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THE AXISYMMETRIC DISPLACEMENT AND ROTATION OF AN ELASTICALLY EMBEDDED DISC INHOMOGENEITY DUE TO THE ACTION OF EXTERNAL LOADS

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SUMMARY

This paper considers the axisymmetric interaction between a rigid circular disc inclusion embedded in bonded contact with an isotropic elastic medium of infinite extent and a concentrated force and a moment which are located at a finite distance from the inclusion. Explicit results are obtained for the displacement and rotation of the disc inclusion.

1. INTRODUCTION

Problems relating to the behaviour of rigid disc inclusions embedded in bonded contact with elastic media have been examined by several investigators including Keer,[1] Kassir and Sih[2] and Selvadurai.[3] A majority of these investigations concentrate on the class of problem wherein the loads are applied directly to the rigid disc inclusion. In this paper we examine two problems in which the loads are located at a finite distance from the plane of the inclusion. In particular, the externally located loadings correspond to a force directed along, and a concentrated moment acting about, the axis of symmetry of the disc inclusion (Fig. 1(a) and Fig. 2(a)). The solution of these problems is facilitated by their reduction to two further sub-problems which reflect the state of symmetry or asymmetry about the plane $z = 0$ (Fig. 1(b) and (c) and Fig. 2(b) and (c)). Since we are primarily concerned with the evaluation of the resultant displacement and rotation of the disc inclusion, it is evident that only two of these sub-problems need be examined (Fig. 1(c) and Fig. 2(b)). Since the disc inclusion is assumed to be in bonded contact with the elastic medium, displacement boundary conditions are prescribed at the interface of the inclusion region $z = 0^{+-}$ and $r \leq a$. The suffixes $(\pm)$ refer to the faces of the inclusion in contact with the halfspace regions $z > 0$ and $z < 0$ respectively.

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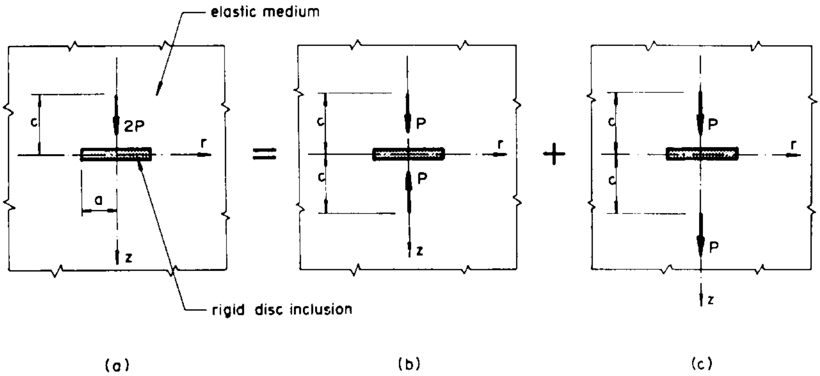

Fig. 1. Embedded disc inclusion subjected to an external force.

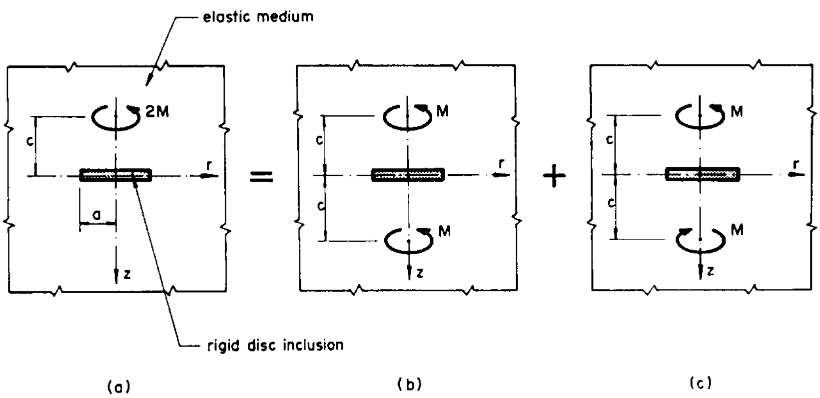

Fig. 2. Embedded disc inclusion subjected to an external couple.

By adopting an integral transform formulation, the analysis of the sub-problem can be reduced to the solution of two sets of dual integral equations related to an equivalent halfspace region. The solution of these dual integral equations is facilitated by the generalised results given by Sneddon.[4,5] Exact closed form expressions are derived for the resultant displacement and rotation of the rigid circular disc inclusion.

2. STRESS FUNCTIONS FOR AXIAL SYMMETRY

In the ensuing we shall consider, separately, the analysis of the sub-problems illustrated in Figs 1(c) and 2(b). The class of problems which exhibit axial symmetry can be analysed by employing Love's strain function approach.[6] This stress function is governed by the differential equation

$$\nabla^2 \nabla^2 \Phi(r, z) = 0 \tag{1}$$

where ∇^2 is the Laplace's operator. The stresses and displacements in the medium can be expressed uniquely in terms of $\Phi(r, z)$. The displacement and stress components of interest to the axisymmetric inclusion problem are

$$2Gu_r = -\frac{\partial^2 \Phi}{\partial r \, \partial z} \qquad 2Gu_z = 2(1-v)\nabla^2\Phi - \frac{\partial^2 \Phi}{\partial z^2} \tag{2}$$

$$\sigma_{zz} = \frac{\partial}{\partial z}\left[(2-v)\nabla^2\Phi - \frac{\partial^2\Phi}{\partial z^2}\right] \tag{3}$$

where G and v are the elastic constants of the medium. Similarly, the class of problem which exhibits a rotational symmetry about the z-axis can be analysed by employing Galerkin's displacement function approach.[7] The differential equation governing this class of problem is

$$\left\{\nabla^2 - \frac{1}{r^2}\right\}\left\{\nabla^2 - \frac{1}{r^2}\right\} \Psi(r, z) = 0 \tag{4}$$

The displacement and stress components in the elastic medium of interest to the analysis of the embedded inclusion are

$$Gu_\theta = (1-v)\left\{\nabla^2 - \frac{1}{r^2}\right\} \Psi \tag{5}$$

$$\sigma_{\theta z} = (1-v)\left[\frac{\partial}{\partial z}(\nabla^2\Psi) - \frac{1}{r^2}\frac{\partial \Psi}{\partial z}\right] \tag{6}$$

3. AXISYMMETRIC TRANSLATION OF THE RIGID DISC INCLUSION

We first examine the problem of an inclusion-free elastic medium of infinite extent which is subjected to a doublet of forces P which are directed along the axis of symmetry. The points of action of these forces are located at $(0, \pm c)$. This particular problem exhibits a state of asymmetry about the plane $z = 0$. The normal stress σ_{zz} and the radial displacement u_r are identically equal to zero on $z = 0$. The non-zero displacement on $z = 0$ due to this doublet of forces is given by

$$u_z(r, 0) = \frac{P}{8\pi G(1-v)}\left[\frac{c^2}{(r^2+c^2)^{3/2}} + \frac{(3-4v)}{(r^2+c^2)^{1/2}}\right] = u_z^0(r) \tag{7}$$

We note that the problem related to the axisymmetric translation of the embedded inclusion acted upon by two concentrated forces which are directed along the z-axis (Fig. 1(c)) also exhibits a state of asymmetry about $z = 0$. Hence the analysis of the bonded disc inclusion problem can be restricted to a single halfspace region ($z \geq 0$) in which the plane $z = 0$ is subjected to the mixed boundary conditions

$$u_r(r, 0) = 0 \qquad r \geq 0 \tag{8}$$

$$u_z(r, 0) = \Delta - u_z^0(r) \qquad 0 \leq r \leq a \tag{9}$$

$$\sigma_{zz}(r, 0) = 0 \qquad a < r < \infty \tag{10}$$

where Δ is the net rigid displacement of the embedded inclusion. The solution of eqn (1) appropriate for the halfspace region $z > 0$ is

$$\Phi(r, z) = \frac{1}{a^2} \int_0^\infty \xi[A(\xi) + B(\xi)z] \mathrm{e}^{-\xi z/a} J_0(\xi r/a) \, \mathrm{d}\xi \tag{11}$$

where $A(\xi)$ and $B(\xi)$ are arbitrary functions. In order to satisfy the symmetry requirement, eqn (8), we require

$$B(\xi) = \frac{\xi}{a} A(\xi) \tag{12}$$

It can be shown that the boundary conditions (9) and (10) are equivalent to the pair of dual integral equations

$$H_0\{\xi^{-1}R(\xi); r\} = f_1(r) \qquad 0 \leq r \leq a \tag{13a}$$

$$H_0\{R(\xi); r\} = 0 \qquad a < r < \infty \tag{13b}$$

where $R(\xi) = \xi^3 A(\xi)$;

$$f_1(r) = -\frac{2Ga^4\delta}{(3-4\nu)} + \frac{Pa^4}{4\pi(1-\nu)(3-4\nu)} \left[\frac{c^2}{(r^2+c^2)^{3/2}} + \frac{(3-4\nu)}{(r^2+c^2)^{1/2}}\right]$$

and the operator H_n ($n = 0, 1$) is defined by

$$H_n\{N(\xi); r\} = \int_0^\infty \xi N(\xi) J_n(\xi r/a) \, \mathrm{d}\xi \tag{14}$$

The solution of the dual system, eqns (13a and b), can be obtained from the generalised results given by Sneddon.[4] It has been shown that by introducing a substitution of the type

$$R(\xi) = \int_0^a \chi(t) \cos(\xi t/a) \, \mathrm{d}t \tag{15}$$

the second equation of the dual system, eqn (13b), is identically satisfied and the first equation can be reduced to the solution of the Abel integral equation. The solution of this latter integral equation yields

$$\chi(t) = -\frac{4G\,\Delta a^4}{\pi(3-4\nu)} + \frac{Pa^4}{\pi^2(1-\nu)(3-4\nu)}\left[\frac{2(1-\nu)c}{(t^2+c^2)} - \frac{t^2c}{(t^2+c^2)^2}\right] \tag{16}$$

The result, eqn (16), can be utilised to develop formal integral expressions for the stresses and displacements in the elastic medium. As outlined previously, in this paper we are primarily interested in establishing the relationship between the applied loads P and the resulting displacement Δ of the disc inclusion. To derive this result we make use of the expression for the traction resultant T_z acting on the interfaces $z=0^{+\,-}$; $r \le a$. From the asymmetry of the problem we note that $\sigma_{zz}(r, 0^+) = -\sigma_{zz}(r, 0^-)$. Since the rigid disc inclusion is free from any force resultant, we require

$$T_z = \frac{4(1-\nu)}{a^4}\int_0^{2\pi}\int_0^a r\int_0^\infty \xi^4 A(\xi) J_0(\xi r/a)\,d\xi\,dr\,d\theta = 0 \tag{17}$$

Evaluating eqn (17) we obtain

$$\Delta = \frac{P}{8\pi Ga(1-\nu)}\left[(3-4\nu)\tan^{-1}\left(\frac{a}{c}\right) + \frac{ac}{(a^2+c^2)}\right] \tag{18}$$

It is of interest to note the following limiting results of the above expression for the resultant displacement:

1. As $c \to 0$, eqn (18) reduces to

$$\Delta = \frac{P(3-4\nu)}{16\pi Ga(1-\nu)} \tag{19}$$

This result represents the rigid displacement of a circular disc inclusion which is embedded in bonded contact with the elastic medium and subjected to a force $2P$ acting at its centre, along the z-direction. It is in agreement with the result obtained by Kanwal and Sharma[8] and Selvadurai[9] who employed singularity methods and spheroidal harmonic function solutions respectively.

2. As $a \to 0$, eqn (18) reduces to

$$\Delta = \frac{P}{2\pi Gc} \tag{20}$$

This result agrees with the expression for the axial displacement $u_z(0, c)$ in an elastic medium subjected to a Kelvin force of magnitude $2P$.

3. As $c \to \infty$, the inclusion experiences no displacement.

Since the action of an opposing set of doublet forces (Fig. 1(b)) causes no resultant displacement, expression (18) represents the net displacement of an elastically embedded rigid circular disc inclusion which is subjected to a resultant force of magnitude $2P$ acting along the z-direction at the location $(0, \pm c)$.

4. AXISYMMETRIC ROTATION OF THE RIGID DISC INCLUSION

The procedure for the analysis of the axisymmetric rotation of the embedded disc inclusion subjected to a concentrated moment $2M$ located at $z = -c$ (Fig. 2(a)) follows a similar pattern. The non-zero displacement component in the inclusion-free elastic medium which is subjected to a doublet of moments of magnitude M, located at $z = \pm c$ along the axis of symmetry and acting in the same direction is

$$u_\theta(r, 0) = \frac{Mr}{4\pi G(r^2 + c^2)^{3/2}} = u_\theta^0(r) \tag{21}$$

The analysis of this inclusion problem can also be referred to a single halfspace region ($z \geq 0$) in which the plane $z = 0$ is subjected to the following mixed boundary conditions:

$$u_\theta(r, 0) = \Omega r - u_\theta^0(r) \qquad 0 \leq r \leq a \tag{22}$$

$$\sigma_{\theta z}(r, 0) = 0 \qquad a < r < \infty \tag{23}$$

where Ω is the resultant rigid rotation of the disc inclusion. The solution of eqn (4) appropriate for the halfspace region $z \geq 0$ is

$$\Psi(r, z) = \frac{1}{a^2} \int_0^\infty \xi \{C(\xi) + D(\xi)z\} e^{-\xi z/a} J_1(\xi r/a)\, d\xi \tag{24}$$

where $C(\xi)$ and $D(\xi)$ are arbitrary functions. Using eqns (5), (6) and (24) it can be shown that the mixed boundary conditions, eqns (22) and (23), are equivalent to the pair of dual integral equations

$$H_1\{\xi^{-1}S(\xi); r\} = f_2(r) \qquad 0 \leq r \leq a \tag{25a}$$

$$H_1\{S(\xi); r\} = 0 \qquad a < r < \infty \tag{25b}$$

where $S(\xi) = \xi^2 D(\xi)$ and

$$f_2(r) = -\frac{Ga^3\Omega r}{2(1-\nu)} + \frac{Ma^3 r}{8\pi(1-\nu)(r^2 + c^2)^{3/2}} \tag{26}$$

The solution of the dual system can be obtained from the generalised results given by Sneddon.[4] The relationship between the externally applied moments M and the resultant rigid rotation can be obtained by computing the total torque T induced on the inclusion. Since the inclusion is torque free we require

$$T = \frac{4(1-\nu)}{a^4} \int_0^{2\pi} \int_0^a r^2 \int_0^\infty \xi^3 D(\xi) J_1(\xi r/a)\, d\xi\, dr\, d\theta \tag{27}$$

Evaluating eqn (27) we obtain

$$\Omega = \frac{3M}{8\pi G a^3}\left\{\tan^{-1}\left(\frac{a}{c}\right) - \frac{ac}{(a^2+c^2)}\right\} \tag{28}$$

The limiting results of expression (28) can be summarised as follows:

1. As $c \to 0$, eqn (28) yields the result for the rigid rotation of an embedded rigid disc inclusion which is subjected to a concentrated moment $2M$ at its centre (see, e.g. Selvadurai[10]).
2. As $a \to 0$, the rotational displacement Ωa reduces to zero.
3. As $c \to \infty$, the inclusion experiences zero rotation.

Furthermore, the disc inclusion which is acted upon by an opposing set of moments of equal magnitude located at $z = \pm c$ (Fig. 2(c)) experiences zero rotation. As such expression (28) represents the resultant rotation of a disc inclusion due to a single concentrated moment of magnitude $2M$ located at a distance $z = \pm c$ from the inclusion.

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