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FLEXURE OF AN INFINITE BEAM EMBEDDED IN A VLAZOV LAYER

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Summary—This paper examines the problem related to the flexure of an infinite beam of finite width embedded in bonded contact with a Vlazov-type elastic layer. In such an elastic layer internal displacement constraints are prescribed *a priori*. The expressions for the deflection, flexural moment, etc. in the infinite beam are presented in integral form. Numerical results are presented for the flexural moment induced in the embedded beam due to the action of a concentrated line load normal to its length.

NOTATION

x, y, z	Cartesian coordinate system
u, v, w	displacement components
$w(x, y)$	surface displacement of the Vlazov layer
$h(z)$	function describing axial variation of $w(x, y)$
$q(x, y), q(x)$	contact stress
C_1, C_2	material constants of the Vlazov layer
H	semi-thickness of the Vlazov layer
h	thickness of beam
b	width of beam
$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$	Laplace's operator
E_0, ν_0	modified elastic constants
E, ν	material elastic constants of the Vlazov layer
E_b, ν_b	elastic constants of the beam material
I	moment of inertia of the beam section
$p(x)$	applied loading
q_0, q_0^*, q'	uniform contact stress
Q	contact force
a	length parameter
w^*	uniform beam displacement
$\bar{w}_0$	non-dimensional beam deflection
$\eta^2 = m^2 + \frac{C_1 a^2}{C_2}$	substitution parameter
$M(x)$	flexural moment in beam
$V(x)$	shear force in beam
P_0	line loading
ϕ	relative rigidity parameter
δ, λ	substitution parameters
m, n	wave length parameters for the harmonic loading

1. INTRODUCTION

The class of problems which deal with the interaction of structural elements embedded in elastic media has several useful engineering applications. Such problems serve as models for the study of interaction between the matrix and fibres of fibre-reinforced materials [1–3], kink band formation in fibrous composites [4, 5] and in the analysis of flexural interaction between pipelines and the surrounding soil medium [6]. In examining these problems it is usually assumed that the matrix surrounding the structural element can be represented by an elastic medium of infinite extent. The structural element is represented by a flat beam (of finite width and thickness) or a cylindrical shell (of finite radius) and infinite length. When these simplifying assumptions are incorporated the analysis of the complete three-dimensional interaction problem requires the solution of a complicated system of coupled integral equations. When examining the flexural interaction of beams embedded in elastic media, the analysis can be further simplified by restricting the flexural behaviour to the longitudinal direction. Despite these simplifications solutions to embedded beams and shells can be obtained only for a limited number of cases.

In this paper we examine the problem of flexure of an infinite beam of finite width

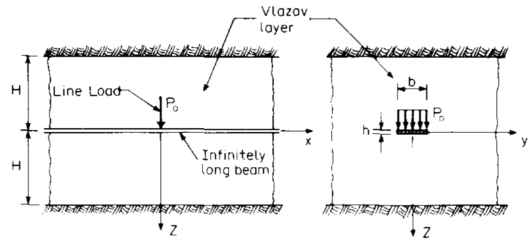

FIG. 1. Geometry of the infinitely long beam of finite width embedded in a Vlazov layer.

embedded in a Vlazov-type elastic layer[7, 8]. The Vlazov-type idealized elastic layer is characterized by its ability to transmit applied loads to adjacent locations of the layer by virtue of continuity of displacement in the medium. This is in contrast to a Winkler-type idealized elastic layer in which the displacement field is essentially discontinuous and the displacements occur only within the loaded region. A comprehensive comparative evaluation of these models is given in [6]. In a Vlazov-type elastic layer the displacement field is prescribed *a priori*. In particular, the non-zero displacement component corresponds to the axial displacement $w(x, y, z)$ in the z-direction (Fig. 1). It is assumed that flexure of the infinite beam of finite width takes place only in the longitudinal direction. When this condition is imposed the solution for the longitudinal flexural interaction can be examined in an exact fashion. Results for the flexural deflection, flexural moment, etc. can be evaluated in integral form. Numerical results are presented for the problem in which the embedded infinite beam is subjected to a concentrated line load which acts normal to its length.

2. ANALYSIS

The displacement field in a Vlazov layer is

$$u(x, y, z) = 0; v(x, y, z) = 0; w(x, y, z) = w(x, y)h(z) \tag{1}$$

where $\{u, v, w\}$ are the components of the displacement vector referred to the rectangular Cartesian coordinates and $h(z)$ is some prescribed function. By considering a variational formulation of the equations of equilibrium for the Vlazov layer[7–9] it can be shown that when the surface of the layer is subjected to a stress $q(x, y)$, the relationship between $w(x, y)$ and $q(x, y)$ can be represented in the form of a differential equation

$$q(x, y) = C_1 w(x, y) - C_2 \nabla^2 w(x, y) \tag{2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{3}$$

is Laplace's operator and the parameters C_1 and C_2 are given by

$$C_1 = \frac{E_0}{(1 - \nu_0^2)} \int_0^H \left(\frac{dh}{dz}\right)^2 dz; \; C_2 = \frac{E_0}{2(1 + \nu_0)} \int_0^H [h]^2 \, dz. \tag{4}$$

In (4), H is the thickness of the Vlazov layer (Fig. 1) and the modified elastic constants E_0 and ν_0 are related to the elastic constants E and ν according to

$$E_0 = \frac{E}{(1 - \nu^2)}; \; \nu_0 = \frac{\nu}{1 - \nu}. \tag{5}$$

Specific expressions for C_1 and C_2 can be obtained by assigning particular variations for $h(z)$. For a linear variation of $h(z)(= \{1 - (z/H)\})$, (4) gives

$$C_1 = \frac{E_0}{H(1 - \nu_0^2)}; \; C_2 = \frac{E_0 H}{6(1 + \nu_0)}. \tag{6}$$

3. THE FLEXURE OF THE EMBEDDED BEAM

We consider the problem of an infinite beam of finite width which is embedded in bonded contact with a Vlazov layer (Fig. 1). We assume that the beam does not experience flexure in the y–z plane. As a consequence, the deflections of the infinite beam and the contact stresses at the beam-Vlazov layer interfaces are independent of the y-coordinate. The differential equation governing the flexure of the embedded beam takes the form

$$E_b I \frac{d^4 w(x)}{dx^4} + b\{q^+(x) - q^-(x)\} = bp(x) \tag{7}$$

where $q^+(x)$ and $q^-(x)$ are the contact stresses at the beam; Vlazov medium interfaces in contact with regions $0 \le z \le H$ and $0 \ge z \ge -H$ respectively; $E_b I$ is the flexural rigidity of the beam; b is the width of the beam and $p(x)$ is the loading applied to the embedded beam. In order to examine the interaction problem it is necessary to establish a relationship between the contact stresses $q^\pm(x)$ and the beam deflection $w(x)$. The kinematics of deformation of the Vlazov layer (equation (1)) ensures that the displacement components u and v are zero on the plane $z = 0$. Furthermore, due to the asymmetry of the displacement field $w(x, y, z)$ about $z = 0$ we have $q(x, y, 0^+) = -q(x, y, 0^-)$. These contact stresses, which occur only within the beam region $|x| < \infty$; $|y| \le b/2$, should give rise to a beam deflection which is constant across the width of the beam. Owing to the asymmetry of the problem, we can restrict our attention to the examination of the region $0 \le z \le H$, of the Vlazov layer. In examining the behaviour of a Vlazov layer it is observed that when a displacement constraint is imposed on the surface of the layer in a region B (with boundary ∂B) the reactive stresses within this area correspond to a uniform stress in B and a concentrated line reaction along ∂B [6–8]. Similar considerations apply to the contact stresses which are developed at the interfaces between the embedded beam and the Vlazov layer. When no flexure takes place in the transverse direction, the contact stress distribution within the beam region is uniform at any transverse cross section and concentrated line reactions are developed at the edges of the embedded beam. In order to examine the interaction problem it is necessary to establish a relationship between the contact stress $q^+(x)$ and the beam deflection $w(x)$. To develop such a relationship we examine the displacement of the region of the Vlazov layer $0 \le z \le H$, which is subjected to the stress distribution

$$q(x, y) = q_0 \cos\left(\frac{mx}{a}\right)\cos\left(\frac{ny}{a}\right) \tag{8}$$

on the plane $z = 0$. The axial displacement on $z = 0$ due to the stress $q(x, y)$ is given by

$$w(x, y) = \frac{q_0 a^2 \cos(mx/a)\cos(ny/a)}{[C_1 a^2 + (m^2 + n^2)C_2]}. \tag{9}$$

The results (8) and (9) can be used to generate the required contact stress patterns within the embedded beam region. When the contact stress distribution within the beam corresponds to

$$q(x, y) = \begin{cases} q' \cos(mx/a); & |y| \le b/2; \quad |x| < \infty \\ 0 & ; \quad |y| \ge b/2; \quad |x| < \infty \end{cases} \tag{10}$$

where q' is a constant, the displacement field within the beam region (on the plane $z = 0$) can be evaluated in the form

$$w(x, y) = \frac{q' a^2}{2C_2\eta^2}\left[2 - \exp\left\{-\left(y - \frac{b}{2}\right)\frac{\eta}{a}\right\} - \exp\left\{-\left(y + \frac{b}{2}\right)\frac{\eta}{a}\right\}\right] \tag{11}$$

where

$$\eta^2 = \left\{m^2 + \frac{C_1 a^2}{C_2}\right\}. \tag{12}$$

Similarly when the loading corresponds to a line loading of the type

$$q(x, y) = \begin{cases} Q' \cos(mx/a); & |y| = b/2; \quad |x| < \infty \\ 0 & ; \quad |y| > b/2; \quad |y| < b/2; \quad |x| < \infty \end{cases} \tag{13}$$

the associated displacement field within the beam region is given by

$$w(x, y) = \frac{Q' a}{2C_2\eta}\left[\exp\left\{-\left(y - \frac{b}{2}\right)\frac{\eta}{a}\right\} + \exp\left\{-\left(y + \frac{b}{2}\right)\frac{\eta}{a}\right\}\right]. \tag{14}$$

In order for the displacement in the beam region to be constant on any cross section $x =$ const., we require $Q' = q'a/\eta$. Furthermore from equilibrium considerations, the uniform displacement (w^*) at any cross section is given by $q' = C_1 w^*$.

We assume that the deflection of the beam can be expressed in the form

$$w(x) = w^* \cos\left(\frac{mx}{a}\right). \tag{15}$$

The associated contact force distribution is represented in the form

$$bq(x) = bq^* \cos\left(\frac{mx}{a}\right). \tag{16}$$

By considering the equilibrium of forces for the entire beam section $|y| \le b/2$ at any location $x =$ const., it can be shown that

$$bq^* = bC_1 w^*\left[1 + \frac{2}{b}\left\{\frac{m^2}{a^2} + \frac{C_1}{C_2}\right\}^{-1/2}\right]. \tag{17}$$

The equations (15)–(17) provided the required relationship between $w(x)$ and $q^+(x)$ $(= -q^-(x))$.

To develop a specific solution for the flexural interaction between the embedded beam and the Vlazov layer, we assume that the loading $p(x)$ can be expressed in the form of a Fourier integral

$$p(x) = \int_0^\infty p^*(m) \cos\left(\frac{mx}{a}\right) \mathrm{d}m \tag{18}$$

where

$$p^*(m) = \int_0^\infty p(\xi) \cos\left(\frac{m\xi}{a}\right) \mathrm{d}\xi. \tag{19}$$

By making use of the results derived previously it can be shown that the solution of the differential equation (7) can be evaluated in the integral form

$$w(x) = \frac{ba^4}{E_b I} \int_0^\infty \frac{p^*(m) \cos(m - x/a)\, \mathrm{d}m}{[m^4 + \Omega(m)]} \tag{20}$$

where

$$\Omega(m) = \frac{2ba^4 C_1}{E_b I}\left[1 + \frac{2}{b}\left\{\frac{m^2}{a^2} + \frac{C_1}{C_2}\right\}^{-1/2}\right]. \tag{21}$$

The integral expressions for the flexural moment $M(x)$ and shear force $V(x)$ in the embedded beam can be evaluated in the form

$$\{M(x);\, V(x)\} = ba \int_0^\infty \frac{p^*(m)\{-am^2\colon m^3\}\, \mathrm{d}m}{[m^4 + \Omega(m)]}. \tag{22}$$

This formally completes the analysis of the embedded beam problem. In the ensuing section we shall develop particular results for the problem where the embedded beam is subjected to a concentrated line load of magnitude P_0 per unit length.

4. NUMERICAL RESULTS

In the particular instance where the embedded beam is subjected to a concentrated line load of magnitude P_0, the expression for the transform equivalent of the load (19) gives $p^*(m) = P_0/\pi a$. This result can be used to generate the appropriate expressions for $w(x)$, $M(x)$ and $V(x)$. For the purpose of illustration we shall restrict our attention to the numerical evaluation of the beam deflection at the point of application of the load. We have

$$w(0) = \frac{P_0 ba^3}{\pi E_b I} \int_0^\infty \frac{\mathrm{d}m}{[m^4 + \Omega(m)]}. \tag{23}$$

Also, the distribution of $w(x, y, z)$ through the depth of the Vlazov layer is assumed to be of a linear form. The appropriate constants C_1 and C_2 are given by (6). The length parameter a is set equal to the half thickness (H) of the Vlazov layer. For this special case

$$w(0) = \frac{12P_0H^3}{\pi E_b h^3} \int_0^\infty \frac{\mathrm{d}m}{\left[m^4 + \phi\left\{1 + \frac{2}{\lambda}(m^2 + \delta)^{-1/2}\right\}\right]} \tag{24}$$

where ϕ is a relative rigidity parameter defined by

$$\phi = \frac{24(1 - \nu)}{(1 + \nu)(1 - 2\nu)}\left\{\frac{EH^3}{E_b h^3}\right\} \tag{25}$$

and

$$\lambda = \frac{b}{h}; \quad \delta = \frac{(1 - \nu)}{(1 - 2\nu)}. \tag{26}$$

From (24) it is evident that as the width of the beam increases, the three dimensional effects in the interaction problem become less significant. In the limit $b \to \infty$, (24) reduces to the result for the flexural deflection of a beam embedded in a Winkler-type stratum, i.e.

$$[w(0)]_{\mathrm{Winkler}} = \frac{12P_0H^3}{\pi E_b h^3} \int_0^\infty \frac{\mathrm{d}m}{[m^4 + \phi]}. \tag{27}$$

This result however does not correspond to plane strain deformations of the embedded beam since the flexural rigidity of the beam in (7) is given by $E_b I = E_b bh^3/12$. The appropriate plane strain result can be obtained by replacing this flexural rigidity by $D^* = E_b bh^3/12(1 - \nu_b^2)$ where ν_b is Poisson's ratio for the beam material. (Alternatively, the value of E_b may be replaced by $E_b/(1 - \nu_b^2)$.)

The integral representation for $w(0)$, given by (24) is numerically evaluated by employing Gaussian quadrature techniques. The Figs. 2–4 illustrate the variation of the normalized beam deflection

$$\bar{w}_0 = \frac{w(0)}{12P_0H^3/\pi E_b h^3} \tag{28}$$

with the spatial and material variables ϕ, λ and δ. As noted previously the solution for the interaction problem corresponding to the limit $\lambda \to \infty$ can be obtained by considering the flexural interaction of a beam (or plate)

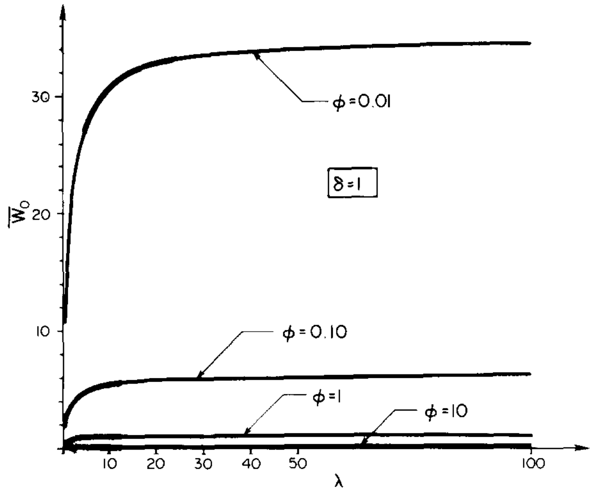

FIG. 2. Variation in the central deflection of the embedded beam.

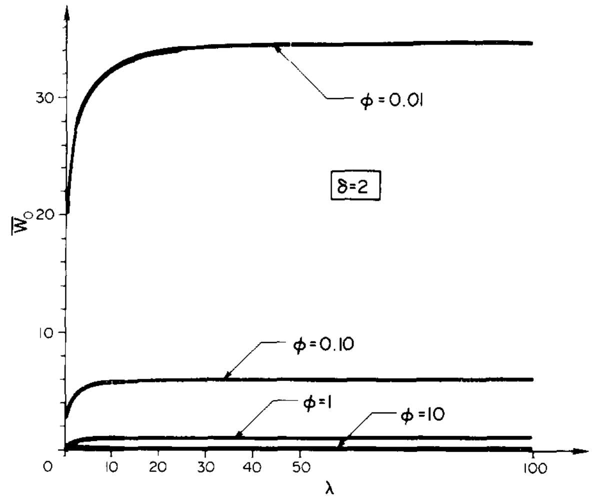

FIG. 3. Variation in the central deflection of the embedded beam.

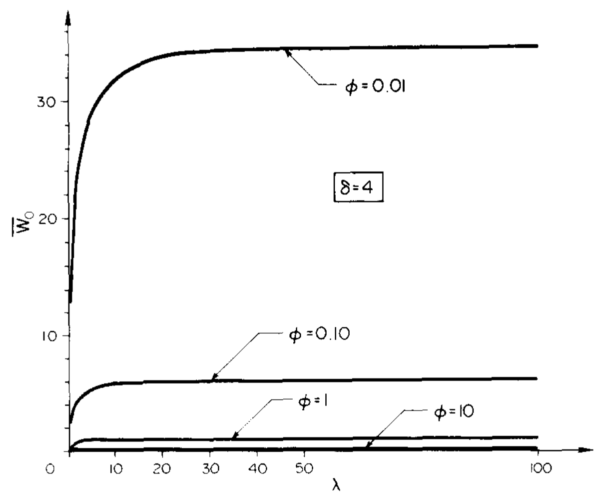

FIG. 4. Variation in the central deflection of the embedded beam.

embedded between two Winkler-type layers. In the particular case when this embedded beam is subjected to a line load P_0, the integral in (27) can be evaluated in the explicit form

$$[\bar{w}_0]_{\text{Winkler}} = \frac{[w(0)]_{\text{Winkler}}}{12P_0H^3/\pi E_b h^3} = \frac{\pi}{2\sqrt{2}\phi^{3/4}}. \tag{29}$$

This exact result provides a useful check on the accuracy of the numerical integration procedure employed in the evaluation of (24).

5. CONCLUSIONS

The present paper develops an analysis of the flexure of an infinite beam of finite width which is embedded in a Vlazov-type elastic layer. It is shown that in instances where the beam experiences flexure only in the longitudinal direction the solution to the above problem can be obtained in integral form. The numerical results presented in the paper indicate that as the width of the beam increases, the edge effects in the three-dimensional problem acquire diminishing importance and the present solution converges to the solution for an infinite beam embedded in a Winkler layer. From the results presented in the paper it would appear that the three dimensional solution converges to the two dimensional plane strain result, for all values of the relative rigidity parameter ϕ when $b/h > 20$. This provides a practical limit which can be used to assess the limits of applicability of the two separate analytical techniques. The basic procedures discussed in this paper can be further extended to examine the flexural behaviour of beams of finite width and length embedded in a Vlazov layer.

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