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Betti's Reciprocal Relationships for the Displacements of an Elastic Infinite Space Bounded Internally by a Rigid Inclusion

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I. INTRODUCTION

This note summarizes certain results pertaining to the reciprocity of displacements in an elastic infinite space bounded internally by a rigid disk inclusion. It is found that the reciprocal theorem of Betti [1] can be used to great advantage in examining the displacements of the embedded inclusion induced by forces which act within the elastic medium. The adjoint or auxiliary solution required to generate the reciprocal relationships can be deduced from the analysis of a mixed boundary value problem related to an elastic half-space region.

II. ANALYSIS

The class of problems which deal with the behavior of inclusions embedded in elastic media is of interest to the study of multiphase materials. The behavior of loaded disk inclusions embedded in bonded contact with the surrounding elastic matrix has been examined by several authors including Collins [2], Kassir and Sih [3], Mura and Lin [4], and Selvadurai [5, 6]. Recently Selvadurai [7] has extended the study of the embedded rigid disk inhomogeneity to include the effect of concentrated loads and couples which are located at a finite distance from the inhomogeneity. The analysis of the latter group of problems is facilitated owing to the possibility of their reduction to mixed boundary value problems associated with an equivalent half-space region. These mixed boundary value problems yield sets of dual integral equations which can be readily solved with the aid of standard techniques (see, for example, Sneddon [8]). In these problems the results for the displacement and rotation of the rigid disk inclusion can be evaluated in exact closed forms. Despite this apparent advantage, the analysis of the embedded, externally loaded disk inclusion can be attempted only in rare cases which involve axisymmetric or asymmetric loadings located along the axes of geometric or loading symmetry. The analysis of problems which involve direct loading of the embedded disk inclusion is, however, relatively straightforward. It is therefore pertinent to enquire whether the solution to the directly loaded inclusion problem could in any way serve as an auxiliary solution which enables the application of Betti's reciprocal theorem. Such a procedure has been used by Shield [9] and Shield and Anderson [10] in connection with the analysis of the load-displacement relationship for simply connected elastic bodies. The primary object of this note is to illustrate (by appeal to some specific examples) that the results for the loaded rigid disk inclusion can be used as the auxiliary solution for the generation of the load-displacement results for the embedded disk inclusion which is subjected to externally applied concentrated loads. The methodology adopted to illustrate the applicability of Betti's reciprocal theorem to the embedded disk inclusion is heuristic rather than rigorous. The results are therefore of insufficient generality. Such a rigorous treatment requires proof of the applicability of Betti's reciprocal theorem for both exterior domains and concentrated loads respectively (cf. Gurtin and Sternberg [11] and Turteltaub and Sternberg [12]).

III. THE AXISYMMETRIC PROBLEM

We first consider the axisymmetric problem in which the rigid circular disk inclusion (of radius a), embedded in bonded contact with the elastic medium of infinite extent, is subjected to a concentrated load at the origin (Fig. 1, left). Due to this axisymmetric deformation, the disk inclusion experiences a rigid displacement w in the z-direction. It is evident that the deformation field induced by the disk inclusion exhibits a state of asymmetry about $z = 0$. Hence the analysis of the bonded disk inclusion problem can be restricted to a single half-space region ($z \geq 0$) in which the plane $z = 0$ is subjected to the mixed boundary conditions

$$\begin{aligned} u_r(r, 0) &= 0, & r &\geq 0 \\ u_z(r, 0) &= w, & 0 &\leq r \leq a \\ \sigma_{zz}(r, 0) &= 0, & a &< r < \infty \end{aligned} \tag{1}$$

By employing a Hankel transform formulation of the governing equations (i.e., the equations governing Love's [13] strain potential approach), it can be shown that the mixed boundary conditions in (1) are equivalent to the system of dual integral equations

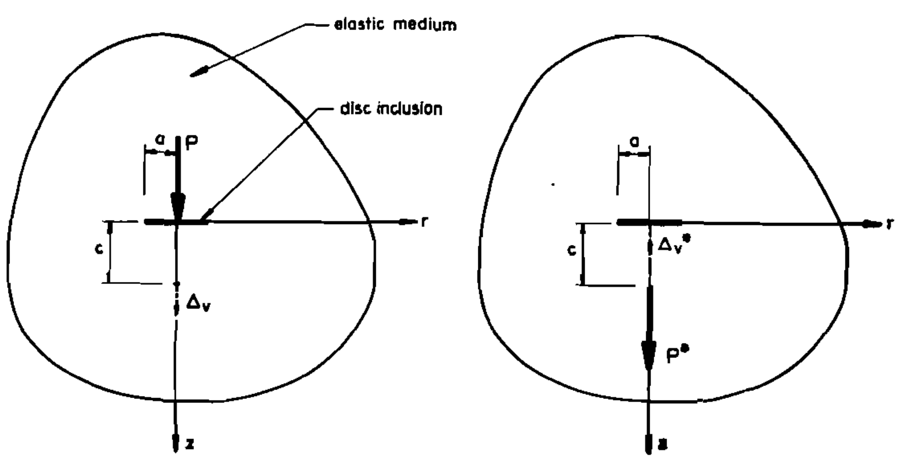

Fig. 1 Embedded inclusion, axisymmetric loading.

$$H_0\{\xi^{-1}R(\xi); r\} = f_1(r) = -\frac{2Ga^4w}{(3-4\nu)}, \qquad 0 \le r \le a$$
$$H_0\{R(\xi); r\} = 0, \qquad a < r < \infty \tag{2}$$

for the unknown function $R(\xi)$. [In (2), G = shear modulus; ν = Poisson's ratio.] The operator H_n $(n = 0, 1)$ is defined by

$$H_n\{N(\xi); r\} = \int_0^\infty \xi N(\xi) J_n(\xi r/a)\, d\xi \tag{3}$$

The solution of the dual system (2) can be readily obtained by making use of the generalized results given by Sneddon [8]. The displacements and stresses in the elastic medium can be expressed, in integral form, in terms of $R(\xi)$. The result of particular interest to this paper is the distribution of axial displacement u_z along the z-axis. By evaluating the tractions on the plane surfaces $z = \pm 0$, it can be shown that the relationship between w and the load P applied to the inclusion is given by $w = P(3-4\nu)/32Ga(1-\nu)$. The displacement $u_z(0, z)$ is given by

$$u_z(0, z) = \frac{P}{16\pi Ga(1-\nu)}\left[(3-4\nu)\tan^{-1}\left(\frac{a}{z}\right) + \frac{az}{(a^2+z^2)}\right] \tag{4}$$

We now consider the auxiliary axisymmetric problem related to the elastic medium bounded internally by a bonded disk inclusion. In the auxiliary problem the disk inclusion is subjected to a concentrated force P^* which acts at a distance c along the z-axis. The solution to this problem can also be reduced to the solution of a system of dual integral equations of the type (2), where the corresponding displacement boundary conditions are such that

$$f_1^*(r) = -\frac{2Ga^4\Delta_v^*}{(3-4\nu)} + \frac{P^*a^4}{4\pi(1-\nu)(3-4\nu)}\left[\frac{c^2}{(r^2+c^2)^{3/2}} + \frac{(3-4\nu)}{(r^2+c^2)^{1/2}}\right] \tag{5}$$

where Δ_v^* is the resulting displacement of the embedded inclusion. It can be shown that the relationship between Δ_v^* and P^* is given by

$$\Delta_v^* = \frac{P^*}{16\pi Ga(1-\nu)}\left[(3-4\nu)\tan^{-1}\left(\frac{a}{c}\right) + \frac{ac}{(a^2+c^2)}\right] \tag{6}$$

We denote the axial displacement at $(0, c)$ due to the loaded disk inclusion by Δ_v. By comparing the resulting expressions (4) and (6), it is evident that the displacements Δ_v and Δ_v^* and the applied forces P and P^* satisfy Betti's reciprocal theorem

$$P\Delta_v^* = P^*\Delta_v \tag{7}$$

IV. THE ASYMMETRIC PROBLEM

In this section we examine the problem where the infinite medium is subjected to a state of asymmetric deformation by the in-plane translation of an embedded disk inclusion. The disk inclusion is subjected to a lateral load K as shown in Fig. 2, left. The analysis of this problem can also be reduced to a mixed boundary value problem associated with a half-space region. The mixed boundary conditions take the form

$$\begin{aligned} &u_r(r, \theta, 0) = \delta \cos \theta, && 0 \le r \le a \\ &u_\theta(r, \theta, 0) = -\delta \sin \theta, && 0 \le r \le a \\ &\sigma_{rz}(r, \theta, 0) \sin \theta + \sigma_{\theta z}(r, \theta, 0) \cos \theta = 0, && r \ge 0 \\ &\sigma_{rz}(r, \theta, 0) \cos \theta - \sigma_{\theta z}(r, \theta, 0) \sin = 0, && a < r < \infty \end{aligned} \tag{8}$$

where δ is the in-plane translation of the embedded inclusion in the radial ($\theta = 0$) direction. A Hankel transform formulation of the governing equations yields the following system of dual integral equations for the above boundary conditions,

$$\begin{aligned} &H_0\{\xi^{-1}S(\xi); r\} = f_2(r) = \frac{-4G\delta a^4}{(7 - 8\nu)}, \qquad 0 \le r \le a \\ &H_0\{S(\xi); r\} = 0 \qquad a < r < \infty \end{aligned} \tag{9}$$

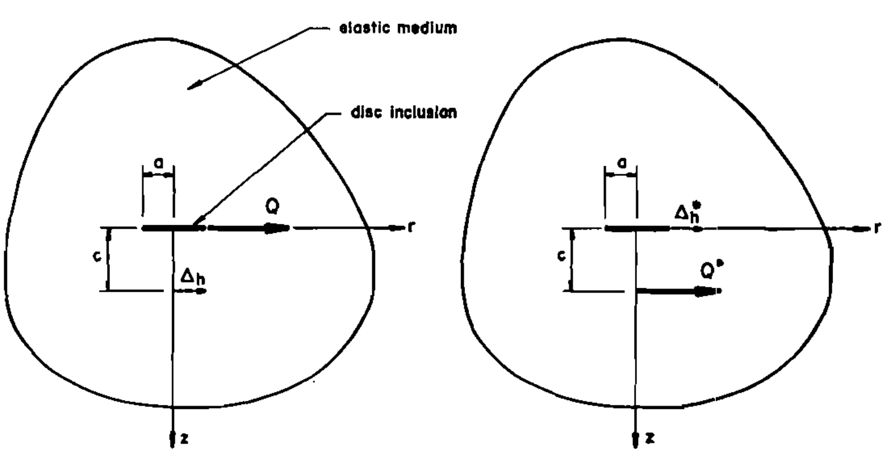

Fig. 2 Embedded inclusion, asymmetric loading.

for the unknown function $S(\xi)$. The displacements and stresses in the elastic medium can be expressed in integral form in terms of $S(\xi)$. By evaluating the resultant traction on the disk inclusion it can be shown that $\delta = Q(7 - 8\nu)/64Ga(1 - \nu)$. Furthermore, by evaluating the integral expression for the radial displacement $u_r(r, \theta, z)$, it can be shown that

$$u_r(0, \theta, z) = \frac{Q \cos \theta}{32\pi Ga(1 - \nu)}\left[(7 - 8\nu)\tan^{-1}\left(\frac{a}{z}\right) = \frac{az}{(a^2 + z^2)}\right] \tag{10}$$

The auxiliary asymmetric problem related to the elastic medium bounded internally by a bonded disk inclusion is that in which the displacement of the inclusion is caused by a concentrated radial force Q^* which acts at a distance c along the z-axis (Fig. 2, right). Here again, the analysis of the problem can be reduced to the solution of the dual system in which

$$f_2^*(r) = -\frac{4Ga^4\Delta_h^*}{(7 - 8\nu)} + \frac{Qa^{*4}}{4\pi(1 - \nu)(7 - 8\nu)}\left[\frac{r^2}{(r^2 + c^2)^{3/2}} + \frac{2(3 - 4\nu)}{(r^2 + c^2)^{1/2}}\right] \tag{11}$$

and Δ_h^* is the resulting rigid displacement. The relationship between the rigid displacement Δ_h^* and the externally applied load Q^* is given by

$$\Delta_h^* = \frac{Q^*}{32\pi Ga(1 - \nu)}\left[(7 - 8\nu)\tan^{-1}\left(\frac{a}{c}\right) - \frac{ac}{(a^2 + c^2)}\right] \tag{12}$$

The radial displacement at the location $(0, 0, c)$ due to the in-plane loading of the disk inclusion is denoted by Δ_h. Again, by comparing the resulting relationship and (12), it can be shown that the displacements Δ_h and Δ_h^* and the applied forces Q and Q^* satisfy Betti's reciprocal theorem

$$Q\Delta_h^* = Q^*\Delta_h \tag{13}$$

V. THE ROTATIONALLY SYMMETRIC PROBLEM

The discussion of the reciprocal behavior between the displacements at a point in the elastic medium due to the loaded disk inclusion and the displacement of the disk inclusion due to a concentrated load located at an exterior point can be extended to include situations which involve rotationally symmetric deformations. We examine the problem in which the disk inclusion

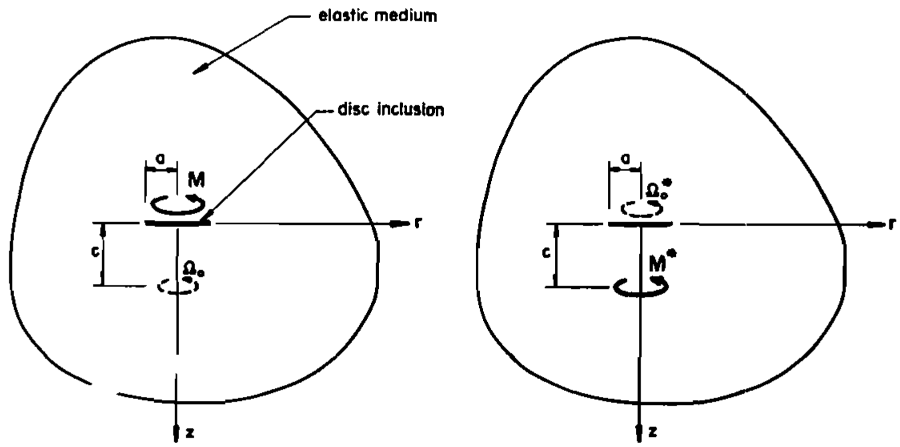

Fig. 3 Embedded inclusion, rotationally symmetric loading.

embedded in bonded contact with the elastic medium is subjected to a couple as shown in Fig. 3, left. The deformation field induced in the elastic medium is rotationally symmetric and it can be examined by recourse to Galerkin's [14] displacement function approach. The symmetry of the problem suggests that the analysis may be restricted to a single half-space region ($z \geq 0$) in which the plane $z = 0$ is subjected to the mixed boundary conditions

$$\begin{aligned} u_\theta(r, 0) &= \omega r, & 0 \leq r \leq a \\ \sigma_{\theta z}(r, 0) &= 0, & a < r < \infty \end{aligned} \tag{14}$$

where ω is the resultant rotation of the disk inclusion. By adopting a Hankel transform formulation of the governing equations it can be shown that the above boundary conditions are equivalent to the system of dual integral equations

$$\begin{aligned} &H_1\{\xi^{-1}T(\xi); r\} = f_3(r) = -\frac{Ga^3\omega r}{2(1-\nu)}, \qquad 0 \leq r \leq a \\ &H_1\{T(\xi); r\} = 0, \qquad a < r < \infty \end{aligned} \tag{15}$$

for the unknown function $T(\xi)$. The solution of the dual system can be used to generate the moment-rotation relationship for the embedded inclusion, i.e., $\omega = 3M/32Ga^3$. The rotation $[\partial u_\theta/\partial r] = \Omega$, at a point $(0, z)$ along the z-axis, is given by

$$\Omega = \frac{3M}{16\pi G a^3}\left[\tan^{-1}\left(\frac{a}{z}\right) - \frac{az}{(a^2 + z^2)}\right] \tag{16}$$

The corresponding auxiliary problem concerns the rotation of an embedded rigid disk inclusion due to a concentrated axial couple M^* applied at a finite distance (c) from the inclusion (Fig. 3, right). The analysis can be reduced to the solution of the dual system (15) where the appropriate function $f_3^*(r)$ is given by

$$f_3^*(r) = -\frac{Ga^3\Omega_0^* r}{2(1-\nu)} + \frac{M^* a^3 r}{8\pi(1-\nu)(r^2 + c^2)^{3/2}} \tag{17}$$

where Ω_0^* is the resulting rigid rotation of the disk inclusion. The relationship between Ω_0^* and M^* takes the form

$$\Omega_0^* = \frac{3M^*}{16\pi G a^3}\left[\tan^{-1}\left(\frac{a}{c}\right) - \frac{ac}{(a^2 + c^2)}\right] \tag{18}$$

We denote the rotation $[\partial u_\theta/\partial r]$ at $(0, c)$ by Ω_0. By comparing the resulting expressions for (16) and (18) it is evident that the two states satisfy Betti's reciprocal theorem

$$M\Omega_0^* = M^*\Omega_0 \tag{19}$$

VI. SUMMARY

In this paper we have examined the solutions to a set of problems which relate to the behavior of a disk inclusion embedded in bonded contact with an elastic medium of infinite extent. The problems examined refer to the specific cases where the disk inclusion is subjected to concentrated loads which act within the disk inclusion region or concentrated loads which are located within the elastic medium. The analysis of these problems is facilitated owing to their reducible character. The reduced problems (a mixed boundary value problem associated with a half-space region) yield systems of dual integral equations which have standard solutions. Upon examination of these results it becomes evident that displacements and forces for the two categories of problem (i.e., the class of problems in which the embedded inclusion is subjected to a load within its region and the class of problems in which the embedded inclusion is subjected to a load in the exterior region) satisfy Betti's reciprocal theorem. Although the results presented here for the disk inclusion lack the degree of generality for the applicability of Betti's reciprocal theorem to all classes of embedded inclusion regions, it would be sufficient to note the following generalized conjecture.

Consider an elastic medium R_1 of infinite extent which is bounded internally by a rigid inclusion region R_2 of smooth boundary ∂R (the body $R = R_1 \cup R_2$). The boundary ∂R represents a bonded contact between the elastic medium and the inclusion. The displacement (u_i) and stress (σ_{ij}) fields induced in the elastic medium either due to loads applied within it or by loads transmitted from the loaded inclusion are such that u_i are $O(1/r)$ and σ_{ij} are $O(1/r^2)$ as $r \to \infty$, where $r = (x_i x_i)^{1/2}$. We suppose that the inclusion region R_2 is subjected to the concentrated forces T_i at A (Fig. 4). The displacements induced in the elastic region R_1 at B are u_i. We next examine the problem in which the region R_1 is subjected to concentrated forces T_i^* at B. The corresponding displacements of A in R_2 are denoted by u_i^*. The two states characterized by (u_i, T_i) and (u_i^*, T_i^*) satisfy Betti's reciprocal relationship $T_i u_i^* = T_i^* u_i$. The applicability of such a reciprocal relationship to the embedded inclusion problem can be used to considerable advantage in the analysis of inclusion problems wherein the loads are applied within the elastic medium.

The auxiliary solution (i.e., the problem related to the directly loaded rigid inclusion) can be examined relatively conveniently by using several analytical techniques such as complex potential function techniques, singularity methods, and direct spheroidal harmonic function techniques (see, e.g., Collins [2], Kanwal and Sharma [15], and Selvadurai [16]). As an example of an application of the reciprocity principle, we consider the problem in

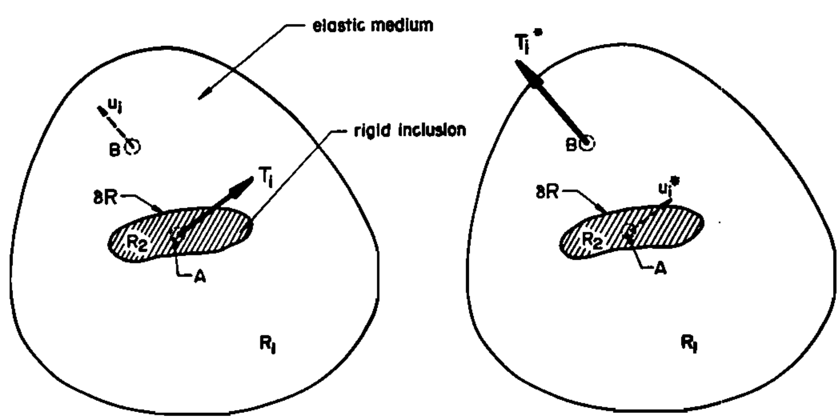

Fig. 4 Inclusion subjected to arbitrary loading, reciprocal states.

which the embedded rigid disk inclusion (of radius a) is displaced by a uniform axisymmetric load of radius b ($<a$) which is located at a finite distance c from the inclusion (Fig. 5). The general analysis of this problem requires the solution of a set of dual integral equations [of the type (2)] in which $f_1(r)$ has a particularly complicated form which involves elliptic integrals. In order to apply the reciprocal relationship to this problem we examine the problem in which the rigid disk inclusion is directly subjected to the uniform circular load of intensity p_0 and radius b. The axial displacement at the plane $z = c$ is given by

$$u_z(r, c) = \frac{p_0 b^2}{8G(1-\nu)a}\int_0^\infty \frac{\sin\xi}{\xi}\left\{(3-4\nu) + \frac{\xi c}{a}\right\}e^{-\xi c/a} J_0(\xi r/a)\,d\xi \quad (20)$$

The displacement of the rigid disk inclusion $\delta_0(c)$ due to the external load p_0 can be expressed in the form

$$\frac{\delta_0(c)}{\delta_0(0)} = \frac{4}{\pi b^2(3-4\nu)}\int_0^b r\int_0^\infty \frac{\sin\xi}{\xi}\left[(3-4\nu) + \frac{\xi c}{a}\right]e^{-\xi c/a} J_0(\xi r/a)\,d\xi\,dr \quad (21)$$

where

$$\delta_0(0) = \frac{p_0\pi b^2(3-4\nu)}{32(1-\nu)Ga} \quad (22)$$

The integrals occurring in (21) can be evaluated in explicit form, i.e.,

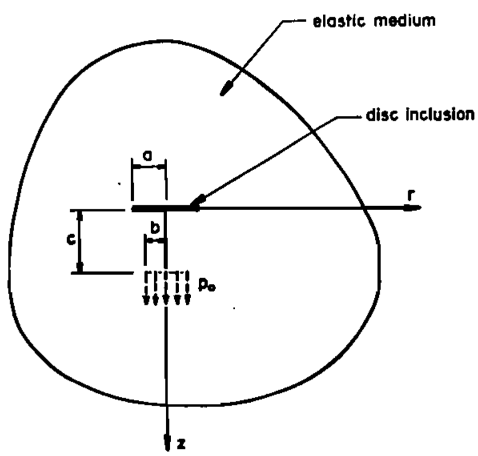

Fig. 5 Inclusion subjected to distributed external loads.

$$\frac{\delta_0(c)}{\delta_0(0)} = \frac{4\lambda}{\pi\zeta^2(3-4\nu)}\left\{1 - R\sin\frac{\theta_1}{2}\right\}$$
$$+ \frac{2}{\pi}\tan^{-1}\left\{\frac{R\sin\left(\frac{\theta_1}{2}\right) + \sqrt{1+\lambda^2}\sin\theta_2}{R\cos\left(\frac{\theta_1}{2}\right) + \sqrt{1+\lambda^2}\cos\theta_2}\right\}$$
$$+ \frac{1}{\pi\zeta^2}\left\{2R\sqrt{1+\lambda^2}\sin\left(\frac{\theta_1+2\theta_2}{2}\right) - (1+\lambda^2)\sin 2\theta_2 - R^2\sin\theta_1\right\} \tag{23}$$

where

$$R^4 = (\zeta^2 + \lambda^2 - 1)^2 + 4\lambda^2, \qquad \lambda = c/a, \qquad \zeta = b/a$$
$$R\sin\left(\frac{\theta_1}{2}\right) = \left[\frac{1}{2}(R^2 + 1 - \zeta^2 - \lambda^2)\right]^{1/2}$$
$$R\cos\left(\frac{\theta_1}{2}\right) = \left[\frac{1}{2}(R^2 - 1 + \zeta^2 + \lambda^2)\right]^{1/2} \tag{24}$$
$$R^2\sin\theta_1 = 2\lambda, \qquad \sin\theta_2 = \frac{1}{\sqrt{1+\lambda^2}}, \qquad \cos\theta_2 = \frac{\lambda}{\sqrt{1+\lambda^2}}$$
$$\sin 2\theta_2 = \frac{2\lambda}{(1+\lambda^2)}$$

It may be noted that as $\lambda \to \infty$, the normalized displacement $[\delta_0(c)/\delta_0(0)] \to 0$. Similarly by letting $\zeta \to 0$, while maintaining $p_0\pi b^2$ finite (= P) the result (23) reduces to

$$\frac{\delta_0^*(c)}{\delta_0^*(0)} = \frac{2}{\pi(3-4\nu)}\left[(3-4\nu)\tan^{-1}\left(\frac{1}{\lambda}\right) + \frac{\lambda}{(\lambda^2+1)}\right] \tag{25}$$

which is in agreement with the result (4). [Also $\delta_0^*(0) = P(3-4\nu)/32Ga(1-\nu)$.]

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