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THE STATICAL REISSNER-SAGOCI PROBLEM FOR AN INTERNALLY LOADED TRANSVERSELY ISOTROPIC HALFSPACE

A.P.S. Selvadurai

Department of Civil Engineering
Carleton University
Ottawa, Ontario, Canada

ABSTRACT

The present paper examines the elastostatic problem related to the axisymmetric rotation of a rigid circular punch which is bonded to the surface of a transversely isotropic elastic halfspace region. The rotation of the punch is induced by a concentrated couple which acts at a finite distance from the punch along the axis of symmetry. The resulting rigid rotation of the punch is evaluated in exact closed form.

INTRODUCTION

The elastostatic problem related to the determination of the components of stress and displacement in the interior of a halfspace when a circular area of the boundary is forced to rotate about an axis which is normal to the undeformed plane surface is usually referred to as the Reissner-Sagoci problem. Reissner and Sagoci [1] employed a method of analysis which utilizes a system of oblate spheroidal coordinates to investigate the state of stress and displacement in the halfspace region. The same problem was re-examined by Sneddon [2,3] and Rostovtsev [4] who utilized a Hankel transform-based dual integral equation formulation. The classical Reissner-Sagoci problem has been extended by several investigators to accomodate a variety of effects. Uflyand [5] and Collins [6] investigated the torsional indentation problem related to a homogeneous isotropic layer. Sneddon et al. [7] examined the torsional indentation problem related to an infinitely long cylinder. Similarly Dhaliwal et al. [8] examined the torsional indentation of a bi-material cylindrical composite halfspace region. The torsional indentation of a halfspace and a two layer system by an embedded rigid cylinder was examined by Freeman and Keer [9] and Luco [10] respectively.

The extension of the Reissner-Sagoci problem to include effects of non-homogeneity is of some interest to geomechanics. The torsional indentation of a non-homogeneous halfspace region, which exhibits exponential or power law variations in the shear modulus, is examined by Protsenko [11,12], Kassir [13], Kolybikhin [14], Singh [15] and Chuapresert and Kassir [16]. Similar problems related to the non-homogeneous elastic layer are examined by Protsenko [17], Dhaliwal and Singh [18,19] and Hassan [20]. Selvadurai [21] has examined the Reissner-Sagoci problem related to a finitely deformed incompressible elastic halfspace, which serves as a useful procedure for the determination of the mechanical properties of rubber-like elastic materials. Inverse problems which result from the torsional indentation of the non-homogeneous halfspace region are examined by Gladwell and Coen [22]. Finally, an informative account of the torsional indentation problem is given by Gladwell [23].

In this paper we examine the Reissner-Sagoci problem related to a transversely isotropic elastic halfspace region in which the rotation of the bonded circular punch is induced by a concentrated couple which is located along the axis of symmetry. A Hankel transform formulation of the governing equations result in a system of dual integral equations which can be solved by employing the procedure outlined by Sneddon [3]. The rotation of the bonded rigid punch due to the internally applied concentrated couple is evaluated in exact closed form.

BASIC EQUATIONS

We shall consider a cylindrical polar coordinate system (r,θ,z) (Fig. 1) where the z-axis is parallel to the material axis of symmetry of the transversely isotropic elastic medium. Furthermore, attention is restricted to the class of rotationally symmetric deformations which are characterized by the non-zero displacement component $u_\theta(r,z)$. It can be shown [24] that in the absence of body forces, the displacement and stress fields associated with this deformation can be expressed in terms of a single stress function $\Psi(r,z)$

which satisfies the equation

$$\{\nabla_1^2 + \frac{\partial^2}{\partial \tilde{z}^2}\} \quad \Psi(r,z) = 0 \tag{1}$$

where

$$\nabla_1^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \quad ; \quad \tilde{z} = \frac{z}{\sqrt{\nu_3}} \quad ; \quad \nu_3 = \frac{2c_{44}}{c_{11}-c_{12}} \tag{2}$$

and c_{ij} are the elastic constants of the transversely isotropic elastic solid. The displacement u_θ and the non-zero stress components $\sigma_{\theta z}$ and $\sigma_{r\theta}$ can be expressed in terms of $\Psi(r,z)$ in the forms

$$u_\theta(r,z) = -\frac{\partial \Psi}{\partial r} \tag{3}$$

$$\sigma_{\theta z}(r,z) = -c_{44}\frac{\partial^2 \Psi}{\partial r \partial z} \quad ; \quad \sigma_{r\theta} = \frac{(c_{11}-c_{12})}{2}\{\frac{2}{r}\frac{\partial \Psi}{\partial r} + \nu_3\frac{\partial^2 \Psi}{\partial z^2}\} \tag{4}$$

Concentrated couple in a halfspace region

Prior to the examination of the title problem it is necessary to develop the solution to the problem in which a concentrated couple acts at a finite distance (z=c) from the surface of the halfspace region. As a step towards the development of the above solution we examine the action of a concentrated couple (about the z-axis) at the origin of coordinates of a transversely isotropic infinite space region. Avoiding details of calculation it can be shown that the displacement and stress fields associated with the action of a concentrated couple (of magnitude M) in the transversely isotropic infinite space can be derived from the stress function

$$\Psi(r,z) = \frac{M\sqrt{\nu_3}}{8\pi c_{44}\{r^2+\tilde{z}^2\}^{1/2}} \tag{5}$$

The associated displacement and stress components are

$$u_\theta(r,z) = \frac{M\sqrt{\nu_3}\; r}{8\pi c_{44}\{r^2+\tilde{z}^2\}^{3/2}} \tag{6}$$

$$\{\sigma_{\theta z}:\sigma_{r\theta}\} = \frac{3M}{8\pi\sqrt{\nu_3}\{r^2+\tilde{z}^2\}^{5/2}} \quad \{-3r\tilde{z}\sqrt{\nu_3}:-3r^2\} \tag{7}$$

We consider a symmetrically placed combination of concentrated couples as shown in Figure 2. By virtue of the symmetry of the state of deformation induced by this doublet it is evident that the plane $z = 0$ of the infinite space is subjected to zero surface traction and hence it corresponds to a free surface. The rotational displacement at the surface of the halfspace region due to the application of a couple at $z = c$ is thus given by

$$u_\theta(r,0) = \frac{M\sqrt{\nu_3}\, r}{4\pi c_{44}(r^2+\tilde{c}^2)^{3/2}} = u_\theta^0(r) \tag{8}$$

where $\tilde{c} = c/\sqrt{\nu_3}$.

<u>The interaction problem</u>

We now consider the problem of the interaction between the bonded rigid circular punch and the concentrated couple which acts at a finite distance from the surface of the halfspace region. The mixed boundary conditions associated with the torsional indentation problem are

$$u_\theta(r,0) = \Omega r - u_\theta^0(r) \quad ; \quad 0 \le r \le a \tag{9}$$

$$\sigma_{\theta z}(r,0) = 0 \quad ; \quad a < r < \infty \tag{10}$$

where Ω is the rigid rotation of the bonded punch. Considering a zeroth-order Hankel transform solution of the governing differential equation it can be shown that the particular form of the stress function $\Psi(r,z)$ appropriate for the halfspace region $z \ge 0$ is

$$\Psi(r,z) = \frac{1}{a^2}\int_0^\infty \xi\, A(\xi) e^{-\eta z} J_0(\xi r/a)\, d\xi \tag{11}$$

where $A(\xi)$ is an arbitrary function and $\eta = \xi/a\sqrt{\nu_3}$. By making use of (3), (4) and (11) the boundary conditions (9) and (10) yield the pair of dual integral equations

$$H_0\ \{\xi^{-1}R(\xi)\ :\ r\} = f(r) \quad ; \quad 0 \le r \le a \tag{12}$$

$$H_0\ \{\ R(\xi)\ :\ r\} = 0 \quad ; \quad a \le r \le \infty \tag{13}$$

where $R(\xi) = \xi^2 A(\xi)$. The zero order Hankel operator H_0 and the function $f(r)$ are given by

$$H_0\{X(\xi)\ :\ r\} = \int_0^\infty \xi\ X(\xi)\ J_0(\xi r/a)\ d\xi \tag{14}$$

$$f(r) = \Omega a^3 r - \frac{M\sqrt{\nu_3}\ a^3 r}{4\pi c_{44}\{r^2+\breve{c}^2\}^{3/2}} \tag{15}$$

The general solution of the dual systems (12)-(13) is given by Sneddon [3] and the details of the method will not be pursued here. It is sufficient to note that by introducing a transformation of the type $aR(\xi) = \int_0^a g(t)\ \sin(\xi t/a)dt$, the equation (13) is identically satisfied. Also (12) is reduced to an integral equation of the Abel type.

The solution of this integral equation yields

$$R(\xi) = \frac{4a^2}{\pi}\int_0^a [\Omega t - \frac{M\sqrt{\nu_3}\ \breve{c}t}{4\pi c_{44}(t^2+\breve{c}^2)^2}]\ \sin(\xi t/a)dt \tag{16}$$

Formal expressions for the displacement and stress components in the transversely isotropic elastic halfspace can now be evaluated in integral form. The result of primary interest to the present paper concerns the rotation of the rigid circular punch due to the application of the internal concentrated couple. To develop this result we evaluate the resultant torque (T) acting on the bonded rigid circular punch due to the action of the stresses $\sigma_{\theta z}$. Since the resultant torque is zero we require

$$T = \frac{2\pi c_{44}}{a^4\sqrt{\nu_3}}\int_0^a r^2\ \{\int_0^\infty \xi^3 A(\xi)\ J_1(\xi r/a)\ d\xi\}\ dr = 0 \tag{17}$$

Evaluating (17) we obtain

$$\Omega = \frac{3M\sqrt{\nu_3}}{8\pi c_{44}a^3}\left[\tan^{-1}\left(\sqrt{\nu_3}\frac{a}{c}\right) - \frac{\sqrt{\nu_3}\,ac}{(\nu_3 a^2+c^2)}\right] \tag{18}$$

CONCLUSIONS

The torque-twist relationship (18) extends the classical Reissner-Sagoci statical problem to include effects of transverse isotropy and internally located concentrated couples.

In the limit as $c \to 0$, (18) gives the result for the rotation experienced by a punch bonded to a transversely isotropic elastic halfspace region due to the direct action of a couple M : i.e.

$$[\Omega(0)]_{\text{Trans. Isotr.}} = \frac{3M}{16\pi c_{44}a^3}\left(\frac{2c_{44}}{c_{11}-c_{12}}\right)^{1/2} \tag{19}$$

Also in the limit of material isotropy $c_{11} = \lambda + 2\mu$; $c_{12} = \lambda$; $c_{44} = \mu$ (where λ and μ are Lame's constants) and (18) gives

$$[\Omega(c)]_{\text{Isotr.}} = \frac{3M}{8\pi\mu a^3}\left[\tan^{-1}\left(\frac{a}{c}\right) - \frac{ac}{(a^2+c^2)}\right] \tag{20}$$

We may further note that as $c \to \infty$, $\Omega(\infty) \to 0$ and when $c = 0$, (20) reduces to the classical Reissner-Sagoci result $3M = 16\pi\mu a^3\Omega$.

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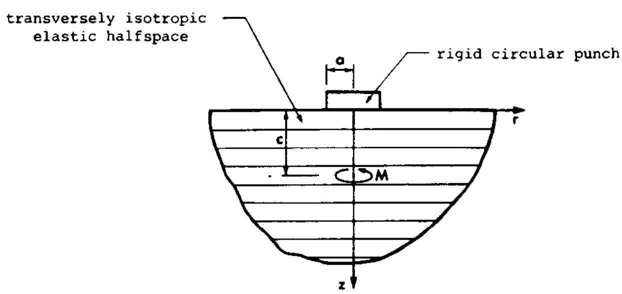

Fig. 1. The Reissner-Sagoci problem for an internally loaded halfspace.

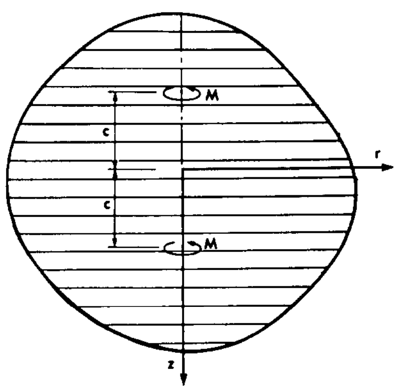

Fig. 2. Internal loading of the transversely isotropic infinite space.