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AXIAL DISPLACEMENT OF A RIGID ELLIPTICAL DISC INCLUSION EMBEDDED IN A TRANSVERSELY ISOTROPIC ELASTIC SOLID

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Abstract

The present paper examines the axial translation of a rigid elliptical disc inclusion which is embedded in bonded contact with a transversely isotropic elastic medium of infinite extent. The load-displacement relationship for the embedded elliptical inclusion is evaluated in explicit closed form.

Introduction

The stress analysis of an elastic infinite medium which is bounded internally by an inhomogeneity is of considerable interest to the study of composite materials. Solutions developed for spheroidal and ellipsoidal inhomogeneities embedded in an isotropic elastic medium are given in the classical works of Eshelby [1], Lur'e [2]; and Edwards [3]. The disc shaped inclusion is a particular simplification of the above class of three-dimensional inhomogeneities. The studies by Collins [4], Kassir and Sih [5], Kanwal and Sharma [6], and Selvadurai [7,8] are primarily concerned with the study of disc inclusions embedded in an isotropic elastic medium of infinite extent. Several authors have extended these solutions to include other features such as transverse isotropy of the elastic medium, flexural behaviour of the disc inclusion and delamination at the inclusion - elastic medium interface. A recent review of inclusion problems in classical elasticity and references to further work in this area are given by Mura [9] and Selvadurai [10].

This paper examines the problem of the axial displacement of an elliptical disc inclusion embedded in bonded contact with a transversely isotropic elastic medium. The plane of the elliptical disc inclusion is assumed to coincide with the plane of transverse isotropy (Fig. 1). The method of analysis follows a complex potential function approach similar to that developed by Green and Sneddon [11] for the analysis of elliptical crack problems in isotropic elastic media.

Fundamental Formulae

Complete accounts of the methods employed in the analysis of three-dimensional problems in transversely isotropic elastic media are given by Elliott [12, 13], Green and Zerna [14] and Kassir and Sih [15]. It can be shown that, in the absence of body forces, the displacement and stress fields can be expressed in terms of two 'harmonic' functions ϕ_1, ϕ_2 which are solutions of

$$\left\{ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_i^2} \right\} \phi_i = 0; \qquad (i = 1,2) \tag{1}$$

where $z_i = z/\sqrt{\nu_i}$ and ν_i are roots of the equation

$$c_{11}\, c_{44}\, \nu^2 + [c_{13}(2c_{44}+c_{13}) - c_{13}c_{33}]\, \nu + c_{33}\, c_{44} = 0 \tag{2}$$

We note that c_{ij} are the elastic constants of the transversely isotropic elastic material and the z-axis is normal to the plane of isotropy. The displacement and stress components relevant for the present problem can be written in the form

$$\{u_x : u_y : u_z\} = \left\{ \frac{\partial}{\partial x}(\phi_1+\phi_2) : \frac{\partial}{\partial y}(\phi_1+\phi_2) : \frac{\partial}{\partial z}(k_1\phi_1 + k_2\phi_2) \right\} \tag{3}$$

$$\sigma_{zz} = (k_1c_{33} - \nu_1c_{13}) \frac{\partial^2 \phi_1}{\partial z^2} + (k_2c_{33} - \nu_2c_{13}) \frac{\partial^2 \phi_2}{\partial z^2} \tag{4}$$

where k_1 and k_2 are given by

$$k_i = \frac{c_{11}\nu_i - c_{44}}{c_{13} + c_{44}}; \qquad (i = 1,2) \tag{5}$$

The elliptical inclusion problem

We consider the problem related to a elliptical disc shaped rigid inclusion which is embedded in bonded contact with the transversely isotropic elastic medium of infinite extent (Fig. 1). For ease of reference we shall adopt the following nomenclature. Referring to the plane $z = 0$ (which contains the inclusion) the inclusion region (i.e. $(x^2/a^2) + (y^2/b^2) \leq 1$: where a and b are the major and minor semi-axis of the ellipse respectively) is denoted by S_i; the region exterior to the inclusion is denoted by S_e; also $S = S_i \cup S_e$. The state of deformation induced in the elastic medium due to the axial displacement of the elliptical inclusion is such that the displacements u_x and u_y and the stress σ_{zz} exhibit a state of asymmetry about the plane $z = 0$. The analysis of the inclusion problem can therefore be restricted to the analysis of a single half-space region in which the plane $z \geq 0$ (also denoted by $z = 0^+$) is subjected to appropriate mixed boundary conditions. Due to the fully bonded conditions at the inclusion region it is evident that u_x and u_y are zero in S_i. Considering the above conditions we have

$$u_x(x, y, 0^+) = u_y(x, y, 0^+) = 0; \qquad (x,y) \in S_e \tag{6}$$

$$\sigma_{zz}(x, y, 0^+) = 0; \qquad (x,y) \in S_e \tag{7}$$

In the inclusion region

$$u_z(x, y, 0^+) = \delta \quad ; \qquad (x,y) \in S_i \tag{8}$$

$$u_x(x, y, 0^+) = u_y(x,y,0^+) = 0; \qquad (x,y) \in S_i \tag{9}$$

From (6) and (9) it is evident that

$$u_x(x, y, 0^+) = u_y(x, y, 0^+) = 0; \qquad (x,y) \in S \tag{10}$$

In order to satisfy the above boundary condition, we select solutions of (1) which are of the form

$$\phi_1 = \Phi(x,y,z_1); \qquad \phi_2 = -\Phi(x,y,z_2) \tag{11}$$

where $\nabla^2 \Phi = 0$ and ∇^2 is Laplace's operator referred to the Cartesian coordinate system. Considering the reduction (11), the expressions for u_z and σ_{zz} yield

$$u_z = \{ \frac{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}}{\sqrt{\nu_1\nu_2}} \} \frac{\partial\Phi}{\partial z} \tag{12}$$

$$\sigma_{zz} = c_{33}\{ \frac{k_1\nu_2 - k_2\nu_1}{\nu_1\nu_2} \} \frac{\partial^2\Phi}{\partial z^2} \tag{13}$$

and the mixed boundary conditions (7) and 8) can be expressed as

$$\frac{\partial\Phi}{\partial z} = \frac{\delta\sqrt{\nu_1\nu_2}}{\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}} \quad ; \qquad (x,y) \in S_i \tag{14}$$

$$\frac{\partial^2\Phi}{\partial z^2} = 0; \qquad (x,y) \in S_e \tag{15}$$

Following Lamb [16], and Green and Sneddon [11] it can be shown that $\partial\Phi/\partial z$ represents the velocity potential of the motion of a perfect fluid flowing through an elliptical aperture in a thin boundary. Thus we have

$$\frac{\partial\Phi}{\partial z} = \frac{\delta a\sqrt{\nu_1\nu_2}}{2\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}K(e_0)} \int_\xi^\infty \frac{ds}{[s(a^2+s)(b^2+s)]^{\frac{1}{2}}} \tag{16}$$

where (ξ, η, ζ) are the ellipsoidal coordinates of the point (x,y,z)

and are the roots of

$$\frac{x^2}{a^2+\theta} + \frac{y^2}{b^2+\theta} + \frac{z^2}{\theta} - 1 = 0 \tag{17}$$

In these coordinates S_i corresponds to $\xi = 0$ and S_e corresponds to $\eta = 0$. In (16), $K(e_0)$ is the complete elliptic integral of the first kind where $e_0 = (a^2 - b^2)/a^2$. It can also be shown that $\xi = a^2(\mathrm{sn}^{-2} u - 1)$ and $E(u) = \int_0^u \mathrm{dn}^2\, t\, dt$, where sn u is the Jacobian elliptic function which has real and imaginary roots 4K and 2iK' respectively, corresponding to the moduli e_0 and e_0' (=b/a). To complete the analysis it is necessary to determine an explicit form for the function $\Phi(x,y,z)$, such that (16) is satisfied. We note, however, that it is possible to obtain u_z and σ_{zz} directly from (16) and the formulae (12) and (13). In particular, the value of σ_{zz} at the bonded faces of the elliptical disc inclusion (in contact with the regions $z \geq 0$ or $z \leq 0$) are given by

$$\sigma_{zz}(x,y,0+) = \mp \frac{(k_1\nu_2 - k_2\nu_1)c_{33}\delta}{\sqrt{\nu_1\nu_2}\,\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}bK(e_0)\left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right]^{\frac{1}{2}}} \tag{18}$$

The force acting on the inclusion is given by

$$P = \iint_{S_i} [\sigma_{zz}(x,y,0^+) - \sigma_{zz}(x,y,0^-)]dx\, dy \tag{19}$$

Evaluating (19) we obtain the force-displacement relationship for the embedded rigid disc inclusion. Assuming that the displacement δ occurs in the direction of the applied force we have

$$P = \frac{4\pi c_{33}a\delta\{k_1\nu_2 - k_2\nu_1\}}{\sqrt{\nu_1\nu_2}\,\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}K(e_0)} \tag{20}$$

Limiting cases

In the limiting case when $\nu_i \to 1$, we recover from (20) the result for the force-displacement relationship for an elliptical disc inclusion embedded in an isotropic elastic medium. We note that as $\nu_i \to 1$,

$$\frac{k_1\nu_2 - k_2\nu_1}{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}} = \frac{2c_{44}}{c_{11} + c_{44}} \tag{21}$$

where $c_{11} = c_{33} = \lambda + 2\mu$; $c_{44} = \mu$ and λ, μ are Lame's constants for the isotropic elastic material. Making use of these results, (20) can be reduced to the form

$$P = \frac{16\pi\mu(1-\nu)a\delta}{(3-4\nu)\ K(e_0)} \tag{22}$$

In the limit as $a \to b$, (22) gives the solution for the force-displacement relationship for a rigid circular inclusion in an isotropic elastic solid i.e.

$$P = \frac{32\mu(1-\nu)a\delta}{(3-4\nu)} \tag{23}$$

Also, as $a \to b$, (20) gives the force-displacement relationship for a penny shaped inclusion in a transversely isotropic elastic solid

$$P = \frac{8c_{33}a\delta\{k_1\nu_2 - k_2\nu_1\}}{\sqrt{\nu_1\nu_2}\ \{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}} \tag{24}$$

The limiting cases derived above (22-24) are in agreement with the appropriate results given by Collins [4], Kassir and Sih [5], Kanwal and Sharma [6] and Selvadurai [7,17].

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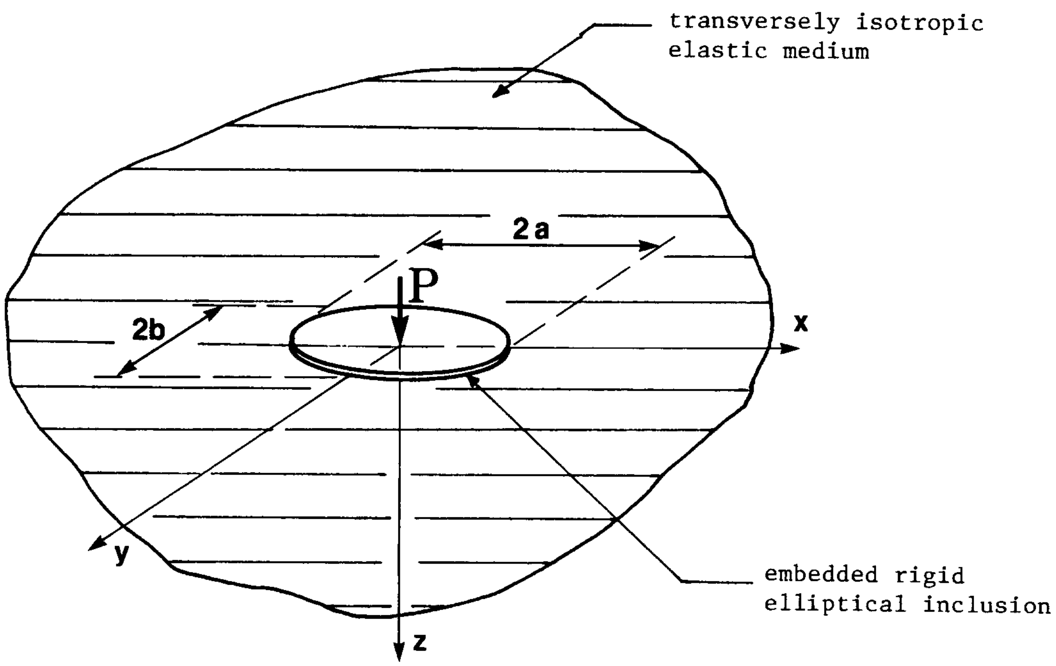

Fig. 1