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# Concentrated body force loading of an elastically bridged penny shaped flaw in a unidirectional fibre reinforced composite

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ABSTRACT

The present paper examines a problem related to the stress concentrations which occur at a penny shaped flaw located in a unidirectional fibre reinforced material. In particular it is assumed that the random array of reinforcing elastic fibres exhibit continuity across the faces of the penny shaped flaw. The process of flaw bridging is analysed by representing the intact fibre reinforced material as a transversely isotropic elastic material. The mathematical analysis of the body force loading of the bridged penny shaped flaw problem can be reduced to the solution of a Fredholm integral equation of the second-kind. The solution of this integral equation directly furnishes the stress intensity factor for the elastically bridged flaw. The numerical results given in the paper illustrate the manner in which the location of the body forces, the flaw geometry, the fibre-matrix modular ratio etc., influence the stress intensity factor for the bridged penny shaped flaw.

## 1. Introduction

A unidirectional fibre reinforced composite essentially consists of a matrix which is reinforced with a random or regular network of closely spaced aligned fibres. The elastic stress analysis of these composites can be performed by idealizing the material as a transversely isotropic elastic medium. The overall elasticity parameters associated with the transversely isotropic elastic idealization can be estimated by appeal to a theory of mixtures. Several investigators have provided theoretical relationships for such estimates. Hashin and Rosen [1], Hill [2], Chen and Cheng [3] and Adams and Doner [4, 5] have derived expressions for the effective elasticity moduli of the transversely isotropic elastic idealization in terms of the mechanical properties of the fibre and matrix materials and their respective volume fractions. These estimates relate to unidirectionally reinforced fibrous composites in which both the matrix and the fibres are elastically isotropic. These analyses have been extended [6–8] to include constituents which display anisotropic elastic properties. Extensive accounts of these developments are given by Broutman and Krock [9], Garg *et al.* [10] and Christensen [11].

The study of fracture processes in such unidirectional fibre reinforced composites is of fundamental importance to their engineering design. There appears to be a multitude of processes, both macroscopic and microscopic in nature, which contribute to fracture processes in such composites. The articles by Kelly [12], Sih [13], Beaumont [14] and Backlund [15] clearly illustrate the various phenomena such as flaw bridging, crack formation, fibre pullout, matrix microcracking, matrix yielding, fibre debonding, delamination etc., which lead to fracture of unidirectionally reinforced composites.

In this paper we shall examine the process of flaw bridging which occurs as a result of continuity of intact fibres across the faces of the flaw. Since fibre reinforced composites are usually composed of slender but elastic fibres it is of interest to examine the overall effect fibre continuity may have on results, such as the intensity factor at the bridged flaw.

This paper attempts to provide a simplified theoretical model for the process of flaw bridging in a unidirectional fibre reinforced composite. It is assumed that a penny shaped flaw (of diameter $2a$ and thickness $2\ell$; $\ell/a \ll 1$) exists in a unidirectional fibre reinforced composite. This definition of the penny shaped flaw is somewhat unconventional in comparison with the classical definition (see e.g. Sneddon [16]) in which $\ell/a \to 0$. Flaws of this nature can be created as a result of entrapped air voids or as a result of rapid changes in the properties of the matrix. The penny shaped idealization is employed for the purpose of the theoretical formulation. There is very little published information which establishes probable limits for $a/\ell$ (see e.g. Barenblatt *et al.* [17], Rice and Johnson [18]). In the present paper we shall ascribe values of $a/\ell$ which range from 10 to $10^4$. Although the flaw thickness $2\ell$ is assumed to be small compared to the flaw diameter $2a$, processes such as delamination of fibres in the vicinity of the faces of the flaw can lead to an increase in the effective bridging length of the fibres (Fig. 2). In the present investigation it is assumed that the plane of symmetry of the penny shaped flaw is normal to the fibre direction. Furthermore it is assumed that the closely spaced group of fibres which bridge the faces of the flaw can be modelled as a series of one dimensional linear elastic elements with independent mechanical action. The latter assumption closely follows the techniques adopted by Barenblatt [19] and Goodier and Kanninen [20] in the modelling of finite and nonlinear cohesive forces at the crack tip. Although continuum models of this type do not take into account the complicated discrete structure at a flaw boundary, they provide useful analogues for the determination of global effects of cohesive forces, flaw bridging etc. on the stress amplification in the continuum region. The analysis of the flaw bridging problem related to axisymmetric body force loading (Fig. 1) follows a standard Hankel transform development of the governing field equations (see e.g. Sneddon [21], Kassir and Sih [22]). The fibre continuity introduces a coupled constraint at the faces of the penny shaped flaw region. The mixed boundary value problem associated with the bridged flaw yields a set of dual integral equations which can be further reduced to a single Fredholm integral equation of the second kind. The solution of the latter integral equation directly furnishes the stress intensity factor for the bridged flaw. Numerical results presented in the paper illustrate the manner in which the location of the body forces, the flaw aspect ratio, the matrix-fibre modular ratio etc., influence the stress intensity factor for the elastically bridged flaw.

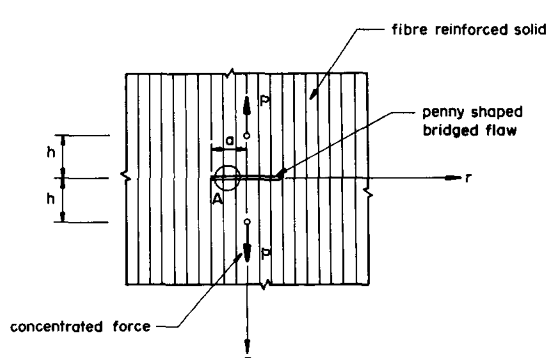


Figure 1. Geometry of the penny shaped flaw and the loading configuration.

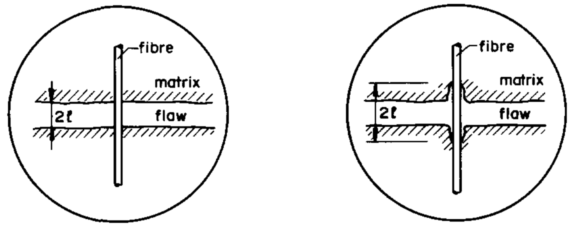


Figure 2. Schematic representation of the detail at A.

## 2. Basic equations

We consider the class of axisymmetric problems related to a transversely isotropic elastic medium in which the $z$-axis of the cylindrical polar coordinate system $(r, \theta, z)$ coincides with the axis of material symmetry. The methods of analysis of this class of axisymmetric problem makes extensive use of the potential function techniques proposed by Elliott [23, 24], and Lekhnitskii [25]. Complete accounts of these techniques are also given by Green and Zerna [26] and Kassir and Sih [22]. It can be shown that in the absence of body forces, the axisymmetric displacement and stress fields can be expressed in terms of two functions $\varphi_i(r, z)$ $(i = 1, 2)$ which are solutions of

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z_i^2}\right\}\varphi_i(r, z) = 0 \tag{1}$$

where $z_i = z/\sqrt{v_i}$ and $v_i$ are roots of the equation

$$c_{11}c_{44}v^2 + \{c_{13}(2c_{44} + c_{13}) - c_{11}c_{33}\}v + c_{33}c_{44} = 0 \tag{2}$$

The roots $v_i$ may be real or complex depending upon the elastic constands $c_{ij}$ (see e.g. [22], [23]). The displacement and stress fields in the transversely isotropic elastic material can be expressed in terms of $\varphi_i(r, z)$. The results of particular interest to the present paper are the following

$$u_z(r, z) = \frac{\partial}{\partial z}\{k_1\varphi_1 + k_2\varphi_2\} \tag{3}$$

$$\sigma_{zz}(r, z) = \{k_1c_{33} - v_1c_{13}\}\frac{\partial^2\varphi_1}{\partial z^2} + \{k_2c_{33} - v_2c_{13}\}\frac{\partial^2\varphi_2}{\partial z^2} \tag{4}$$

$$\sigma_{rz}(r, z) = c_{44}\left\{(1 + k_1)\frac{\partial^2\varphi_1}{\partial r\,\partial z} + (1 + k_2)\frac{\partial^2\varphi_2}{\partial r\partial z}\right\} \tag{5}$$

and

$$k_i = \left\{\frac{c_{11}v_i - c_{44}}{c_{13} + c_{44}}\right\}. \tag{6}$$

The elastic constants $c_{ij}$ can be expressed in terms of the isotropic elastic constants for the fibre (suffix $f$) and matrix (suffix $m$) phases and their respective volume fractions (i.e. $E_f$, $v_f$; $E_m$, $v_m$ and $V_f$, $V_m$ respectively.) These expressions are given by Hashin and Rosen [1], Hill [2] and Rosen [27]. The results given by Rosen [27] are summarized in Appendix 1 for completeness.

## 3. Body force loading of the intact solid

Before examining the bridged penny shaped flaw problem related to the transversely isotropic elastic medium it is necessary to record here some salient results which relate to the axisymmetric loading of the intact solid by symmetrically placed body forces. Body forces of magnitude $P$ act at the locations $z = \pm h$ on the $z$-axis. They are directed in the positive and negative $z$-directions respectively. The analysis of this problem reduces to the determination of a fundamental solution for a transversely isotropic elastic infinite space. The solution for this problem is given by several investigators including Michell [28], Elliott [23], Shield [29] and Pan and Chou [30]. The result of primary interest to this paper is the distribution of normal stress on the plane of symmetry $z = 0$. By virtue of the symmetry, $\sigma_{rz}(r,0) = 0$. The normal stress on the plane of symmetry is given by

$$\sigma^B_{zz}(r,0) = \frac{-P(c_{13}+c_{44})\nu_1\nu_2}{2\pi c_{33}c_{44}(\nu_1-\nu_2)} \times \left[\frac{(k_1c_{33}-\nu_1c_{13})h}{(\nu_1r^2+h^2)^{3/2}} - \frac{(k_2c_{33}-\nu_2c_{13})h}{(\nu_2r^2+h^2)^{3/2}}\right] \tag{7}$$

It may be easily verified that as $\nu_1, \nu_2 \to 1$, the result (7) reduces to the isotropic result that can be obtained by combining a dipole of Kelvin forces.

## 4. Body force loading of the bridged penny shaped flaw

We now examine the axisymmetric problem related to the transversely isotropic infinite space which is bounded internally by a penny shaped flaw which occupies the region $z = \pm\ell; r \leqslant a$ such that $a/\ell \ll 1$. The flaw is bridged by unidirectional fibres which are randomly or regularly distributed in its entire region. The bridged flaw is subjected to axisymmetric body forces of magnitude $P$ which are located on the $z$ axis at $z = \pm h$, and directed away from the surfaces of the flaw (Fig. 1). Since the problem exhibits a state of symmetry in $\sigma_{zz}$ and $u_z$ about $z = 0$ (and since $a/\ell \ll 1$) we can restrict the analysis to a single half-space region ($z \geqslant 0$) in which the plane $z = 0$ is subjected to the mixed boundary conditions

$$\sigma_{rz}(r,0) = 0; \quad r \geqslant 0 \tag{8}$$

$$u_z(r,0) = 0; \quad a \leqslant r \leqslant \infty \tag{9}$$

$$\sigma_{zz}(r,0) = \frac{P}{\pi a^2}\left[\frac{\zeta_1 a^2 h_1}{(r^2+h_1^2)^{3/2}} - \frac{\zeta_2 a^2 h_2}{(r^2+h_2^2)^{3/2}}\right] + \frac{\tilde{E}_f}{\ell}u_z(r,0); \quad r < a \tag{10}$$

where

$$\zeta_i = \frac{(k_ic_{33}-\nu_ic_{13})(c_{13}+c_{44})\nu_1\nu_2}{2c_{33}c_{44}(\nu_1-\nu_2)\nu_i}; \; h_i = \frac{h}{\sqrt{\nu_i}}; \; \tilde{E}_f = E_fV_f \tag{11}$$

where $E_f$ is the modulus of elasticity of the fibres and $2\ell$ is the effective length of the fibres which bridge the flaw region.

For the analysis of this problem, we select solutions of (1) appropriate for the half-space region $z \geqslant 0$ (i.e. both $\sigma_{ij}$ and $u_i$ should tend to zero as $R\,(=\sqrt{r^2+z^2}) \to \infty$). The appropriate solutions for $\varphi_i(r,z)$ are

$$\varphi_i(r,z) = \frac{1}{a^2}\int_0^\infty \xi A_i(\xi)\,e^{-\lambda_i z}J_0(\xi r/a)\,d\xi \tag{12}$$

where $A_i(\zeta)$ are unknown functions and $\lambda_i = \zeta/a\sqrt{v_i}$. From (5) and (12) we note that in order to satisfy the boundary condition (8) we require

$$\sqrt{v_2}\, A_1(\xi)(1 + k_1) = -\sqrt{v_1}\, A_2(\xi)(1 + k_2) \tag{13}$$

Making use of this result and Eqns. (3), (4) and (12) it can be shown that the boundary conditions (9) and (10) are equivalent to the system of dual integral equations:

$$\int_0^\infty \xi B(\xi) F(\xi) J_0(\xi r/a)\, d\xi = \frac{p(r)}{2\mu^*}; \quad 0 \leqslant r \leqslant a \tag{14}$$

$$\int_0^\infty B(\xi) J_0(\xi r/a)\, d\xi = 0; \quad a < r < \infty \tag{15}$$

where

$$\begin{aligned}
&B(\xi) = \xi^2 A_2(\xi); \quad F(\xi) = 1 - \frac{\psi}{\xi} \\
&p(r) = p_0\left[\frac{\zeta_1 a^2 h_1}{(r^2 + h_1^2)^{3/2}} - \frac{\zeta_2 a^2 h_2}{(r^2 + h_2^2)^{3/2}}\right] \\
&\psi = \frac{\tilde{E}_f a\sqrt{v_1 v_2}\,(k_1 - k_2)}{E_m \ell \Omega^*}; \quad \mu^* = \frac{c_{44}\Omega}{2a^4 v_2\sqrt{v_1}\,(1 + k_1)} \\
&\Omega = \sqrt{v_1}\,(1 + k_1)\left\{\frac{k_2 c_{33} - v_2 c_{13}}{c_{44}}\right\} - \sqrt{v_2}\,(1 + k_2)\left\{\frac{k_1 c_{33} - v_1 c_{13}}{c_{44}}\right\} \\
&\Omega^* = \frac{\Omega c_{44}}{E_m}; \quad p_0 = \frac{P}{\pi a^2}
\end{aligned} \tag{16}$$

and $E_m$ is the modulus of elasticity of the matrix material. This system of dual integral equations can be transformed to a single Fredholm integral equation of the second kind by employing the transformation

$$B(\xi) = \frac{p_0}{\pi\mu^*}\int_0^1 \Phi^*(t)\sin(\xi t)\, dt \tag{17}$$

Consequently, we obtain the Fredholm integral equation of the second kind

$$\Phi^*(t) - \frac{\psi}{\pi}\int_0^1 K(t,\tau)\Phi^*(\tau)\, d\tau = \frac{v_1\zeta_1 t}{(v_1 t^2 + \eta^2)} - \frac{v_2\zeta_2 t}{(v_2 t^2 + \eta^2)} = g(t) \tag{18}$$

where the kernel function $K(t, \tau)$ is given by

$$K(t,\tau) = 2\int_0^\infty \xi^{-1}\sin(\xi t)\sin(\xi\tau)\, d\xi = \ln\left\{\frac{t + \tau}{t - \tau}\right\} \tag{19}$$

and $\eta = h/a$.

The mathematical analysis of the elastically bridged penny shaped flaw loaded by symmetrically placed body forces is now reduced to the solution of the integral equation (18). The solution of (18) gives, formally, all results concerning the distribution of displacements and stresses within the elastic medium and in the bridged flaw region. The ensuing discussions however, will be restricted to the examination of the influence of flaw bridging on the stress intensity factor for the penny shaped flaw.

## 5. The stress intensity factor

A result of primary interest to linear elastic fracture mechanics of the fibre reinforced composite concerns the distribution of stress in the vicinity of the boundary of the bridged flaw region. This state of stress is characterized by the stress intensity factor $K_{\mathrm{I}}$ (for the flaw opening mode) defined by

$$K_{\mathrm{I}} = \lim_{r \to a^+} [2(r-a)]^{\frac{1}{2}} \sigma_{zz}(r,0) \tag{20}$$

By employing the results derived in the previous section it can be shown that

$$[K_{\mathrm{I}}]_{\text{bridged flaw}} = \frac{P}{\pi^2 a^{3/2}} \{2\Phi^*(1)\} \tag{21}$$

In the limiting case when the elasticity of the bridging fibres $E_f$ reduces to zero, $\psi = 0$, and (18) gives the result

$$[K_{\mathrm{I}}]_{\text{unbridged flaw}} = \frac{P}{\pi^2 a^{3/2}} \frac{(c_{13}+c_{44})\nu_1\nu_2}{c_{33}c_{44}(\nu_1-\nu_2)} \times \times \left[\frac{(k_1c_{33}-\nu_1c_{13})}{(\nu_1+\eta^2)} - \frac{(k_2c_{33}-\nu_2c_{13})}{(\nu_2+\eta^2)}\right] \tag{22}$$

In the limit material isotropy $\nu_1, \nu_2 \to 1$ and

$$c_{11} = c_{33} = \lambda + 2\mu; \quad c_{13} = \lambda; \quad c_{44} = \mu \tag{23}$$

where $\lambda$ and $\mu$ are the classical Lamé constants. Thus for the isotropic case of the body force loading of a penny shaped flaw

$$[K_{\mathrm{I}}]^{\text{isotropic}}_{\text{unbridged flaw}} = \frac{P}{\pi^2 a^{3/2}} \left[\frac{(1-\nu)+\eta^2(2-\nu)}{(1-\nu)(1+\eta^2)^2}\right] \tag{24}$$

This is in agreement with the result given by Kassir and Sih [22] and Barenblatt [31]. Also as $\eta \to 0$, the results (22) and (24) both yield the same result

$$\underset{\eta=0}{[K_{\mathrm{I}}]^{\text{isotropic}}_{\text{unbridged flaw}}} = \underset{\eta=0}{[K_{\mathrm{I}}]^{\text{transversely isotropic}}_{\text{unbridged flaw}}} = \frac{P}{\pi^2 a^{3/2}} \tag{25}$$

In the limiting case where the fibre reinforced composite is composed of inextensible fibres, the factor $\psi \to \infty$ and the integral equation (18) reduces to

$$\int_0^1 \ln\left\{\frac{t+\tau}{t-\tau}\right\} \Phi^*(\tau)\,\mathrm{d}\tau = 0 \tag{26}$$

This integral equation has a trivial solution $\Phi^*(t) = 0$. Consequently

$$[K_{\mathrm{I}}]^{\text{inextensible fibres}}_{\text{bridged flaw}} \to 0 \tag{27}$$

When examining the stress analysis of fibre reinforced materials the limit of fibre inextensibility has to be approached with caution. The possibility of the occurrence of boundary layers or stress channelling at the flaw boundary cannot be excluded (see e.g. Morland [32], Spencer [33] and Pipkin [34]).

## 6. Numerical results

For arbitrary fibre elasticity, $K_I$ can be evaluated by a numerical solution of the integral equation (18) [To the author's knowledge there appears to be no formal closed form solution of this integral equation]. The various techniques that can be adopted in the approximate solution of the Fredholm integral equation (18) are well summarized in the texts by Atkinson [35] and Baker [36]. In this paper we adopt a discretization method which involves the application of Gaussian quadrature formulae to approximate the integrals. Briefly, the integral equation (18) can be reduced to a form

$$\Phi^*(t) - \frac{\psi}{\pi}\sum_{j=1}^{n} \omega_j K(t,\tau_j)\Phi^*(\tau_j) = g(t) \tag{28}$$

where $0 \leqslant t \leqslant 1$, and $\omega_j$ are the Gaussian weights. The functional equation (28) can be solved by setting $t = \tau_j (j = 1,\ldots,n)$; The resulting system of equations can be written in the matrix form

$$\mathbf{\Phi}^* = \mathbf{Q}^{-1}\mathbf{G}; \qquad \mathbf{Q} = \left[\mathbf{I} - \frac{\psi}{\pi}\boldsymbol{\omega}\mathbf{K}\right] \tag{29}$$

where $\mathbf{I}$ is the identity matrix; $\boldsymbol{\omega}$ is the vector of Gaussian weights, etc. The accuracy of the approximate method can be established by evaluating the condition number $E = \|\mathbf{Q}\|\,\|\mathbf{Q}^{-1}\|$ or by comparison with certain known exact solutions. The numerical scheme adopted here uses a 20 point Gaussian quadrature which yields satisfactory results, for known exact solutions, to within an accuracy of 1 percent [37]. The elastic constants $c_{ij}$ of the transversely isotropic elastic idealization are obtained by using the composite cylinder assemblage model proposed by Hashin and Rosen [1] and Rosen [27]. This model incorporates randomness of structure and permits the derivation of simple closed form expressions for the elastic moduli. The final results are summarized in Appendix 1. For the purpose of the numerical evaluation

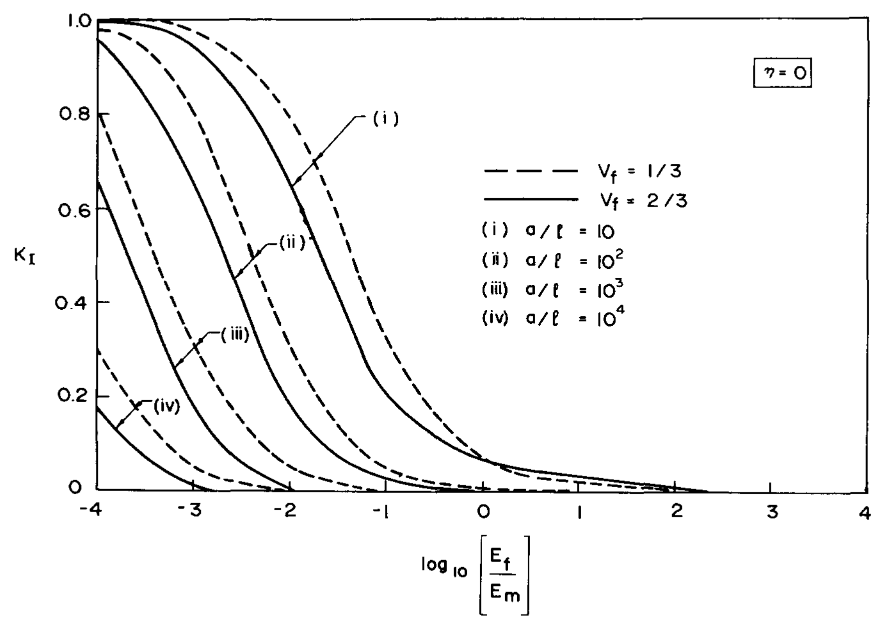


Figure 3. The normalized stress intensity factor $\bar{K}_I$ for the bridged flaw.

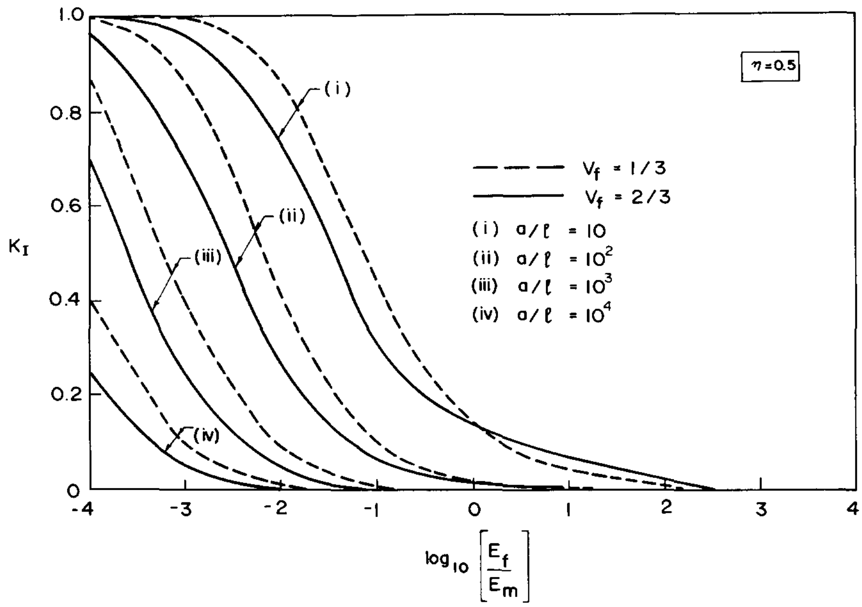


Figure 4. The normalized stress intensity factor $\bar{K}_{\rm I}$ for the bridged flaw.

the elastic properties of the matrix are taken as follows: $E_m = 30\,\mathrm{GN/m^2}$, $\nu_m = 0.35$. The modular ratio $M(=E_f/E_m)$ is assigned values which vary from $10^{-4}$ to $10^4$. Poisson's ratio for the fibre material is taken as $\nu_f = 0.20$. The fibre volume fractions are $V_f = 0.33, 0.67$. As discussed previously, the value of the flaw aspect ratio is open to conjecture; in the absence of specific information the value of $a/\ell$ is assumed to vary between 10 to $10^4$. For conciseness we introduce a normalized stress intensity factor $\bar{K}_{\rm I}$ defined by

$$\bar{K}_{\rm I} = K_{\rm I}\frac{\pi^2 a^{3/2}}{P} \tag{30}$$

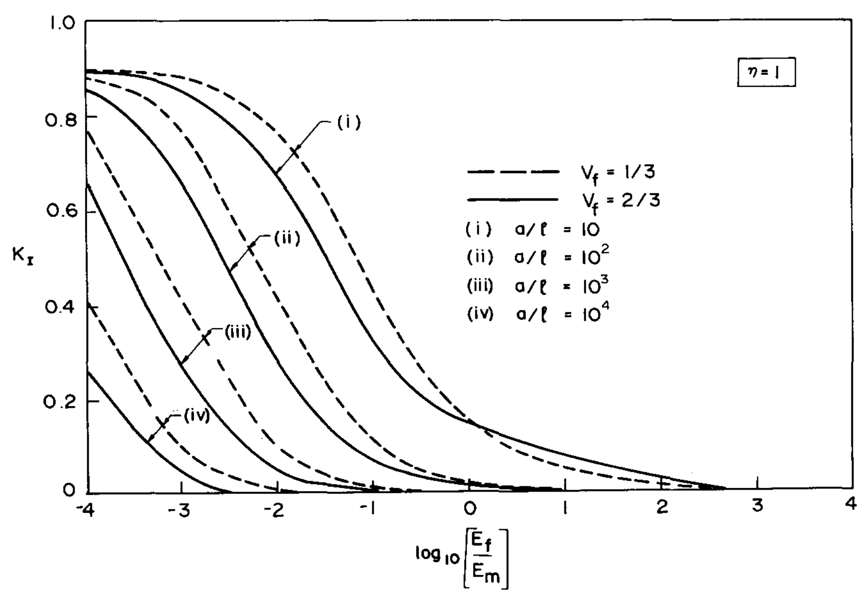


Figure 5. The normalized stress intensity factor $\bar{K}_{\rm I}$ for the bridged flaw.

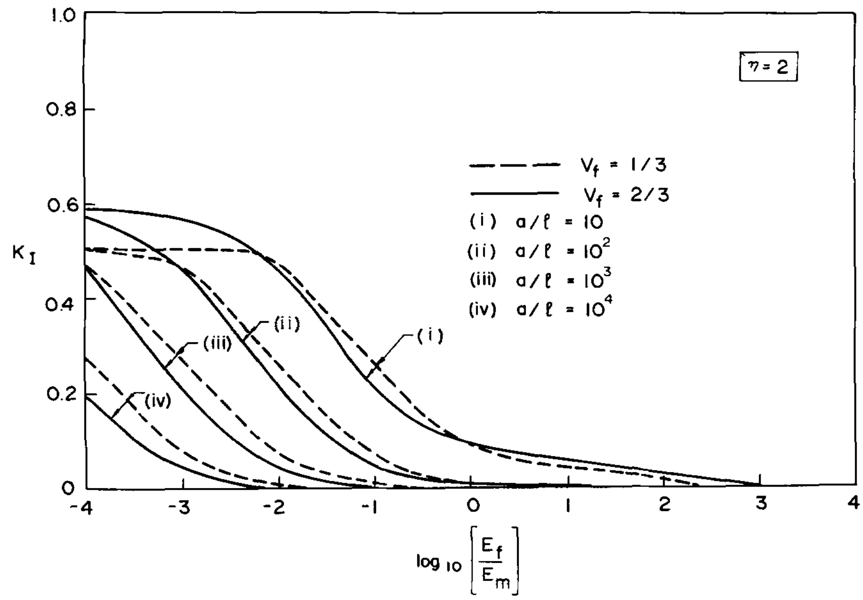


Figure 6. The normalized stress intensity factor $\bar{K}_I$ for the bridged flaw.

The Figs. 3–6 illustrate the manner in which the various elasticity parameters, flaw geometry parameters and the location of the symmetric body forces influence the stress intensity factor of a bridged penny shaped flaw. These results exhibit trends that have been identified previously by limiting considerations. The elasticity of the bridging fibres has a very pronounced influence in the suppression of the stress intensity at the flaw boundary.

## 7. Conclusions

This paper presents a relatively simplified theoretical model for the process of flaw bridging which can occur in unidirectional fibre reinforced material. The bridging action provides a displacement dependent traction condition in the flaw region. The mathematical analysis of the bridged flaw problem indicates that the relative stiffness of the bridging fibres has a considerable influence on the stress intensity factor for the crack opening mode induced by a pair of symmetrically placed body forces. It should be noted that the bridging action of the slender fibres would be effective only when the body forces induce tensile stresses in the penny shaped flaw region. The overall influence of this bridging action on the fracture toughness of the composite merits further study. The model proposed here can be further extended to cover flaw regions with a spheroidal shape geometry, which could adequately account for finite dimensions of the bridging region.

## Appendix 1

The constants $c_{ij}$ can be expressed in terms of the elastic constants $E_1$, $\nu_1^*$, $G_1$, $G_{23}$ and $K_{23}$

(where the subscript 1 refers to the fibre direction and the subscripts 2 and 3 refer to the transversely isotropic plane) in the form

$$c_{11} = K_{23} + G_{23}$$
$$c_{33} = E_1 + 4(\nu_1^*)^2 K_{23}$$
$$c_{13} = 2\nu_1^* K_{23}$$
$$c_{12} = K_{23} - G_{23}$$
$$c_{44} = G_1$$

The relationships between $E_1$, $\nu_1^*$, $G_1$ etc., and the properties of the fibre ($f$) and matrix ($m$) constituents ($E_f$, $E_m$, $\nu_f$, $\nu_m$) and the respective volume fractions ($V_f$, $V_m$) are given below. The expression for $G_{23}$ is equivalent to the upper bound for the assemblage, as found by Hashin and Rosen [1].

$$K_{23} = \left\{\frac{\zeta(1 + 2\nu_m V_f) + 2\nu_m V_m}{\zeta V_m + V_f + 2\nu_m}\right\}(\lambda_m + G_m)$$

$$G_{23} = G_m\left\{\frac{(\alpha + \beta_m V_f)(1 + \rho V_f^3) - 3V_f V_m^2 \beta_m^2}{(\alpha - V_f)(1 + \rho V_f^3) - 3V_f V_m^2 \beta_m^2}\right\}$$

$$\nu_1^* = \left\{\frac{V_f E_f L_1 + V_m E_m L_2 \nu_m}{V_f E_f L_3 + V_m E_m L_2}\right\}$$

$$G_1 = G_m\left\{\frac{\eta(1 + V_f) + V_m}{\eta V_m + V_f + 1}\right\}; \quad E_1 = V_f E_f + V_m E_m$$

where

$$L_1 = 2\nu_f(1 - \nu_m^2)V_f + V_m \nu_m(1 + \nu_m); \quad L_2 = 2V_f(1 - \nu_f^2)$$
$$L_3 = 2(1 - \nu_m^2)V_f + (1 + \nu_m)V_m$$
$$\zeta = \frac{\lambda_f + G_f}{\lambda_m + G_f}; \quad \alpha = \frac{\eta + \beta_m}{\eta - 1}; \quad \rho = \frac{\beta_m - \eta\beta_f}{1 + \eta\beta_f}$$
$$\eta = \frac{G_f}{G_m}; \quad V_m + V_f = 1;$$
$$G_i = \frac{E_i}{2(1 + \nu_i)}; \quad \lambda_i = \frac{\nu_i E_i}{(1 + \nu_i)(1 - 2\nu_i)}; \quad \beta_i = (3 - 4\nu_i)^{-1}, \quad (i = m, f)$$

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## RÉSUMÉ

Le mémoire examine un problème relatif aux concentrations de contrainte qui sont rencontrées dans un défaut en forme d'ongle noyé dans un matériau renforcé de fibres unidirectionelles. En particulier, on suppose que les fibres élastiques constituant le renforcement sont distribuées de manière arbitraire et traversent les faces du défaut sans solution de continuité. On analyse le processus de pontage du défaut en représentant la fibre intacte de matériau de renforcement comme un matériau élastique transversalement isotrope. L'analyse mathématique de la distribution des forces dans le problème de la fissure en ongle noyée et pontée peut être réduite à la solution d'une intégrale de Fredholm du second ordre. La solution de cette équation intégrale fournit directement le facteur d'intensité de contrainte pour le défaut ponté de manière élastique. Le résultat numérique produit dans le mémoire illustre la manière selon laquelle la position des forces agissant sur le corps, la géométrie du défaut, le rapport modulaire fibre/matrice, etc.... influencent le facteur d'intensité de contrainte d'un défaut en ongle ponté.