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On the interaction between an elastically embedded rigid inhomogeneity and a laterally placed concentrated force

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1. Analysis

The problem related to an inclusion reinforced elastic medium is of some interest to the study of composite materials [1–7]. For this reason, several aspects pertaining to the elastostatics of embedded inhomogeneities have received considerable attention. The disc inclusion is a particular simplification of the more general class of three dimensional inhomogeneity. The axisymmetric problem related to a penny shaped inclusion embedded in bonded contact with an elastic medium was first examined by Collins [8]. The asymmetric problem concerning the lateral translation of the embedded circular disc inclusion was investigated by Keer [9]. Kassir and Sih [10] subsequently generalized these results to include the behaviour of embedded elliptical disc inclusions. Selvadurai [11–16] has also examined both axisymmetric and asymmetric problems related to rigid circular disc inclusions embedded in isotropic and transversely isotropic elastic media. Other classes of problems such as partially bonded penny shaped inclusions and flexible inclusions embedded in elastic media of infinite extent are discussed by Keer [17] and Selvadurai [18]. The displacement behaviour of a rigid circular inclusion embedded in an elastic halfspace is also discussed by Hunter and Gamblen [19] and Butterfield and Banerjee [20]. The solutions to these disc inclusion problems can also be recovered as limiting cases of results derived for ellipsoidal or spheroidal rigid inclusions [21, 22]. A variety of analytical techniques have been used in the examination of these inclusion problems. Where the inhomogeneity corresponds to a disc inclusion, methods such as complex potential function techniques [23] and dual integral equation methods [24, 25] have been used in the examination of the problem. Similarly, the spheroidal and ellipsoidal inclusion problems have been examined by using singularity methods and direct spheroidal and ellipsoidal harmonic function techniques. Other generalized formulations of problems related to disc inclusions are given by Gladwell and Selvadurai [26]. A recent review of the subject of in-

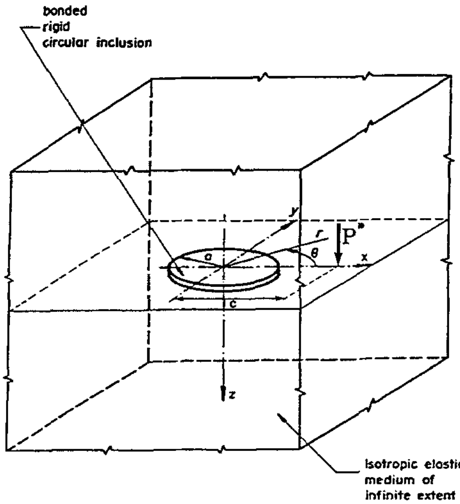

Fig. 1
Geometry of the embedded rigid inclusion and the external loading.

clusions and inhomogeneities embedded in elastic media is given by Mura [27]. It is of interest to note that for a majority of these disc inclusion problems, the rotational and translational stiffnesses can be evaluated in exact closed forms. Such results are of particular interest, for example, in technological problems related to bonded rubber mountings.

In a majority of the investigations related to the embedded disc inclusions it is usually assumed that the loads are applied directly to the rigid inclusion region in a symmetric or asymmetric fashion. In a recent paper Selvadurai [11] examined a different category of inclusion problem wherein the displacements of the rigid disc inclusion are caused by loads which act at a point exterior to the inclusion region. This investigation shows that in the particular case when an external concentrated force acts at a point along the axis of symmetry of the disc inclusion, the mathematical analysis of the inclusion problem can be reduced to the solution of a system of dual integral equations. Here again the resultant displacements of the rigid circular inclusion can be evaluated in exact closed form. In the present paper we examine the problem of the interaction between an elastically embedded disc inclusion and a concentrated force, which is located at an external point and which acts normal to the plane containing the rigid disc inclusion (Fig. 1). It is shown that the mathematical analysis of the problem can be formulated in relation to a mixed boundary value problem associated with a halfspace region. The results for the displacement and rotation of the embedded rigid disc inclusion can be evaluated in exact closed form. Furthermore, it is shown that these resultant displacements can also be evaluated by appeal to Betti's reciprocal theorem. The auxiliary solution required for the application of Betti's reciprocal theorem is obtained from an analysis of the directly loaded rigid disc inclusion.

2. The directly loaded inclusion

Prior to examining the externally loaded rigid disc inclusion it is instructive to record certain results which relate to the directly loaded inclusion. We consider the problem of a rigid circular disc inclusion (of radius a) which is embedded in bonded contact with the surrounding elastic medium. The rigid inclusion is acted upon by an eccentric load P which acts at the location $r = \zeta a$ and $\theta = 0$ (Fig. 2). The analysis of this problem is achieved by combining the separate solutions developed for the elastically embedded rigid circular inclusion which is subjected to a central force (P) and a moment ($M = P \zeta a$) about a horizontal axis. The analysis of these problems is given by several investigators ([8, 10, 11, 13, 21, 22]). We shalll record here only the salient results which relate to the dual integral equation formulation of the separate problems. A Hankel transform development of the separate problems yields two sets of dual integral equations of the type

$$\begin{aligned} H_0\{\xi^{-1} D_1(\xi); r\} &= -\frac{2 G \delta a^4}{(3-4\nu)}; \quad && 0 \leqq r \leqq a \\ H_0\{D_1(\xi); r\} &= 0; && a < r < \infty \end{aligned} \tag{1}$$

and

$$\begin{aligned} H_1\{\xi^{-1} D_2(\xi); r\} &= -\frac{2 G \Omega a^4 r}{(3-4\nu)}; \quad && 0 \leqq r \leqq a \\ H_1\{D_2(\xi); r\} &= 0; && a < r < \infty \end{aligned} \tag{2}$$

for the unknown functions $D_n(\xi)$, $(n = 1, 2)$. In (1) and (2) δ and Ω are, respectively, the rigid translation and rotation of the embedded disc inclusion; G is the linear elastic shear modulus; ν is Poisson's ratio and H_n is the nth order Hankel operator defined by

$$H_n\{g(\xi); r\} = \int_0^\infty \xi\, g(\xi)\, J_n(\xi r/a)\, d\xi . \tag{3}$$

The solution of these dual systems is given, among others, by Sneddon [24] and Selvadurai [13]. The result of primary interest to this paper concerns the

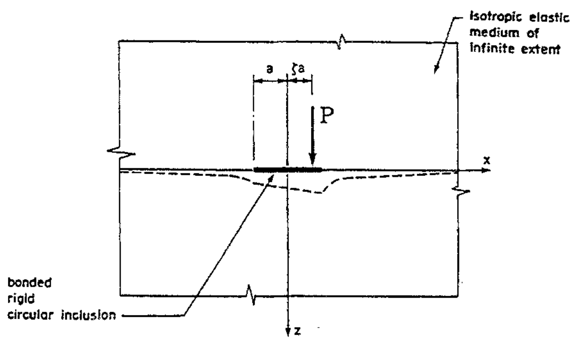

Fig. 2
Direct loading of the embedded circular inclusion.

distribution of displacement $u_z(r, \theta, 0)$ on the plane containing the disc inclusion. The displacements at the interior and exterior regions are given by

$$u_z(r, \theta, 0) \frac{P(3-4\nu)}{32\,G\,a(1-\nu)}\left[1+\frac{3\,\zeta\,\varrho}{2}\cos\theta\right]; \quad 0 \leqq \varrho \leqq 1 \tag{4}$$

$$u_z(r, \theta, 0) = \frac{\mathrm{P}(3-4\nu)}{32\,G\,a(1-\nu)}\left[\frac{2}{\pi}\sin^{-1}\left(\frac{1}{\varrho}\right)\right.$$

$$\left.+\frac{3\,\zeta\,\varrho}{2}\cos\theta\left\{1-\frac{2}{\pi}\tan^{-1}\sqrt{\varrho^2-1}-\frac{2}{\pi}\frac{\sqrt{\varrho^2-1}}{\varrho^2}\right\}\right]; \quad 1 \leqq \varrho \leqq \infty \tag{5}$$

where $\varrho = r/a$.

3. The interaction between the external load and the disc inclusion

Consider the problem of a rigid circular disc inclusion which is embedded in bonded contact with an elastic medium of infinite extent. The disc inclusion is located in the plane $z = 0$. The disc inclusion is perturbed by an external concentrated force P^* which acts at the location $(c, 0, 0)$ (see Fig. 1). As a consequence, the inclusion will experience a rigid central displacement δ^* and a rigid rotation Ω^*. To examine the embedded inclusion problem, we shall subject the inclusion to a "corrective" central force $\bar{P}$ and a "corrective" moment $\bar{M}$ (about the y-axis) such that the inclusion experiences zero axial or radial displacement in the bonded region (see Fig. 3). In this "corrected state", the displacements and stress components in the elastic medium exhibit a state of asymmetry about $z = 0\pm$ (the $+ve$ and $-ve$ sign refer to the regions of the elastic medium with the halfspace regions $z \geqq 0$ and $z \leqq 0$ respectively). In the ensuing development, we shall therefore restrict our attention to the examination of a single halfspace region $z \geqq 0$. In the "corrected state", the plane $z = 0+$ is thus subjected to the following boundary conditions: (i) in the bonded inclusion region

$$u_r(r, \theta, 0^+) = 0; \quad r \leqq a; \quad 0 \leqq \theta \leqq 2\pi \tag{6a}$$

$$u_\theta(r, \theta, 0^+) = 0; \quad r \leqq a; \quad 0 \leqq \theta \leqq 2\pi \tag{6b}$$

$$u_z(r, \theta, 0^+) = 0; \quad r \leqq a; \quad 0 \leqq \theta \leqq 2\pi \tag{6c}$$

and (ii) in the exterior region

$$u_r(r, \theta, 0^+) = 0; \quad a \leqq r \leqq \infty; \quad 0 \leqq \theta \leqq 2\pi \tag{7a}$$

$$u_\theta(r, \theta, 0^+) = 0; \quad a \leqq r \leqq \infty; \quad 0 \leqq \theta \leqq 2\pi \tag{7b}$$

$$\sigma_{zz}(r, \theta, 0^+) = -p(r, \theta); \quad a < r < \infty; \quad 0 \leqq \theta \leqq 2\pi \tag{7c}$$

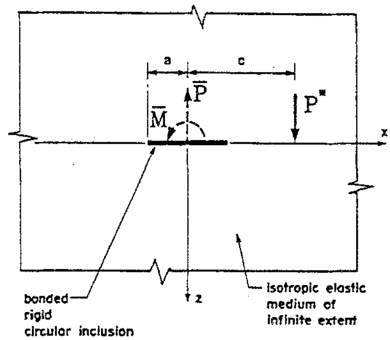

Fig. 3
External loading of the rigid disc inclusion and the associated corrective forces

where $p(r, \theta)$ is an even function of θ. By combining the sets of boundary conditions (6) and (7) it is evident that the plane containing the restrained inclusion is subjected to the mixed boundary conditions

$$u_r(r, \theta, 0^+) = 0; \quad 0 \leqq r \leqq \infty; \quad 0 \leqq \theta \leqq 2\pi \tag{8a}$$

$$u_\theta(r, \theta, 0^+) = 0; \quad 0 \leqq r \leqq \infty; \quad 0 \leqq \theta \leqq 2\pi \tag{8b}$$

$$u_z(r, \theta, 0^+) = 0; \quad 0 \leqq r \leqq a; \quad 0 \leqq \theta \leqq 2\pi \tag{8c}$$

$$\sigma_{zz}(r, \theta, 0^+) = -p(r, \theta); \quad a < r < \infty; \quad 0 \leqq \theta \leqq 2\pi \tag{8d}$$

To examine the mixed boundary value problem defined by (8) it is convenient to adopt the Fourier series-Hankel transform development of the equations of elastic equilibrium given by Muki [28]. The expressions for u_r, u_θ, u_z and σ_{zz} take the forms

$$u_r(r, \theta, z) = \frac{1}{2} \sum_{m=0}^{\infty} \left[\int_0^\infty \frac{\xi^2}{a} \frac{dF_m}{dz} \{J_{m+1}(\xi r/a) - J_{m-1}(\xi r/a)\} d\xi \right] \cos m\theta \tag{9}$$

$$u_\theta(r, \theta, z) = \frac{1}{2} \sum_{m=0}^{\infty} \left[\int_0^\infty \frac{\xi^2}{a} \frac{dF_m}{dz} \{J_{m+1}(\xi r/a) - J_{m-1}(\xi r/a)\} d\xi \right] \sin m\theta \tag{10}$$

$$u_z(r, \theta, z) = \sum_{m=0}^{\infty} \left[\int_0^\infty \left\{ (1-2\nu) \frac{d^2 F_m}{dz^2} - 2(1-\nu) \frac{\xi^2}{a^2} F_m \right\} \xi J_m(\xi r/a) \, d\xi \right] \cos m\theta \tag{11}$$

and

$$\sigma_{zz}(r, \theta, z) = 2G \sum_{m=0}^{\infty} \left[\int_0^\infty \left\{ (1-\nu) \frac{d^3 F_m}{dz^3} - (2-\nu) \frac{\xi^2}{a^2} \frac{dF_m}{dz} \right\} , \cdot \xi J_m(\xi r/a) \, d\xi \right] \cos m\theta \tag{12}$$

respectively. The function $F_m(z)$ appropriate for the region $z \geqq 0$ is

$$F_m(z) = a^3 \left[C_m(\xi) + D_m(\xi) \frac{z}{a} \right] e^{-\xi z/a} \tag{13}$$

and $C_m(\xi)$ and $D_m(\xi)$ are arbitrary functions. In order to satisfy the boundary conditions (8a) and (8b) we require $D_m(\xi) = \xi\, C_m(\xi)$. Using this result, the expression for u_z and σ_{zz}, evaluated on $z = 0$, can be written in the form

$$u_z(r, \theta, 0^+) = -\frac{(3-4\nu)\, a}{2(1-\nu)} \sum_{m=0}^{\infty} [H_m\{\xi^{-2}\, \Psi_m(\xi); r\}] \cos m\theta \tag{14}$$

$$\sigma_{zz}(r, \theta, 0^+) = 2\, G \sum_{m=0}^{\infty} [H_m\{\xi^{-1}\, \Psi_m(\xi); r\}] \cos m\theta \tag{15}$$

where $\Psi_m(\xi)\, [= 2(1-\nu)\, \xi^4\, C_m(\xi)]$ are arbitrary functions which are to be determined by satisfying the mixed boundary conditions (8c) and (8d). Assuming that $p(r, \theta)$ can be expressed in the form

$$p(r, \theta) = 2\, G \sum_{m=0}^{\infty} p_m(r) \cos m\theta \tag{16}$$

the mixed boundary conditions (8c) and (8d) can be reduced to the system of dual integral equations

$$\begin{aligned} &H_m\{\xi^{-2}\, \Psi_m(\xi); r\} = 0; \qquad 0 \leqq r \leqq a \\ &H_m\{\xi^{-1}\, \Psi_m(\xi); r\} = p_m(r); \quad a < r < \infty \end{aligned} \tag{17}$$

The solution of the system of dual integral equations (17) is given by several authors including Noble [29] and Sneddon and Lowengrub [30]. We shall record here only the results which are of immediate interest to the present paper. We note that for the concentrated external loading of the infinite space region

$$[p_0(r); p_m(r)] = \frac{P^*(1+\nu)\, [1:2]\, \Delta(r-c)}{8\, \pi^2\, E\, r} \tag{18}$$

where $\Delta(r-c)$ is the Dirac delta function, P^* is the magnitude of the external load and $(c, 0, 0)$ is the location of the load. The normal stress σ_{zz} in the bonded region $r \leqq a$ is given by

$$\sigma_{zz}(r, \theta, 0^+) = \frac{P^*}{\pi^2} \sum_{m=0}^{\infty} \left(\frac{r}{c}\right)^m \cos m\theta \int_a^{\infty} \frac{t\, H(c-t)\, \mathrm{d}t}{\sqrt{c^2 - t^2}\, (t^2 - r^2)^{3/2}} \tag{19}$$

where $H(c-t)$ is the Heaviside step function. The contact stresses can be reduced to the explicit form

$$\sigma_{zz}(r, \theta, 0) = \frac{P^* \sqrt{\eta^2 - 1}}{2\, \pi^2\, a^2 \sqrt{1-\varrho^2}\, \{\eta^2 + \varrho^2 - 2\, \varrho\, \eta \cos\theta\}}; \qquad 0 < \varrho < 1 \tag{20}$$

where $\varrho = r/a$ and $\eta = c/a$. The result (20) can now be used to evaluate the "corrective" force and moment resultants $\bar{P}$ and $\bar{M}$ required to maintain zero displacements in the bonded inclusion region. The force and moment resultants are obtained from the integrals

$$\bar{P}=\int_0^a\int_0^{2\pi}[\sigma_{zz}(r,\theta,0^+)-\sigma_{zz}(r,\theta,0^-)]\,r\,\mathrm{d}r\,\mathrm{d}\theta \tag{21}$$

$$\bar{M}=\int_0^a\int_0^{2\pi}[\sigma_{zz}(r,\theta,0^+)-\sigma_{zz}(r,\theta,0^-)]\,r^2\cos\theta\,\mathrm{d}r\,\mathrm{d}\theta \tag{22}$$

We note that owing to the asymmetry of the state of deformation in the infinite space

$$\sigma_{zz}(r,\theta,0^+)=-\sigma_{zz}(r,\theta,0^-) \tag{23}$$

By evaluating the above integrals we obtain

$$\bar{P}=\frac{2P^*}{\pi}\sin^{-1}\left(\frac{1}{\eta}\right) \tag{24}$$

$$\bar{M}=P^*a\,\eta\left\{1-\frac{2}{\pi}\tan^{-1}\sqrt{\eta^2-1}-\frac{2}{\pi}\,\frac{\sqrt{\eta^2-1}}{\eta^2}\right\} \tag{25}$$

By subjecting the rigid circular inclusion to force and moment resultants in the opposite sense we can render the inclusion free from external force (i.e. $\iint_{S_0}\sigma_{ij}n_j\,\mathrm{d}s=0$; where S_0 is any closed surface which includes the rigid circular inclusion but excludes the concentrated force P^*; and n_j are the direction cosines of the outward normal to S_0). By combining the results (4) and (5) derived for the directly loaded inclusion with (24) and (25) we can obtain the displacements of the inclusion induced solely by the externally placed load P^*. It can be shown that in the region $0\leqq\varrho\leqq 1$

$$u_z(r,\theta,0)=\frac{P^*(3-4\nu)}{32\,G\,a(1-\nu)}\left[\frac{2}{\pi}\sin^{-1}\left(\frac{1}{\eta}\right)+\frac{3\,\eta\,\varrho}{2}\cos\theta\cdot\left\{1-\frac{2}{\pi}\sqrt{\eta^2-1}-\frac{2}{\pi}\,\frac{\sqrt{\eta^2-1}}{\eta^2}\right\}\right] \tag{26}$$

It may be verified that as $\eta\to\infty$, $u_z(r,\theta,0)$ (defined by (26)) reduces to zero.

4. Reciprocal states

Consider the displacement $u_z^*(r,\theta,z)$ of the rigid circular inclusion at the location $(\zeta\,a,0,0)$ due to the externally placed lateral load P^* applied at $(\eta\,a,0,0)$. From the result (26) we have

$$u_z^*(\zeta\,a,0,0)=\frac{P^*(3-4\nu)}{32\,G\,a(1-\nu)}R(\zeta,\eta) \tag{27}$$

where

$$R(\zeta, \eta) = \frac{2}{\pi} \sin^{-1}\left(\frac{1}{\eta}\right) + \frac{3\,\zeta\,\eta}{2}\left\{1 - \frac{2}{\pi}\sqrt{\eta^2-1} - \frac{2}{\pi}\frac{\sqrt{\eta^2-1}}{\eta^2}\right\} \tag{28}$$

Similarly, consider the displacement $u_z(r, \theta, z)$ at the exterior point $(\eta\, a, 0, 0)$ due to a load P applied at $(\zeta\, a, 0, 0)$. From the result (5) we have

$$u_z(\eta\, a, 0, 0) = \frac{P(3-4\,\nu)}{32\,G\,a(1-\nu)}\,R(\zeta, \eta)\,. \tag{29}$$

By comparing (27) and (29) it is evident that the load-displacement relations associated with the externally loaded inclusion and the directly loaded inclusion satisfy Betti's reciprocal theorem

$$u_z\,P^* = u_z^*\,P\,. \tag{30}$$

The reciprocal relationship developed above can be generalized to include arabitrary locations for the points of application of the loads and for the positions at which the displacements are evaluated. This amounts to a straightforward change in the frame of reference. Consider a rigid circular inclusion embedded in bonded contact with an isotropic elastic infinite space and subjected to an internal concentrated force P at the location $A\,(\varrho\, a, \theta, 0)$. The displacement at the location $B\,(\lambda\, a, \varphi, 0)$ is denoted by $\bar{u}_z$ (Fig. 4). Similarly, the displacement of the rigid circular inclusion at $A\,(\varrho\, a, \theta, 0)$ due to the external load P^* applied at $B\,(\lambda\, a, \varphi, 0)$ is denoted by $\bar{u}_z^*$. Betti's recipro-

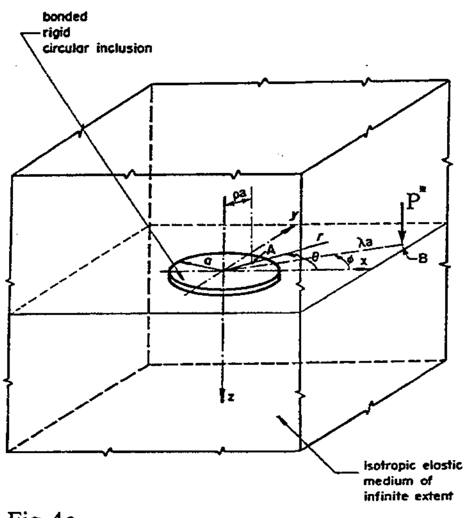

Fig. 4a
Reciprocal states for the embedded rigid circular inclusion.

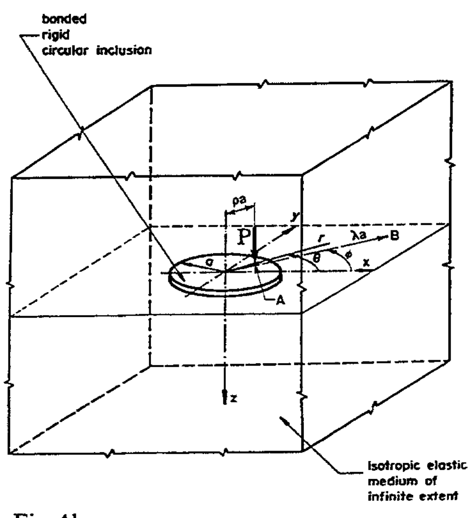

Fig. 4b
Reciprocal states for the embedded rigid circular inclusion.

cal relationship yields

$$\frac{\bar{u}_z}{P}=\frac{\bar{u}_z^*}{P^*}=\frac{(3-4\nu)}{32\,G\,a\,(1-\nu)}\left[\frac{2}{\pi}\sin^{-1}\left(\frac{1}{\lambda}\right)+\frac{3\,\lambda\,\varrho}{2}\cos(\theta-\varphi)\cdot\left\{1-\frac{2}{\pi}\tan^{-1}\sqrt{\lambda^2-1}-\frac{2}{\pi}\frac{\sqrt{\lambda^2-1}}{\lambda^2}\right\}\right]. \tag{31}$$

5. Conclusions

In the present paper we have examined the problem concerning the interaction between a rigid circular disc inclusion embedded in bonded contact with an elastic medium and a concentrated load which is located at an exterior point. The results for the displacement and rotation of the disc inclusion induced by the external load are evaluated in exact closed form. It is shown that these results can also be evaluated by appeal to Betti's reciprocal theorem. The auxiliary solution required for the application of the reciprocal theorem is derived from an analysis of the directly loaded rigid circular disc inclusion.

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Abstract

In this paper we examine the problem of the interaction between a rigid circular disc inclusion embedded in bonded contact with an isotropic elastic infinite space and a concentrated force which acts at an exterior point on the plane of the inclusion. The results for the displacement and rotation of the disc inclusion induced by the laterally placed load are evaluated in exact closed form. Furthermore, it is shown that these results may also be recovered by appeal to Betti's reciprocal theorem in which the auxiliary problem corresponds to the directly loaded inclusion.

Zusammenfassung

Es wird das folgende Problem untersucht: Eine starre runde Scheibe ist in festem Kontakt in einem unbegrenzten isotopen elastischen Raum eingeschlossen; eine konzentrierte Kraft greift an einem äußeren Punkt an in der Ebene der Scheibe. Die Verschiebung und die Drehung der Scheibe (für seitlich verschobene Belastung) werden in geschlossener Form angegeben. Weiterhin wird gezeigt, daß die Resultate auch mit Hilfe von Betti's Reziprozitäts-Theorem erhalten werden können, wobei das Hilfsproblem dem direkt belasteten Fall entspricht.

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