a}{h}\Gamma\Omega_2(\lambda) = 0 \quad (43)$$

in which $\Omega_1(\lambda) = \lambda^4 + \frac{\lambda^3}{\pi} + \lambda^2\left(\frac{1}{\pi^2} + \frac{3}{8}\right)$

$$+ \lambda\left(\frac{2}{3\pi} + \frac{1}{\pi^3}\right) + \left(\frac{23}{24\pi^2} + \frac{15}{64} + \frac{1}{\pi^4}\right)$$

$$\Omega_2(\lambda) = \lambda^6 - \frac{4}{\pi^2}\lambda^5 - \frac{16}{\pi^4}\lambda^4 - \lambda^3\left(\frac{1}{8} + \frac{64}{\pi^6}\right) - \lambda^2\left[\frac{16}{3\pi^4} + \frac{4}{\pi^2}\left(\frac{1}{24} - \frac{8}{9\pi^2}\right.\right.$$

$$\left.\left. + \frac{64}{\pi^6} + \frac{4}{9\pi^3}\right)\right] - \lambda\left[\frac{16}{\pi^4}\left(\frac{1}{24} + \frac{64}{\pi^6} - \frac{8}{9\pi^3} + \frac{8}{9\pi^2}\right) + \frac{256}{9\pi^6} - \frac{4}{15\pi^2}\right]$$

$$\lambda = \frac{b}{a}; \quad \Gamma = \frac{2\sigma_0(1-\nu)}{G} \quad (44)$$

The radius of the zone of separation is given by lowest root of Eq. 43 which satisfies the constraint $\lambda > 1$. Eq. 43 can be numerically evaluated for various values of a/h and Γ to generate the required results. Fig. 4 shows the variation of b/a for various values of Γ and a/h. It is evident that the precompression stress σ_0 has a significant influence on the zone of separation induced by the disk inclusion. As $2\sigma_0(1-\nu)/G \to 1$, the value of b/a approaches unity and the development of a zone of separation is suppressed. Since the solutions for K_I^h and $K_I^{\sigma_0}$ are developed in series forms which involve the parameter $c = a/b$, the accuracy of the solution diminishes as λ (or c) approaches unity. The results developed, however, indicate a realistic trend for all $\Gamma < 10^{-1}$. The results obtained by Eq. 43 is considered to be accurate for all values of $b/a > 1.5$. Also the geometric aspect ratio of the embedded rigid disk inclusion also has a significant influence on the radius of the zone of separation. Computations carried out indicate that when $a/h = 100$, a zone of separation will develop only when Γ is in the order of 10^{-2}.

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Appendix II.—Notation

The following symbols are used in this paper:

$A(\xi), B(\xi), C(\xi), D(\xi)$ = arbitrary functions;
a = radius of the disk inclusion;
b = radius of separation zone or radius of annular crack;
c = a/b

$f_1(r), f_2(r)$ = arbitrary functions;
G = linear elastic shear modulus;
H_n = Hankel operator;
h = thickness of disk inclusion;
h^* = $-Gh/(1-\nu)$;
J_0, J_1 = Bessel functions;
K_I^* = $K_I^h + K_I^{\sigma_0}$;
$K_I^{\sigma_0}$ = stress intensity factor at outer boundary of annular crack;
K_I^h = stress intensity factor for indentation of penny-shaped crack by disk inclusion;
$K(\xi_1, \eta_1)$ = kernel function;
$L(\eta, r)$ = arbitrary function;
(r, θ, z) = cylindrical polar coordinate system;
$\underset{\sim}{u}$ = displacement vector;
u_r, u_θ, u_z = radial, tangential and axial displacements;
$\sigma_{rr}, \sigma_{\theta\theta}, \sigma_{zz}, \sigma_{rz}$ = components of $\underset{\sim}{\sigma}$ referred to the cylindrical polar coordinates;
$\underset{\sim}{\sigma}$ = Cauchy stress tendor;
ν = Poisson's ratio;
∇^2 = Laplace's operator;
Φ = Love's strain potential;
σ_0 = compressive stress;
ρ = r/a;
γ = r/b;
λ = b/a;
Γ = $2\sigma_0(1-\nu)/G$; and
$\Omega_1(\lambda), \Omega_2(\lambda)$ = arbitrary function.