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PLANE STRAIN PROBLEMS IN SECOND-ORDER ELASTICITY THEORY

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Abstract—The equations of second-order elasticity are developed in polar coordinates R, Θ for plane strain deformations of incompressible isotropic elastic materials. By considering a 'displacement function' the second-order problem is reduced to the solution of an equation of the form $\nabla^4\Psi = g(R, \Theta)$ where ∇^2 is Laplace's differential operator and $g(R, \Theta)$ depends only on the first-order solution. The displacement function technique is then applied to obtain a second-order solution to the problem of an elastic body contained between two concentric rigid circular boundaries, when the outer boundary is held fixed and the inner is subjected to a rigid body translation.

1. INTRODUCTION

THE APPROXIMATE solution of problems in finite elasticity by the method of successive approximations is well established. A comprehensive exposition of this method is given in Green and Adkins [1], Truesdell and Noll [2] and in a recent article by Spencer [3]. The fundamental assumption of this method is that the displacement and stress fields can be expanded as power series in terms of a small non-dimensional parameter ϵ. The basic non-linear equations of finite elasticity can then be reduced to sets of linear equations from which the successive terms in the power series expansions can be determined.

The successive approximation procedure has been applied to several problems of plane finite elasticity by Adkins, Green and Shield [4], Adkins, Green and Nicholas [5], Carlson and Shield [6] and others who make efficient use of the complex variable method. An alternative formulation of the plane problem employs the Airy stress function approach used in classical elasticity.

This paper deals with the application of a "displacement function" approach to plane strain problems in second order elasticity, for an incompressible elastic material. In the context of an incompressible elastic material the use of a displacement function in contrast to an Airy stress function has received only limited attention. The formulation of the linear elastic problem in terms of the displacement function closely resembles the formulation of plane flows of a Newtonian viscous fluid in terms of Stokes' stream function. The use of a displacement function is particularly suitable for problems where displacement boundary conditions are specified but it can also be applied to problems where traction boundary conditions are prescribed.

As an illustration of this method we consider the problem of a cylindrical elastic annulus contained between two rigid boundaries, when the outer boundary is held fixed and the inner boundary is subjected to a rigid body translation. The displacement function method for incompressible elastic materials has been used by Selvadurai [7, 8] to obtain second-order solutions to wedge, dislocation, infinite plane and torsion problems, and by Selvadurai and Spencer [9] to axially symmetric three dimensional problems.

2. FINITE PLANE STRAIN—PLANE POLAR COORDINATES

The position of a generic particle in the undeformed configuration is referred to a set of rectangular Cartesian coordinates $X_A (A = 1, 2, 3)$. After a deformation, the current position of the same particle is given by $x_i(X_A)$ $(i = 1, 2, 3)$. The cylindrical polar coordinates, (R, Θ, Z) of particles in the undeformed body and (r, θ, z) of particles in the deformed body are given by

$$\begin{aligned} X_1 &= R\cos\Theta, \qquad X_2 = R\sin\Theta, \qquad X_3 = Z, \\ x_1 &= r\cos\theta, \qquad x_2 = r\sin\theta, \qquad x_3 = z. \end{aligned} \tag{2.1}$$

For plane strain deformations

$$r = r(R, \Theta), \qquad \theta = \theta(R, \Theta), \qquad z = Z. \tag{2.2}$$

The matrix $\boldsymbol{F}$ of deformation gradients in the R, Θ directions is

$$\boldsymbol{F} = \begin{bmatrix} \dfrac{\partial r}{\partial R} & \dfrac{\partial r}{R\partial\Theta} \\ \dfrac{r\partial\theta}{\partial R} & \dfrac{r}{R}\dfrac{\partial\theta}{\partial\Theta} \end{bmatrix}, \tag{2.3}$$

We restrict our attention to incompressible materials for which

$$\det \boldsymbol{F} = \frac{r}{R}\left(\frac{\partial r}{\partial R}\frac{\partial\theta}{\partial\Theta} - \frac{\partial r}{\partial\Theta}\frac{\partial\theta}{\partial R}\right) = 1. \tag{2.4}$$

The Cauchy–Green strain matrix referred to the (r, θ) coordinates is

$$\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T = \begin{bmatrix} \left(\dfrac{\partial r}{\partial R}\right)^2 + \left(\dfrac{1}{R}\dfrac{\partial r}{\partial\Theta}\right)^2 & r\dfrac{\partial\theta}{\partial R}\dfrac{\partial r}{\partial R} + \dfrac{r}{R^2}\dfrac{\partial r}{\partial\Theta}\dfrac{\partial\theta}{\partial\Theta} \\ r\dfrac{\partial\theta}{\partial R}\dfrac{\partial r}{\partial R} + \dfrac{r}{R^2}\dfrac{\partial r}{\partial\Theta}\dfrac{\partial\theta}{\partial\Theta} & r^2\left(\dfrac{\partial\theta}{\partial R}\right)^2 + \dfrac{r^2}{R^2}\left(\dfrac{\partial\theta}{\partial\Theta}\right)^2 \end{bmatrix}. \tag{2.5}$$

The matrix $\boldsymbol{S}$ of the physical components of the symmetrical Cauchy stress tensor (given by resolving the force per unit area in the deformed body in the directions of the axes in the deformed body), referred to the current configuration is

$$\boldsymbol{S} = \begin{bmatrix} S_{rr} & S_{r\theta} \\ S_{r\theta} & S_{\theta\theta} \end{bmatrix}. \tag{2.6}$$

The constitutive equation for an incompressible isotropic elastic material expressed in plane polar coordinates can be stated in the form (e.g. Carlson and Shield [6])

$$\boldsymbol{S} = -p\boldsymbol{I} + \mu\boldsymbol{B}, \tag{2.7}$$

where p is an intermediate scalar pressure and μ is the linear elastic shear modulus.

In the absence of body forces, the equations of equilibrium in the deformed body are

$$\frac{\partial S_{rr}}{\partial r} + \frac{1}{r}\frac{\partial S_{r\theta}}{\partial \theta} + \frac{S_{rr} - S_{\theta\theta}}{r} = 0,$$
$$\frac{\partial S_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial S_{\theta\theta}}{\partial \theta} + \frac{2S_{r\theta}}{r} = 0. \tag{2.8}$$

The components of surface traction in the r and θ directions on a surface $f(r, \theta) = 0$ in the deformed body, are

$$F_r = n_r S_{rr} + n_\theta S_{r\theta}, \qquad F_\theta = n_r S_{r\theta} + n_\theta S_{\theta\theta}, \tag{2.9}$$

where

$$n_r^2 + n_\theta^2 = 1, \qquad \frac{n_r}{n_\theta} = \frac{\partial f/\partial r}{\partial f/r\partial \theta}. \tag{2.10}$$

3. SECOND-ORDER ELASTICITY

We now assume that the polar coordinates (r, θ) of a particle in the current configuration, whose equivalent coordinates in the undeformed configuration are R, Θ, can be expressed as a power series in terms of a small dimensionless parameter ϵ, as follows

$$r = R + \sum_{n=1}^{\infty} \epsilon^n r_n(R, \Theta), \qquad \theta = \Theta + \sum_{n=1}^{\infty} \epsilon^n \theta_n(R, \Theta). \tag{3.1}$$

We also assume that the displacement components u and v in the R and Θ directions can be expressed as power series in the form

$$u = \sum_{n=1}^{\infty} \epsilon^n u_n(R, \Theta), \qquad v = \sum_{n=1}^{\infty} \epsilon^n v_n(R, \Theta). \tag{3.2}$$

By considering the deformation of a generic particle it can then be shown that, to the second order in ϵ, (3.1) can be expressed as

$$r = R + \epsilon u_1 + \epsilon^2\left(u_2 - \frac{v_1^2}{2R}\right),$$
$$\theta = \Theta + \epsilon\frac{v_1}{R} + \epsilon^2\left(\frac{v_2}{R} - \frac{u_1 v_1}{R^2}\right). \tag{3.3}$$

By substituting (3.3) into (2.4) we obtain the first-order incompressibility condition as

$$\frac{\partial u_1}{\partial R} + \frac{u_1}{R} + \frac{1}{R}\frac{\partial v_1}{\partial \Theta} = 0, \tag{3.4}$$

and the second-order incompressibility condition as

$$\frac{\partial u_2}{\partial R} + \frac{u_2}{R} + \frac{1}{R}\frac{\partial v_2}{\partial \Theta} = H(R, \Theta), \tag{3.5}$$

where

$$H(R,\Theta)=\left(\frac{\partial u_1}{\partial R}\right)^2+\frac{\partial v_1}{\partial R}\left(\frac{1}{R}\frac{\partial u_1}{\partial\Theta}-\frac{v_1}{R}\right). \tag{3.6}$$

Also to the second-order in ϵ, the Cauchy–Green strain matrix $\boldsymbol{B}$ can be expressed in the form

$$\boldsymbol{B}=\boldsymbol{I}+\epsilon\boldsymbol{B}^{(1)}+\epsilon^2\boldsymbol{B}^{(2)}, \tag{3.7}$$

where the components of $\boldsymbol{B}^{(1)}$ and $\boldsymbol{B}^{(2)}$ are

$$\begin{aligned}
&B_{11}^{(1)}=2\frac{\partial u_1}{\partial R}; \qquad B_{22}^{(1)}=\frac{2}{R}\frac{\partial v_1}{\partial\Theta}+2\frac{u_1}{R}; \qquad B_{12}^{(1)}=\frac{1}{R}\frac{\partial u_1}{\partial\Theta}+\frac{\partial v_1}{\partial R}-\frac{v_1}{R},\\
&B_{11}^{(2)}=2\frac{\partial u_2}{\partial R}+B_{11}^{*}; \qquad B_{22}^{(2)}=\frac{2}{R}\frac{\partial v_2}{\partial\Theta}+2\frac{u_2}{R}+B_{22}^{*},\\
&B_{12}^{(2)}=\frac{1}{R}\frac{\partial u_2}{\partial\Theta}+\frac{\partial v_2}{\partial R}-\frac{v_2}{R}+B_{12}^{*},
\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}
B_{11}^{*}&=\left(\frac{\partial u_1}{\partial R}\right)^2+\left(\frac{1}{R}\frac{\partial u_1}{\partial\Theta}\right)^2+2\frac{v_1}{R}\frac{\partial v_1}{\partial R}-\frac{v_1^2}{R^2},\\
B_{22}^{*}&=\left(\frac{1}{R}\frac{\partial v_1}{\partial\Theta}+\frac{u_1}{R}\right)^2+\left(\frac{\partial v_1}{\partial R}-\frac{v_1}{R}\right)^2-2\frac{v_1}{R^2}\frac{\partial u_1}{\partial\Theta}+\frac{v_1^2}{R^2},\\
B_{12}^{*}&=\frac{u_1v_1}{R^2}-\frac{v_1}{R}\frac{\partial u_1}{\partial R}+\frac{\partial u_1}{\partial R}\left(\frac{\partial v_1}{\partial R}-\frac{v_1}{R}\right)+\frac{v_1}{R^2}\frac{\partial v_1}{\partial\Theta}+\frac{1}{R}\frac{\partial u_1}{\partial\Theta}\left(\frac{1}{R}\frac{\partial v_1}{\partial\Theta}+\frac{u_1}{R}\right).
\end{aligned}$$

If we further assume that the Cauchy stress matrix $\boldsymbol{S}$ and the scalar pressure p can be expanded as power series in the forms

$$\boldsymbol{S}=\sum_{n=0}^{\infty}\epsilon^n\boldsymbol{S}^{(n)}, \qquad p=\sum_{n=0}^{\infty}\epsilon^n p_{(n)}, \tag{3.9}$$

then from (2.7), we obtain the first and second-order constitutive equations as

$$\boldsymbol{S}^{(1)}=-p_{(1)}\boldsymbol{I}+\mu\boldsymbol{B}^{(1)}, \qquad \boldsymbol{S}^{(2)}=-p_{(2)}\boldsymbol{I}+\mu\boldsymbol{B}^{(2)}. \tag{3.10}$$

The components of $\boldsymbol{S}^{(1)}$ are

$$\begin{aligned}
&S_{rr}^{(1)}=-p_{(1)}+\mu\left(2\frac{\partial u_1}{\partial R}\right); \qquad S_{\theta\theta}^{(1)}=-p_{(1)}+\mu\left(\frac{2}{R}\frac{\partial v_1}{\partial\Theta}+2\frac{u_1}{R}\right),\\
&S_{r\theta}^{(1)}=\mu\left(\frac{1}{R}\frac{\partial u_1}{\partial\Theta}+\frac{\partial v_1}{\partial R}-\frac{v_1}{R}\right),
\end{aligned} \tag{3.11}$$

and for conciseness the components of $\boldsymbol{S}^{(2)}$ can be written in the form

$$\begin{aligned}
S_{rr}^{(2)}&=-p_{(2)}+\mu\left(2\frac{\partial u_2}{\partial R}\right)+T_{rr},\\
S_{\theta\theta}^{(2)}&=-p_{(2)}+\mu\left(\frac{2}{R}\frac{\partial v_2}{\partial\Theta}+\frac{2u_2}{R}\right)+T_{\theta\theta},
\end{aligned} \tag{3.12}$$

$$S_{r\theta}^{(2)} = \mu\left(\frac{1}{R}\frac{\partial u_2}{\partial \Theta} + \frac{\partial v_2}{\partial R} - \frac{v_2}{R}\right) + T_{r\theta},$$

where

$$T_{rr} = \mu B_{11}^{*}; \qquad T_{\theta\theta} = \mu B_{22}^{*}; \qquad T_{r\theta} = \mu B_{12}^{*}. \tag{3.13}$$

By expressing the differential operators $\partial/\partial r$ and $\partial/\partial\theta$ in terms of power series expansions of ϵ, the equations of equilibrium in the deformed configuration (2.8) can be reduced to

$$\begin{aligned}
&\frac{\partial S_{rr}^{(1)}}{\partial R} + \frac{1}{R}\frac{\partial S_{r\theta}^{(1)}}{\partial \Theta} + \frac{S_{rr}^{(1)} - S_{\theta\theta}^{(1)}}{R} = 0,\\
&\frac{\partial S_{r\theta}^{(1)}}{\partial R} + \frac{1}{R}\frac{\partial S_{\theta\theta}^{(1)}}{\partial \Theta} + 2\frac{S_{r\theta}^{(1)}}{R} = 0,
\end{aligned} \tag{3.14}$$

in the first order, and in the second order to

$$\begin{aligned}
&\frac{\partial S_{rr}^{(2)}}{\partial R} + \frac{1}{R}\frac{\partial S_{r\theta}^{(2)}}{\partial \Theta} + \frac{S_{rr}^{(2)} - S_{\theta\theta}^{(2)}}{R} = N_1(R, \Theta),\\
&\frac{\partial S_{r\theta}^{(2)}}{\partial R} + \frac{1}{R}\frac{\partial S_{\theta\theta}^{(2)}}{\partial \Theta} + 2\frac{S_{r\theta}^{(2)}}{R} = N_2(R, \Theta),
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
N_1(R,\Theta) &= \frac{\partial u_1}{\partial R}\frac{\partial S_{rr}^{(1)}}{\partial R} + \frac{1}{R}\left(\frac{\partial v_1}{\partial R} - \frac{v_1}{R}\right)\frac{\partial S_{rr}^{(1)}}{\partial \Theta} - \frac{1}{R}\frac{\partial u_1}{\partial R}\frac{\partial S_{r\theta}^{(1)}}{\partial \Theta} + \frac{1}{R}\frac{\partial u_1}{\partial \Theta}\frac{\partial S_{r\theta}^{(1)}}{\partial R} + \frac{u_1}{R}\left(\frac{S_{rr}^{(1)} - S_{\theta\theta}^{(1)}}{R}\right),\\
N_2(R,\Theta) &= -\frac{1}{R}\frac{\partial u_1}{\partial R}\frac{\partial S_{\theta\theta}^{(1)}}{\partial \Theta} + \frac{1}{R}\frac{\partial u_1}{\partial \Theta}\frac{\partial S_{\theta\theta}^{(1)}}{\partial R} + \frac{\partial u_1}{\partial R}\frac{\partial S_{r\theta}^{(1)}}{\partial R} + \frac{1}{R}\left(\frac{\partial v_1}{\partial R} - \frac{v_1}{R}\right)\frac{\partial S_{r\theta}^{(1)}}{\partial \Theta} + 2\frac{u_1}{R^2}S_{r\theta}^{(1)}.
\end{aligned} \tag{3.16}$$

4. DISPLACEMENT FUNCTIONS

We observe that the first-order incompressibility condition (3.4) can be identically satisfied by the introduction of a displacement function Ψ_1 such that

$$u_1 = \frac{1}{R}\frac{\partial \Psi_1}{\partial \Theta}, \qquad v_1 = -\frac{\partial \Psi_1}{\partial R}. \tag{4.1}$$

By making use of the first-order constitutive equations (3.11), the first-order equations of equilibrium (3.14) can then be expressed in the form

$$\begin{aligned}
&-\frac{\partial p_{(1)}}{\partial R} + \frac{\mu}{R}\frac{\partial}{\partial \Theta}[\nabla^2 \Psi_1] = 0,\\
&-\frac{1}{R}\frac{\partial p_{(1)}}{\partial \Theta} - \mu\frac{\partial}{\partial R}[\nabla^2 \Psi_1] = 0,
\end{aligned} \tag{4.2}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial \Theta^2}, \tag{4.3}$$

is the Laplacian in plane polar coordinates referred to the undeformed body.

By eliminating $p_{(1)}$ and Ψ_1 in turn from (4.2) we have

$$\nabla^4 \Psi_1 = 0, \tag{4.4}$$

and

$$\nabla^2 p_{(1)} = 0. \tag{4.5}$$

We adopt a similar procedure in the case of the second-order equations. It may be verified that the second-order incompressibility condition (3.5) is identically satisfied if we express the displacement components u_2 and v_2 in terms of a second-order displacement function Ψ_2 by the relations

$$\begin{aligned} u_2 &= \frac{1}{R}\frac{\partial \Psi_2}{\partial \Theta} + u_2', \\ v_2 &= -\frac{\partial \Psi_2}{\partial R} + v_2', \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} u_2' &= -\frac{1}{2}\frac{u_1^2}{R} - \frac{1}{2}\frac{v_1^2}{R} + \frac{v_1}{R}\frac{\partial u_1}{\partial \Theta}, \\ v_2' &= -v_1\frac{\partial u_1}{\partial R}. \end{aligned} \tag{4.7}$$

We note that it is possible to obtain alternative expressions for u_2' and v_2', but for the present purposes it is sufficient to use the values of u_2' and v_2' as defined by (4.7).

If we make use of (4.6) and the second-order constitutive equations (3.12), the second-order equations of equilibrium can be reduced to the form

$$\begin{aligned} -\frac{\partial p_{(2)}}{\partial R} + \frac{\mu}{R}\frac{\partial}{\partial \Theta}[\nabla^2 \Psi_2] &= F_1(R, \Theta), \\ -\frac{\partial p_{(2)}}{\partial \Theta} - \mu R\frac{\partial}{\partial R}[\nabla^2 \Psi_2] &= F_2(R, \Theta), \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} F_1(R, \Theta) &= -\left[\frac{\partial T_{rr}}{\partial R} + \frac{1}{R}\frac{\partial T_{r\theta}}{\partial \Theta} + \frac{T_{rr} - T_{\theta\theta}}{R} + \mu\left\{\frac{\partial H}{\partial R} + \nabla^2 u_2' - \frac{u_2'}{R^2} - \frac{2}{R^2}\frac{\partial v_2'}{\partial \Theta}\right\} - N_1(R, \Theta)\right], \\ \frac{F_2}{R}(R, \Theta) &= -\left[\frac{\partial T_{r\theta}}{\partial R} + \frac{1}{R}\frac{\partial T_{\theta\theta}}{\partial \Theta} + \frac{2T_{r\theta}}{R} + \mu\left\{\frac{1}{R}\frac{\partial H}{\partial \Theta} + \nabla^2 v_2' - \frac{v_2'}{R^2} + \frac{2}{R^2}\frac{\partial u_2'}{\partial \Theta}\right\} - N_2(R, \Theta)\right]. \end{aligned} \tag{4.9}$$

By eliminating $p_{(2)}$ and Ψ_2 in turn from (4.8) we obtain

$$\nabla^4 \Psi_2 = \frac{1}{\mu R}\left[\frac{\partial F_1}{\partial \Theta} - \frac{\partial F_2}{\partial R}\right], \tag{4.10}$$

$$\nabla^2 p_{(2)} = -\left[\frac{\partial F_1}{\partial R} + \frac{F_1}{R} + \frac{1}{R^2}\frac{\partial F_2}{\partial \Theta}\right]. \tag{4.11}$$

The procedure for determining a second-order solution is as follows: From a knowledge of the first-order solutions for the displacement and stress components, the right hand side of (4.10) can be found. A solution of (4.10) consists of a particular integral and solutions of the homogeneous equation $\nabla^4\Psi_2 = 0$. The correct homogeneous solution is that which satisfies the relevant boundary conditions of the problem. The particular solution together with the correct homogeneous solution then forms a complete solution of the second-order problem for an incompressible isotropic elastic material.

To the second-order in ε the displacement and stress components are given by

$$u = \varepsilon u_1 + \varepsilon^2 u_2; v = \varepsilon v_1 + \varepsilon^2 v_2; S = \varepsilon S^{(1)} + \varepsilon^2 S^{(2)}. \tag{4.12}$$

5. A CYLINDRICAL ANNULUS PROBLEM

A rigid cylinder of radius $a(a > 0)$ lies inside and concentric with a rigid cylindrical region of radius b and the space between contains an incompressible elastic annulus which completely adheres to the rigid surfaces (Fig. 1). The outer boundary is held fixed and the inner boundary is subjected to a rigid body displacement δ in the X_1-direction. We assume perfect bonding at the elastic material rigid medium interfaces thereby prescribing displacement boundary conditions at these surfaces.

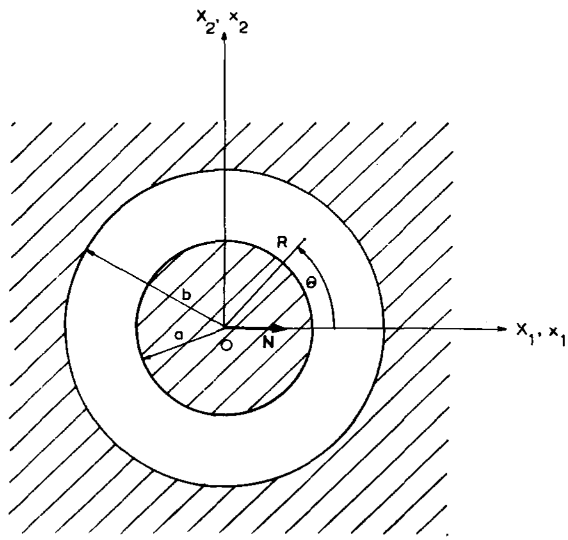

FIG. 1. Geometry of the cylindrical annulus

A first-order displacement function of the form

$$\varepsilon\Psi_1 = \frac{\delta}{a}\left[\alpha_1 R \ln R^* - \alpha_2 R^3 + \alpha_3 R + \frac{\alpha_4}{R}\right]\sin\Theta, \tag{5.1}$$

provides the first-order displacement components which are

$$\varepsilon u_1 = \frac{\delta}{a}\left[\alpha_1 \ln R^* - \alpha_2 R^2 + \alpha_3 + \frac{\alpha_4}{R^2}\right]\cos\Theta,$$

$$\varepsilon v_1 = \frac{\delta}{a}\left[-\alpha_1 \ln R^* + 3\alpha_2 R^2 + \alpha_5 + \frac{\alpha_4}{R^2}\right]\sin\Theta,$$

where

$$\alpha_1 = \frac{a^3}{\eta}\left(1 + \frac{b^2}{a^2}\right); \qquad \alpha_3 = \frac{a^3}{2\eta}\left(\frac{b^2}{a^2} - 1\right); \qquad \alpha_5 = -\frac{a^3}{2\eta}\left(3\frac{b^2}{a^2} + 1\right.$$

$$\alpha_2 = \frac{a}{2\eta}; \qquad \alpha_4 = \frac{a^3 b^2}{2\eta}; \qquad R^* = \frac{R}{b},$$

$$\eta = (a^2 + b^2)\ln\frac{a}{b} - a^2 + b^2,$$

and from the first-order incompressibility condition we note that $\alpha_1 + \alpha_3 +$
The first-order stress components are

$$\varepsilon\frac{S_{rr}^{(1)}}{\mu} = \frac{\delta}{a}\left[4\frac{\alpha_1}{R} + 4\alpha_2 R - 4\frac{\alpha_4}{R^3}\right]\cos\Theta,$$

$$\varepsilon\frac{S_{\theta\theta}^{(1)}}{\mu} = \frac{\delta}{a}\left[12\alpha_2 R + 4\frac{\alpha_4}{R^3}\right]\cos\Theta.$$

$$\varepsilon\frac{S_{r\theta}^{(1)}}{\mu} = \frac{\delta}{a}\left[4\alpha_2 R - 4\frac{\alpha_4}{R^3}\right]\sin\Theta.$$

The small real dimensionless parameter ε is chosen to be equal to δ/a.

We note that owing to the symmetry of the problem the inner cylinder e
resultant moment about the origin. On evaluating the resultant force on any c
in the elastic material it may be shown that the force (N) that has to be applied
of the inner cylinder in the X_1-direction to maintain the rigid body translatio

$$N = \frac{4\pi\mu\delta(a^2 + b^2)}{(a^2 + b^2)\ln(a/b) - a^2 + b^2}.$$

If we let the outer boundary $R = b$ recede to infinity we obtain the first-order s
problem of an infinite plane containing a bonded rigid circular insert loade
by a concentrated force. The corresponding first-order displacement and stres
can be obtained by setting

$$\alpha_1 = a; \qquad \alpha_4 = \frac{a^3}{2}; \qquad \alpha_5 = -a,$$

$$\alpha_2 = \alpha_3 = 0, \qquad R^* = \frac{R}{a} = R',$$

in equations (5.2) and (5.4).

If the expressions (5.2) and (5.4) for the first-order displacement and stres
are substituted in Equations (4.8), the inhomogeneous differential equation f
order displacement function, (4.10), reduces to

$$\nabla^4\Psi_2 = \left[\frac{16\alpha_1\alpha_2}{R^2} + \frac{1}{R^4}\left\{8\alpha_1^2\ln R^* - 2\alpha_1^2 + 4\alpha_1\alpha_3 - 4\alpha_1\alpha_5\right\} - 96\frac{\alpha_4^2}{R^8}\right]\sin 2($$

A particular integral of (5.7) is

$$\Psi_{2p} = \left[-\alpha_1\alpha_2 R^2 \ln R^* + \frac{\alpha_1^2}{4}(\ln R^*)^2 + \frac{1}{4}(\alpha_1\alpha_3 - \alpha_1\alpha_5)\ln R^* - \frac{1}{4}\frac{\alpha_4^2}{R^4}\right]\sin 2\Theta. \tag{5.8}$$

The displacement components derived from the particular solution (5.8) and the Equations (4.6) are

$$\begin{aligned} u_{2p} = {} & \left[\frac{\alpha_2^2}{2}R^3 + R(2\alpha_2\alpha_3 + \alpha_2\alpha_5 - \alpha_1\alpha_2 \ln R^*) + \right. \\ & \left. + \frac{1}{R}\{\tfrac{1}{2}\alpha_1^2 \ln R^* + 3\alpha_2\alpha_4 - \tfrac{1}{4}\alpha_3^2 + \tfrac{1}{4}\alpha_5^2 + \tfrac{1}{2}\alpha_3\alpha_5) + \frac{1}{R^3}(\alpha_4\alpha_5 - \alpha_1\alpha_4 \ln R^*)\right]\cos 2\Theta \\ & + \left[-\alpha_2^2 R^3 + \alpha_1\alpha_2 R + \frac{1}{R}(-2\alpha_2\alpha_4 - \tfrac{1}{2}\alpha_3\alpha_5 - \tfrac{1}{4}\alpha_3^2 - \tfrac{1}{4}\alpha_5^2) + \frac{\alpha_1\alpha_4}{R^3} - \frac{\alpha_4^2}{R^5}\right], \end{aligned} \tag{5.9}$$

$$\begin{aligned} v_{2p} = {} & \left[3\alpha_2^2 R^3 + R(-\tfrac{1}{2}\alpha_1\alpha_2 + \alpha_2\alpha_5 + \alpha_1\alpha_2 \ln R^*) + \right. \\ & \left. + \frac{1}{R}(\tfrac{1}{4}\alpha_1^2 + 4\alpha_2\alpha_4) + \frac{1}{R^3}(-\tfrac{1}{2}\alpha_1\alpha_4 + \alpha_4\alpha_5 - \alpha_1\alpha_4 \ln R^*)\right]\sin 2\Theta. \end{aligned}$$

By considering the kinematics of deformation of the inner and outer elastic material—rigid medium interfaces it can be shown that the second-order displacement components must satisfy the boundary conditions

$$u_2(R, \Theta) = v_2(R, \Theta) = 0, \tag{5.10}$$

on $R = a$ and $R = b$.

On applying the boundary conditions (5.10) to the displacement components (5.9) we obtain

$$\begin{aligned} u_{2p}(a, \Theta) &= \rho_1 \cos 2\Theta, & v_{2p}(a, \Theta) &= \rho_3 \sin 2\Theta, \\ u_{2p}(b, \Theta) &= \rho_2 \cos 2\Theta, & v_{2p}(b, \Theta) &= \rho_4 \sin 2\Theta, \end{aligned} \tag{5.11}$$

where the constants $\rho_1, \rho_2, \ldots, \rho_4$ are implied from (5.9).

To satisfy the boundary conditions (5.10), we therefore require four solutions of the homogeneous equation $\nabla^4\Psi_2 = 0$. These solutions are

$$\Psi_{2H} = \left(\beta_1 R^2 + \frac{\beta_2}{R^2} + \beta_3 + \beta_4 R^4\right)\sin 2\Theta, \tag{5.12}$$

where $\beta_1, \beta_2, \ldots, \beta_4$ are arbitrary constants.

By making use of the boundary conditions (5.10), and (5.11) we obtain

$$\beta_1 = \frac{\rho_1}{\lambda^3}(a^3b^2 + a^5 + ab^4) + \frac{\rho_2}{\lambda^3}(-a^4b - a^2b^3 - b^5) + \frac{\rho_3}{2\lambda^2}(a^3 + 2ab^2) + \frac{\rho_4}{2\lambda^2}(b^3 + 2a^2b),$$

$$\beta_2 = \frac{\rho_1}{\lambda^3}a^5b^4 - \frac{\rho_2}{\lambda^3}a^4b^5 + \frac{\rho_3}{2\lambda^2}a^3b^4 + \frac{\rho_4}{2\lambda^2}a^4b^3,$$

$$\beta_3 = \frac{\rho_1}{2\lambda^3}(-4a^5b^2 - a^3b^4 - ab^6) + \frac{\rho_2}{2\lambda^3}(a^6b + a^4b^3 + 4a^2b^5)$$
$$+ \frac{\rho_3}{2\lambda^2}(-ab^4 - 2a^2b^2) + \frac{\rho_4}{2\lambda^2}(-a^4b - 2a^2b^2),$$
$$\beta_4 = -\frac{a\varphi\rho_1}{2\lambda^3} + \frac{b\varphi\rho_2}{2\lambda^3} - \frac{a\rho_3}{2\lambda^2} - \frac{b\rho_4}{2\lambda^2}, \tag{5.13}$$

where

$$\left.\begin{matrix}\varphi\\ \lambda\end{matrix}\right\} = (a^2 \pm b^2).$$

The final expressions for the second-order displacement and stress components are

$$u_2 = \Big[R^3\{\tfrac{1}{2}\alpha_2^2 + 2\beta_4\} + R(2\alpha_2\alpha_3 + \alpha_2\alpha_5 - \alpha_1\alpha_2 \ln R^* + 2\beta_1)$$
$$+ \frac{1}{R}(\tfrac{1}{2}\alpha_1^2 \ln R^* + 3\alpha_2\alpha_4 + \tfrac{1}{2}\alpha_3\alpha_5 - \tfrac{1}{4}\alpha_3^2 + \tfrac{1}{4}\alpha_5^2 + 2\beta_3)$$
$$+ \frac{1}{R^3}(\alpha_4\alpha_5 - \alpha_1\alpha_4 \ln R^* + 2\beta_2)\Big]\cos 2\Theta$$
$$+ \left[-\alpha_2^2R^3 + \alpha_1\alpha_2R + \frac{1}{R}(-2\alpha_2\alpha_4 - \tfrac{1}{4}\alpha_1^2) + \frac{\alpha_1\alpha_4}{R^3} - \frac{\alpha_4^2}{R^5}\right], \tag{5.14}$$

$$v_2 = \Big[R^3(3\alpha_2^2 - 4\beta_4) + R(-\tfrac{1}{2}\alpha_1\alpha_2 + \alpha_2\alpha_5 + \alpha_1\alpha_2 \ln R^* - 2\beta_1)$$
$$+ \frac{1}{R}(\tfrac{1}{4}\alpha_1^2 + 4\alpha_2\alpha_4) + \frac{1}{R^3}(-\tfrac{1}{2}\alpha_1\alpha_4 + \alpha_4\alpha_5 - \alpha_1\alpha_4 \ln R^* + 2\beta_2)\Big]\sin 2\Theta,$$

and

$$\frac{S_{rr}^{(2)}}{\mu} = \Big[4\alpha_2^2R^2 + 4\alpha_2\alpha_3 - 2\alpha_2\alpha_5 + 2\alpha_1\alpha_2 \ln R^* + 4\beta_1$$
$$+ \frac{1}{R^2}(-2\alpha_1^2 \ln R^* - 4\alpha_2\alpha_4 - 2\alpha_3\alpha_5 + \tfrac{1}{2}\alpha_1^2 - \alpha\beta_5 + \alpha_3^2 - 8\beta_3)$$
$$+ \frac{1}{R^4}(2\alpha_1\alpha_4 \ln R^* - 2\alpha_4\alpha_5 - 12\beta_2) + 4\frac{\alpha_4^2}{R^6}\Big]\cos 2\Theta$$
$$+ \Big[-2\alpha_2^2R^2 - 2\alpha_1\alpha_2 + 4\alpha_2\alpha_5 + \frac{1}{R^2}(4\alpha_2\alpha_4 + \tfrac{3}{2}\alpha_1^2)$$
$$+ \frac{1}{R^4}(-5\alpha_1\alpha_4 + 4\alpha_3\alpha_4 + 4\alpha_1\alpha_4 \ln R^*) + 6\frac{\alpha_4^2}{R^6}\Big],$$

$$\frac{S^{(2)}_{\theta\theta}}{\mu} = \left[R^2(22\alpha_2^2 - 24\beta_4) + 6\alpha_2\alpha_5 - 2\alpha_1\alpha_2 \ln R^* - 4\beta_1\right.$$

$$\left. + \frac{1}{R^2}(24\alpha_2\alpha_4 + \tfrac{1}{2}\alpha_1^2) + \frac{1}{R^4}(4\alpha_3\alpha_4 + 6\alpha_4\alpha_5 - 2\alpha_1\alpha_4 \ln R^* + 12\beta_2) - 4\frac{\alpha_4^2}{R^6}\right]\cos 2\Theta$$

$$+ \left[-6\alpha_2^2 R^2 + 2\alpha_1\alpha_2 + 4\alpha_2\alpha_5 + 8\alpha_2\alpha_3 + 8\alpha_1\alpha_2 \ln R^*\right.$$

$$\left. + \frac{1}{R^2}(-4\alpha_2\alpha_4 + \tfrac{1}{2}\alpha_1^2) + \frac{1}{R^4}(3\alpha_1\alpha_4 + 4\alpha_4\alpha_5 - 4\alpha_1\alpha_4 \ln R^*) + 2\frac{\alpha_4^2}{R^6}\right], \tag{5.15}$$

$$\frac{S^{(2)}_{r\theta}}{\mu} = \left[R^2(9\alpha_2^2 - 12\beta_4) + \alpha_1\alpha_2 + 2\alpha_2\alpha_5 - 4\alpha_2\alpha_3 - 2\alpha_1\alpha_2 \ln R^* - 4\beta_1\right.$$

$$+ \frac{1}{R^2}(\alpha_1^2 \ln R^* - 6\alpha_2\alpha_4 + \alpha_1\alpha_3 - \tfrac{1}{2}\alpha_1^2 + \tfrac{1}{2}\alpha_3^2 + \tfrac{1}{2}\alpha_5^2 - 4\beta_3)$$

$$\left. + \frac{1}{R^4}(\alpha_1\alpha_4 - 2\alpha_4\alpha_5 + 2\alpha_1\alpha_4 \ln R^* - 12\beta_2) + 4\frac{\alpha_4^2}{R^6}\right]\sin 2\Theta,$$

where $\alpha_1, \alpha_2, \ldots, \alpha_5, \beta_1, \beta_2, \ldots, \beta_4$ are defined by (5.3), (5.11) and (5.13). A relation between the applied force N and the rigid body translation δ for the second-order problem can be obtained by considering the resultant force in the X_1-direction transmitted across any contour $r = r_0$ in the deformed body. This force resultant is

$$P_{x1} = \int_C (S_{rr} \cos\theta - S_{r\theta} \sin\theta)\, r_0\, \mathrm{d}\theta \tag{5.16}$$

where, on the contour C

$$r_0 = R + \epsilon u_1(r_0, \Theta), \qquad \theta = \Theta + \frac{\epsilon v_1(r_0, \Theta)}{r_0}, \tag{5.17}$$

to order ϵ.

Since the stress components $S^{(1)}_{rr}, S^{(2)}_{rr}, \ldots$, etc., are in terms of polar coordinates of the undeformed configuration, it is convenient to express (5.16) also in terms of R and Θ. The equation (5.16) can then be written as

$$P_{x1} = \epsilon P^{(1)}_{x1} + \epsilon^2 P^{(2)}_{x1}, \tag{5.18}$$

to order ϵ^2, where

$$P^{(1)}_{x1} = \int_{-\pi}^{\pi} (S^{(1)}_{rr} \cos\Theta - S^{(1)}_{r\theta} \sin\Theta)\, r_0\, \mathrm{d}\Theta, \tag{5.19}$$

and

$$P^{(2)}_{x1} = \int_{-\pi}^{\pi} \left[S^{(2)}_{rr} \cos\Theta - S^{(2)}_{r\theta} \sin\Theta - u_1\left(\frac{\partial S^{(1)}_{rr}}{\partial R}\cos\Theta - \frac{\partial S^{(1)}_{r\theta}}{\partial R}\sin\Theta\right)\right.$$

$$\left. - \frac{v_1}{R}(S^{(1)}_{rr}\sin\Theta + S^{(1)}_{r\theta}\cos\Theta) + \frac{1}{R}\frac{\partial v_1}{\partial\Theta}(S^{(1)}_{rr}\cos\Theta - S^{(1)}_{r\theta}\sin\Theta)\right] r_0\, \mathrm{d}\Theta. \tag{5.20}$$

On evaluating the expressions for $P_{x1}^{(1)}$ and $P_{x1}^{(2)}$ we have

$$\epsilon P_{x1}^{(1)} = \frac{\delta}{a}\left[\frac{4\pi\mu a(a^2+b^2)}{(a^2+b^2)\ln(a/b) - a^2 + b^2}\right], \qquad P_{x1}^{(2)} = 0. \tag{5.21}$$

It is of interest to note that there is no second-order contribution to P_{x1}. This confirms the fact that from symmetry considerations, for large deformations, N must be an odd function of δ. Therefore the applied force N, given by (5.5), is valid to order ϵ^3. The second-order solution to the problem of the infinite elastic plane loaded by a bonded rigid circular inclusion can now be obtained by using the values of $\alpha_1, \alpha_2 \ldots \alpha_5$ and R^* given by (5.6) in equations (5.8) to (5.15).

For this particular case, the second-order displacement and stress components (5.14) and (5.15) reduce to

$$\left.\begin{aligned} u_2 &= \left(-\frac{1}{2}\frac{a^4}{R^3} + \frac{1}{2}\frac{a^2}{R}\right)\ln R'\cos 2\Theta + \left(-\frac{1}{4}\frac{a^6}{R^5} + \frac{1}{2}\frac{a^4}{R^3} - \frac{1}{4}\frac{a^2}{R}\right), \\ v_2 &= \left(-\frac{1}{4}\frac{a^4}{R^3} + \frac{1}{4}\frac{a^2}{R} - \frac{1}{2}\frac{a^4}{R^3}\ln R'\right)\sin 2\Theta, \end{aligned}\right\} \tag{5.22}$$

$$\left.\begin{aligned} \frac{S_{rr}^{(2)}}{\mu} &= \left[\frac{a^6}{R^6} - 2\frac{a^4}{R^4} + \frac{1}{2}\frac{a^2}{R^2} + \left(\frac{a^4}{R^4} - 2\frac{a^2}{R^2}\right)\ln R'\right]\cos 2\Theta + \\ &\quad + \left(\frac{3}{2}\frac{a^6}{R^6} - \frac{5}{2}\frac{a^4}{R^4} + \frac{3}{2}\frac{a^2}{R^2} + 2\frac{a^4}{R^4}\ln R'\right), \\ \frac{S_{\theta\theta}^{(2)}}{\mu} &= \left(-\frac{a^6}{R^6} + \frac{1}{2}\frac{a^2}{R^2} - \frac{a^4}{R^4}\ln R'\right)\cos 2\Theta + \left(\frac{1}{2}\frac{a^6}{R^6} - \frac{1}{2}\frac{a^4}{R^4} + \frac{1}{2}\frac{a^2}{R^2} - 2\frac{a^4}{R^4}\ln R'\right), \\ \frac{S_{r\theta}^{(2)}}{\mu} &= \left[\frac{a^6}{R^6} - \frac{3}{2}\frac{a^4}{R^4} + \frac{1}{2}\frac{a^2}{R^2} + \left(\frac{a^4}{R^4} + \frac{a^2}{R^2}\right)\ln R'\right]\sin 2\Theta. \end{aligned}\right\} \tag{5.23}$$

To the second-order in ϵ, the applied force N (per unit length) is given by $4\pi\mu\delta$. These results are in agreement with the solution, given by Carlson and Shield [6], to the problem where a stationary rigid circular inclusion is subjected to a resultant force at infinity.

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Résumé—On développe les équations d'élasticité du second ordre dans les coordonnées polaires R, θ pour les déformations planes de matériaux élastiques isotropes incompressibles. En considérant une "fonction déplacement" le problème du second ordre est ramené à la résolution d'une équation de la forme $\nabla^4\Psi = g(R, \theta)$ où ∇^2 est le Laplacien et $g(R, \theta$ dépend seulement de la solution du premier ordre. On applique ensuite la technique de la fonction déplacement pour obtenir une solution du second ordre au problème d'un corps élastique contenu entre deux limites rigides circulaires concentriques lorsque la limite extérieure est maintenue fixe et que la limite interne est soumise à une translation rigide.

Zusammenfassung—Die Gleichungen der Elastizität zweiter Ordnung werden für Verformungen mit ebener Dehnung eines inkompressiblen isotropischen elastischen Stoffes in Polarkoordinaten R, θ hergeleitet. Unter Verwendung einer "Versetüngsfunktion" wird das Problem zweiter Ordnung auf die Lösung einer Gleichung der Form $\nabla^4\Psi = g(R, \theta)$ zurückgeführt, wobei ∇^2 den Laplaceschen Differentialoperator darstellt und $g(R, \theta)$ nur von der Lösung erster Ordnung abhängt. Das Verfahren wird dann angewendet, um die Lösung zweiter Ordnung für den Fall eines elastischen Körpers, der zwischen zwei konzentrischen starren kreisförmigen Begrenzungen eingeschlossen ist, zu bestimmen, wobei die äussere Begrenzung festgehalten wird während die innere Begrenzung eine starre Translation erfährt.

Аннотация—Для плоского деформированного состояния несжимаемого изоторопного упругого материала выведены в рамках теории упругости второго порядка уравнения в полярных координатах R, Θ. Путем введения «функции перемещений» задача второго порядка сводится к решению уравнения типа $\nabla^4\Psi = g(R, \Theta)$, где ∇^2 обозначает дифференциальный оператор Лапласа, а функция $g(R, \Theta)$ зависит лишь от решения первого порядка. Метод функции перемещений применяется затем для получения решения второго порядка в задаче об упругом теле, заключенном между двумя концентрическими жесткими цилиндрическими границами, причем внешний контур закреплен, а внутренний подвержен жесткому смещению.