Approximations for the low frequency response of a rigid plate embedded in an infinite space

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This paper is concerned with the low frequency harmonic oscillation of rigid rectangular foundations deeply embedded in an isotropic elastic soil media. It is shown that a solution based on matched asymptotic expansion techniques can be combined with certain approximate solutions developed for the static compliance of a rigid rectangular plate embedded in an elastic infinite space, to yield a set of bounds for the dynamic response of an embedded rigid rectangular foundation.

INTRODUCTION

The dynamic interaction between deeply embedded foundations and the surrounding soil medium is of some interest in the geotechnical design of footings subjected to earthquake forces or loads induced by oscillatory machines. In particular a correct assessment of the dynamic steady-state force-displacement relationship at the foundation level is fundamental to its proper design. The term 'deeply embedded' is intended to signify foundation systems for which the surrounding soil medium can be approximated by an elastic medium of infinite extent. It has been shown[1] that this assumption is a satisfactory idealization for foundations in which the depth of embedment exceeds approximately $10a$ where a is the largest dimension of the foundation and where $a^2/(\mu/\rho\omega^2) < 1$. ($\mu$ is the isentropic linear elastic shear modulus; ρ is the mass density and ω is the frequency of oscillation.) The idealization of the surrounding soil medium as an elastic infinite space has distinct advantages in the analysis of boundary value problems associated with embedded foundations.[2-6] Despite these advantages, analytical solutions have been obtained for only a limited number of problems associated with embedded foundations. The steady state responses of three-dimensional objects such as spheres and ellipsoids embedded in isotropic elastic media have been examined by Chadwick and Trowbridge,[7] Williams[8] and Datta and Kanwal.[9] The dynamic interaction between an embedded disc inclusion and a normally incident plane harmonic wave was also examined by Mal *et al.*,[10] Datta[11] and Mal.[12] The axisymmetric and asymmetric dynamic responses of a rigid circular disc embedded in bonded contact with an isotropic elastic medium of infinite extent were also examined by Selvadurai.[5,6] Also, of particular interest is the approximate (low frequency) solution to the embedded disc inclusion problem developed by Kanwal[13] by using matched asymptotic expansion techniques. As is evident, much of the analytical results concentrate on simplified geometries of the embedded foundation (i.e. the boundary of the foundation usually conforms to a smooth, regular convex surface; e.g. a circle, sphere, ellipse, ellipsoid, etc.). To date no analytical solutions are available for the dynamic response of rectangular or other irregular shaped foundations deeply embedded in elastic media. The problem of the dynamic behaviour of a rectangular foundation deeply embedded in an elastic medium is of intrinsic engineering interest. Furthermore, any analytical solution developed for this problem would provide a valuable means for the assessment of numerical methods such as finite element, boundary element or boundary integral techniques which may be used in the analysis of dynamic soil-foundation interaction problems with infinite domains.

In this paper an approximate solution is developed for the low frequency dynamic behaviour of a rigid rectangular disc shaped foundation embedded in bonded contact with an elastic medium of infinite extent (Fig. 1). The approximate nature of the solution stems from the utilization of the following two results: (i) a matched asymptotic expansion solution which is valid for low frequency responses of the embedded foundation and (ii) a set of bounds for the static load-displacement response of a rigid rectangular foundation embedded in bonded contact with an isotropic elastic solid. By using these two solutions, certain bounds are developed for the low frequency response of a rigid rectangular foundation of finite mass embedded in bonded contact with an elastic medium. Also, these bounds for the dynamic response of the rectangular foundation are compared with a set of approximate solutions derived via a dynamic Kelvin force analogue wherein the contact stresses at the interface are prescribed *a priori*. Numerical results presented illustrate the correlation between the various estimates for the dynamic behaviour of rectangular foundations deeply embedded in elastic media.

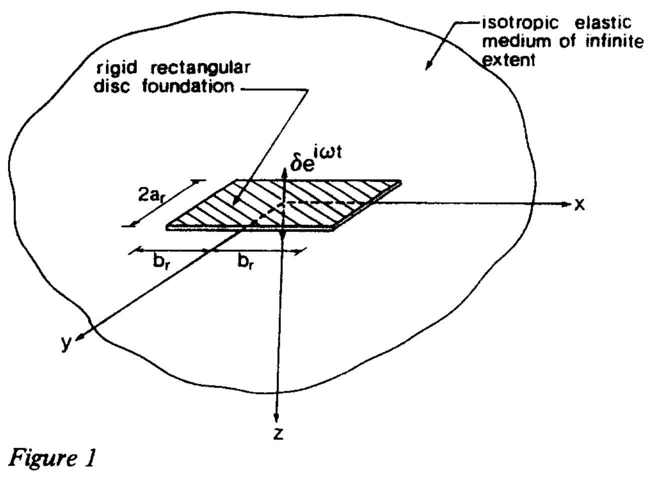

Figure 1

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LOW FREQUENCY SOLUTION BASED ON MATCHED ASYMPTOTIC EXPANSIONS

An analysis of the displacements induced in an infinite elastic medium due to the periodic excitation of a weightless axisymmetric body of arbitrary shape was presented by Kanwal.[13] The approximate nature of the solution stems directly from the use of matched asymptotic expansion techniques for the analysis of the problem. Matched asymptotic expansion techniques have been used quite successfully in the analysis of boundary layer phenomena encountered in fluid dynamics (see e.g. Van Dyke[14]). In this method of analysis two expansions of the displacement field are sought: an inner expansion valid close to the oscillating body and an outer expansion valid in the region remote from the body. The inner expansion is constrained to satisfy the boundary conditions at the surface of the rigid body but not at infinity, whereas the outer expansion is generated to satisfy the conditions at infinity but not at the surface of the body. The successive terms in each expansion are determined by matching the two expansions at some intermediate region. The problem examined by Kanwal[13] has been extended and corrected by Williams.[8] The results which are of primary interest in the vibration of the embedded rectangular foundation are summarized here. The embedded weightless body is subjected to a harmonic axial displacement (Fig. 1), $\delta \exp(i\omega t)$ in the z-direction. The periodic force $P(t)$ required to maintain this oscillation is given by

$$P(t) = P_0 \left\{1 + \frac{ikP_0(2+\gamma^3)}{12\pi\mu\delta a}\right\} \exp(i\omega t) + 0(k^2) \qquad (1)$$

where P_0 is the static load required to produce the displacement δ; $k^2 = \rho\omega^2 a^2/\mu$ is a non-dimensional frequency; μ is the isentropic linear elastic shear modulus; a is a characteristic length parameter associated with the geometry of the axisymmetric solid; $\gamma^2 = (1-2\nu)/(2-2\nu)$ and ν is Poisson's ratio. It should be noted that the result in equation (1) is valid to $0(k^2)$. The accuracy of the matched asymptotic expansion technique can be improved by including higher order terms in the parameter k. The approximation in equation (1) is sufficient for the present purposes of examining the low frequency approximation. Also analytical results derived for the dynamic behaviour of a rigid circular foundation embedded in an elastic medium (i.e. a solution of the associated dual integral equations) indicate that the approximation in equation (1) gives results which are almost identical for the frequency range $k \leqslant 0.5$.[5] The low frequency dynamic response for the embedded foundation can be estimated from equation (1) provided the static load-displacement relationship is known.

STATIC LOAD-DISPLACEMENT RELATIONSHIP FOR THE EMBEDDED RIGID RECTANGULAR FOUNDATION

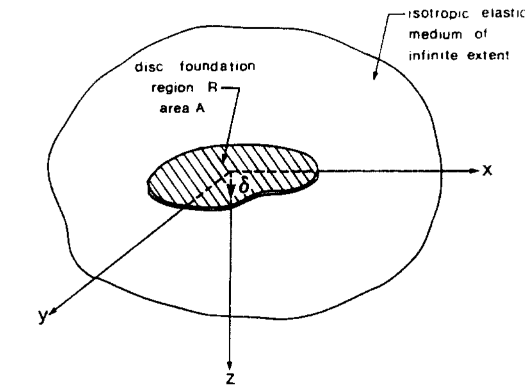
The static elasticity problem concerning the axisymmetric (or axial) displacement of a rigid body embedded within an elastic medium of infinite extent has been examined by several authors including Kanwal and Sharma[15] and Selvadurai.[16,17] These authors examine the class of problems in which the embedded inclusion has the shape of an ellipsoid or a spheroid. The load-deformation responses of other types of plate like circular or elliptical inclusions embedded in elastic media have been examined by Collins,[18] Keer,[19] Kassir and Sih,[20] Selvadurai[21-23] and Gladwell and Selvadurai.[24] From these investigations it becomes clearly evident that analytical solutions for the load-displacement relationship of rigid objects embedded in an elastic medium can be obtained only in situations where the boundary of the inclusion corresponds to a continuous smooth curve. To date there appears to be no explicit analytical solution which furnishes the static load-displacement relationship for rectangular inclusions embedded in an elastic medium. Approximate numerical schemes have, however, been developed (Rowe and Booker[25]) for the examination of the rectangular inclusion problem. The primary limitation of such techniques is that the load-displacement relationship has to be evaluated for each aspect ratio of the embedded rectangular foundation and for a specified Poisson's ratio. An alternative approach would be to develop certain bounds for the static load-displacement relationship of the embedded rectangular foundation by utilizing similar relationships derived for other shapes of embedded objects such as elliptical inclusions. It is of interest to note that the development of bounds for the load-displacement relationship of rigid foundations of arbitrary shape, resting in frictionless contact on the *surface of a half-space* was first considered by Galin.[26] In this approach, the primary objective is to develop upper and lower bounds for the static force P required for a given displacement δ of a rigid foundation with a flat base, which remains parallel to the underformed surface. For the upper bound, the actual contact region is replaced by a circumscribing ellipse of such an orientation so as to have the smallest area. The lower bound is provided by employing one of the symmetrization inequalities established by Polya and Szego[27] for the electrostatically charged lamina. (The electrostatic analogy[28] of the frictionless indentation problem makes the capacitance of the thin lamina analogous to the force-displacement relationship.) Inequalities similar to those developed by Galin[26] for the half-space problem can be developed for the problem dealing with a rigid disc shaped foundation embedded within an elastic medium of infinite extent (Selvadurai[29]). In the ensuing the results of primary interest to this paper will be summarized: a flat rigid disc shaped foundation (region R) (Fig. 2) of an arbitrary plan shape and area A is embedded in bonded contact with an isotropic elastic medium of infinite extent. The load P applied to it causes a rigid body translation δ normal to the plane of the foundation. The bounds for the static load-displacement relationship for the embedded foundation are as follows:

$$\sqrt{\frac{4A}{\pi a_0^2}} < \frac{P(3-4\nu)}{16\delta\mu a_0(1-\nu)} < \frac{\pi b^*}{a_0 K(\pi/2, e)} \qquad (2)$$

In equation (2), a_0 is a geometric length parameter of the problem; b^* is the semi-major axis of the ellipse with the minimum area circumscribing the region R (Fig. 2); e is the eccentricity of the circumscribed ellipse (i.e.: $e = \sqrt{1-(a^*/b^*)^2}$; a^* and b^* are defined in Fig. 2); $K(\pi/2, e)$ is the complete elliptic integral of the first kind defined by

$$K\left(\frac{\pi}{2}, e\right) = \int_0^{\pi/2} \frac{d\theta}{[1 - e^2 \sin^2\theta]^{1/2}} \qquad (3)$$

In the case of rectangular foundations, the upper bound of equation (2) can be further improved by selecting the translational stiffness of an elliptical embedded foundation with



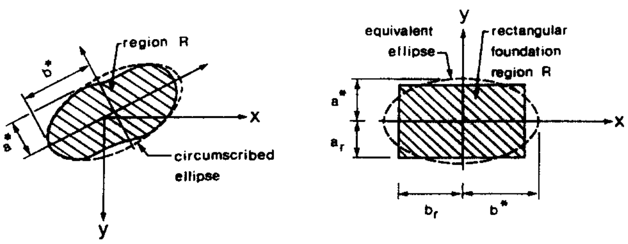


Figure 2

the same plan area and the same aspect ratio. For the specific case of a rectangular foundation, the resulting bounds can be written in the form:

$$\frac{2\sqrt{\Phi}}{\sqrt{\pi}} < \frac{P(3-4\nu)}{32\delta\mu a_r(1-\nu)} < \frac{\Phi\sqrt{\pi}}{K(\pi/2, e)} \tag{4}$$

where $2a_r$ is the width of the rectangular foundation, $2b_r$ is its length; $\Phi = (b_r/a_r) > 1$; $e = \sqrt{1-\Phi^{-2}}$.

For the specific case of an embedded square foundation (i.e. $\Phi = 1$), the bounds converge to the single result

$$\frac{P(3-4\nu)}{32\delta\mu a_r(1-\nu)} = \frac{2}{\sqrt{\pi}} \tag{5}$$

BOUNDS FOR THE AMPLITUDE-FREQUENCY RESPONSE OF EMBEDDED RECTANGULAR FOUNDATIONS

The asymptotic solution in equation (1) can be combined with equation (4) to obtain upper and lower bounds for the low frequency dynamic behaviour of a deeply embedded massless rectangular foundation in bonded contact with an elastic medium. These results can also be generalized to include the mass of the rectangular foundation. Consider a deeply embedded rigid rectangular disc shaped foundation (of dimensions $2a_r \times 2b_r$) which is subjected to a periodic displacement $\delta\exp(i\omega t)$, in the axial direction. The maximum amplitude of the periodic force $Q(t)$ required to maintain the steady oscillation is denoted by Q^*. The low frequency dynamic response of the embedded foundation can be represented in terms of the inequalities

$$\frac{(3-4\nu)}{32(1-\nu)\,[\{c_u - \Delta\xi_1\}^2 + \{\xi_2 c_u^2\}^2]^{1/2}} < \frac{\delta\mu a_r}{Q^*} < \frac{(3-4\nu)}{32(1-\nu)[\{c_l - \Delta\xi_1\}^2 + \{\xi_2 c_l^2\}^2]^{1/2}} \tag{6}$$

where

$$\xi_1 = \frac{k^2(3-4\nu)}{8(1-\nu)^2}$$

$$\xi_2 = \frac{8k(1-\nu)(2+\gamma^3)}{3\pi(3-4\nu)} \tag{7}$$

$$c_l = \frac{2\sqrt{\Phi}}{\sqrt{\pi}}; \quad c_u = \frac{\Phi\sqrt{\pi}}{K(\pi/2, e)}; \quad e^2 = \frac{\Phi^2 - 1}{\Phi^2}$$

and

$$\Delta = M(1-\nu)/4\rho a_r^3$$

is the mass ratio.

NUMERICAL RESULTS

In this section we present numerical results which illustrate the influence of the mass ratio (Δ) the foundation aspect ratio (Φ) and the frequency (k) on the dynamic compliance as estimated from the bounds developed previously. Since the solution is intended to model the low frequency response of the embedded rectangular foundation, the frequency range chosen is restricted to values of $k < 0.50$. In order to provide some basis for the comparison of these results, the following approximate analogue for the embedded rectangular foundation is developed. Firstly, we consider the problem of the harmonic Kelvin force solution related to an isotropic elastic medium. (This problem refers to the dynamic internal loading of an infinite space region by a concentrated harmonic force.) The solution to this problem was developed by Eason *et al.*[30] By utilizing this fundamental solution it is possible to develop the dynamic displacement field in the infinite space region which is subjected to a parabolic stress distribution of the form

$$\sigma_{zz}(x, y, 0, t) = \frac{P\exp(i\omega t)}{\pi^2[(b_r^2 - x^2)(a_r^2 - y^2)]^{1/2}} \tag{8}$$

in the rectangular foundation region $|x| \leqslant b_r$; $|y| \leqslant a_r$. The stress distribution in equation (8) is such that the resultant contact force acting on the foundation is $P(t) = P\exp(i\omega t)$. The contact stresses exhibit singular behaviour along the boundary of the foundation region. We further denote the weighted average $(\bar{u}_z)_R$ of the displacements in the foundation region by $\delta\exp(i\omega t)$: i.e.

$$(\bar{u}_z)_R = \frac{\iint_R u_z(x, y, 0, t)\,dA}{\iint_R dA} = \delta\exp(i\omega t) \tag{9}$$

The dynamic compliance of the rectangular foundation (region) can be expressed in the form

$$\frac{\delta\mu a_r}{Q^*} = \frac{1}{4\pi\left[\left\{g_1 - \dfrac{k^2\Delta}{\pi(1-\nu)}\right\}^2 + g_2^2\right]^{1/2}} \tag{10}$$

where g_1 and g_2 are derived from the integral relationship

$$(g_1 + ig_2) = \left[\frac{1}{\Phi^2 k^2}\int_0^{x=1}\int_0^{y=\Phi}\left\{\int_0^{\Phi}\int_0^{1}\sigma_i(\xi, \eta)\right.\right.$$

$$\times\left\{\exp(-ikl)\left[\frac{k^2}{l}-\frac{1}{l^3}-\frac{ik}{l^2}\right]\right.$$

$$\left.\left.\left.-\exp(-ik\gamma l)\left[\frac{1}{l^3}+\frac{ik\gamma}{l^2}\right]\right\}\mathrm{d}\xi\,\mathrm{d}\eta\right\}\mathrm{d}x\,\mathrm{d}y\right]^{-1} \quad (11)$$

where

$$\sigma_i(\xi,\eta)=\frac{4\Phi}{\pi^2[(\Phi^2-\xi^2)(1-\eta^2)]^{1/2}} \quad (12)$$

$$l^2=(x-\xi)^2+(y-\eta)^2$$

and

$$\Phi=b_r/a_r$$

is the aspect ratio.

Figure 3 shows the variation of the normalized dynamic compliance $\Omega(k)/\Omega(0)$ with the non-dimensional frequency, for the square foundation. Analogous results developed for the rectangular foundation ($\Phi=2$) are shown in Figs. 4 and 5. The normalizing constants $\Omega(0)$ for the various solutions are listed in Table 1. From the results given in Figs. 3-5 it is evident that the mass ratio Δ has a significant influence on the normalized dynamic compliance $\Omega(k)/\Omega(0)$.

CONCLUSIONS

This paper presents an approximate analysis of the dynamic behaviour of an embedded disc shaped rectangular foundation which is subjected to low frequency slow oscillations normal to its plane. A set of bounds are developed for the dynamic compliance of the embedded foundation by combining a solution based on matched asymptotic expansion techniques and a set of inequalities developed for the static compliance. These bounds for the dynamic compliance compare favourably with equivalent results derived from approximate solutions which utilize the dynamic Kelvin force solution. The solutions given in this paper provide useful first approximations for the low frequency dynamic response of embedded rectangular foundations.

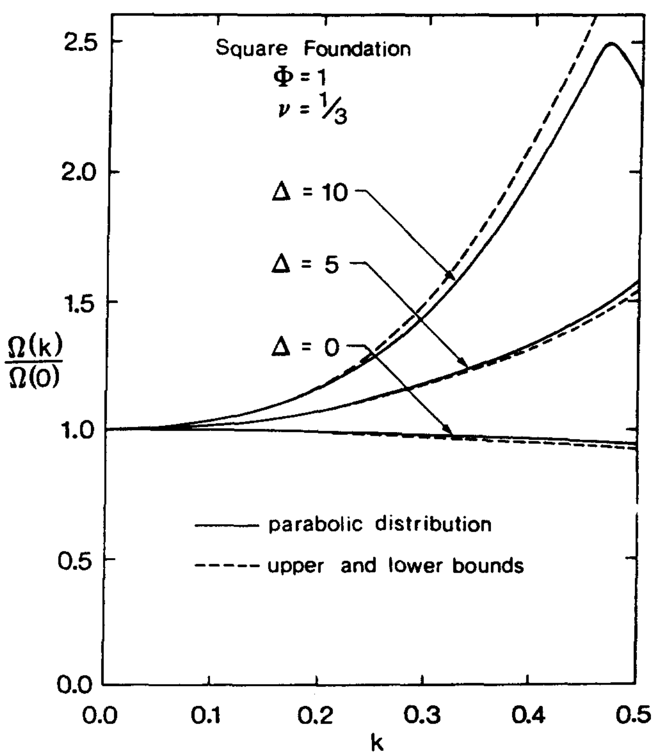


Figure 3

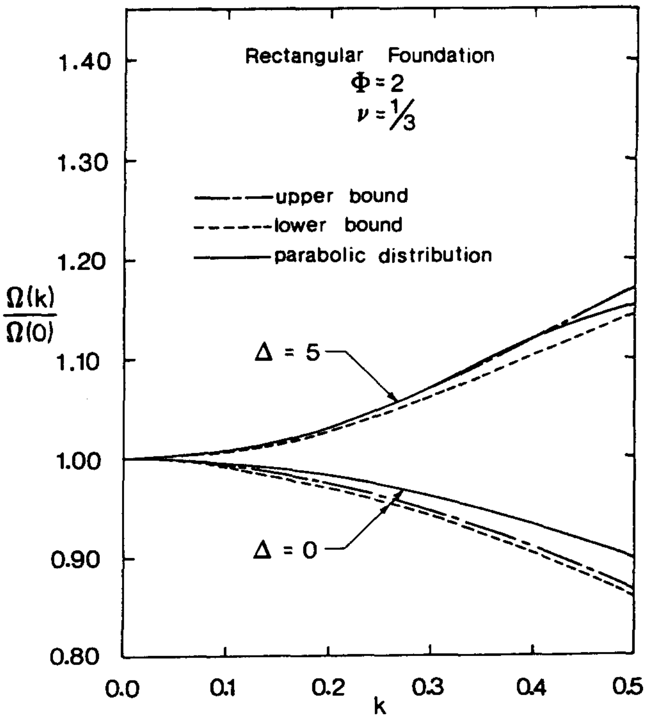


Figure 4

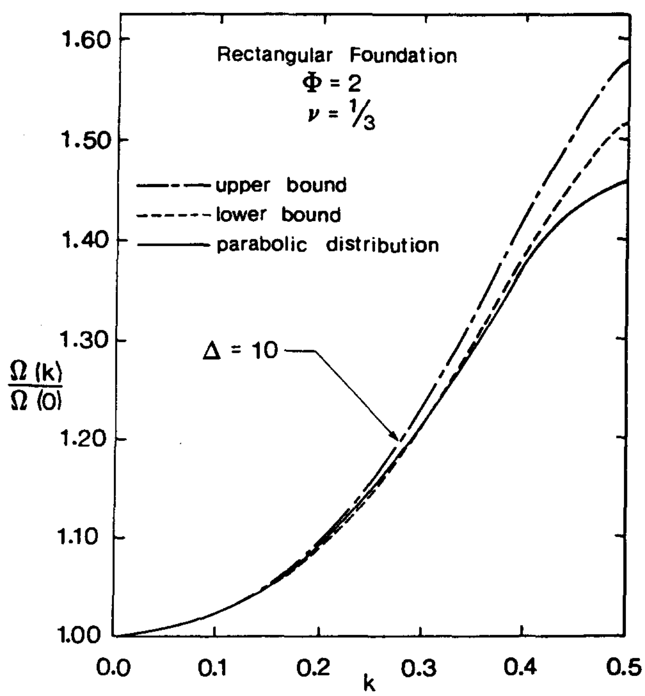


Figure 5

Table 1. The normalizing constants $\Omega(0)$

	$\Phi=1$	$\Phi=2$
Parabolic	0.0692	0.0432
Upper bound	0.0630	0.0489
Lower bound	0.0630	0.0475

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