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THE DISPLACEMENTS OF A FLEXIBLE INHOMOGENEITY EMBEDDED IN A TRANSVERSELY ISOTROPIC ELASTIC MEDIUM

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SUMMARY

This paper examines the flexural behaviour of a disc-shaped elastic inhomogeneity embedded in bonded contact with a transversely isotropic elastic medium. The analysis of the problem employs a variational approach wherein the deflected shape of the inhomogeneity is prescribed a priori. *The variational solution developed for the deflections of the flexible inhomogeneity compare accurately with existing solutions for rigid and perfectly flexible disc-shaped inclusions.*

1. INTRODUCTION

The problem of the determination of the displacements and stress fields in an elastic medium perturbed by a material inhomogeneity is of interest to several problems in engineering. These solutions have applications in the study of stress concentration, fracture and deformability of inhomogeneous composite materials. The present paper deals with the analysis of the flexure of a circular plate-like inhomogeneity embedded in bonded contact with the surrounding transversely isotropic elastic medium. The plane of the disc-shaped inhomogeneity is assumed to coincide with the plane of transverse isotropy of the elastic medium. In particular, the flexural behaviour of the plate inhomogeneity is governed by the Poisson–Kirchhoff thin plate theory. The plate inhomogeneity is subjected to an axisymmetric distributed load which acts at its mid-plane (Fig. 1). An exact formulation of the problem yields a set of integro-differential equations which are yet to be solved. However, the variational formulation proposed in this paper overcomes this difficulty by providing an approximate solution to this complex interaction problem. In the variational formulation it is assumed that the deflected shape of the disc inhomogeneity can be approximated by a set of elementary functions which satisfy

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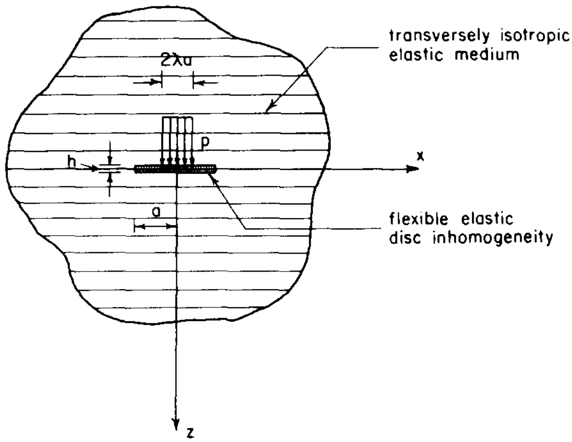

Fig. 1. Geometry of the embedded flexible inhomogeneity.

the kinematic constraints of the deformation and are indeterminate to within a set of constants. The total potential energy of the inhomogeneity–elastic medium–external load system can thus be determined by considering; (i) the elastic strain energy of the infinite space region, (ii) the flexural energy of the inhomogeneity, and (iii) the potential energy of the loads causing flexure. To determine the elastic energy of the transversely isotropic elastic medium it is necessary to compute the tractions at the elastic medium–inhomogeneity interface. The determination of these tractions is facilitated by the analysis of a mixed boundary value problem obtained by considering the asymmetry of the deformation induced in the infinite space. This mixed boundary value problem is further reduced to the solution of a standard pair of dual integral equations. The total potential energy functional is minimised to determine uniquely the constants characterising the assumed displacement field of the disc-shaped inhomogeneity. The analysis outlined above provides an approximate solution for the displacement field of the disc-shaped inhomogeneity which compares accurately with known exact solutions. This problem is also of interest in connection with the static performance of rubber mounts or in the geotechnical study of anchors embedded in rock strata.

2. FUNDAMENTAL FORMULAE

Detailed accounts of the methods of stress analysis of transversely isotropic elastic media are given by Elliot,[1,2] Lekhnitski[3] and Kassir and Sih.[4] Only the salient results will be briefly summarised here. It can be shown that in the absence of body forces, the axisymmetric displacement and stress field in a transversely isotropic elastic material can be expressed in terms of two harmonic functions $\phi_i(r, z)$ which are solutions of the equations

$$\left\{\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \nu_i \frac{\partial^2}{\partial z^2}\right\}\phi_i(r, z) = 0 \qquad (i = 1, 2) \tag{1}$$

where ν_i are the roots of the equation

$$c_{11}c_{44}\nu^2 + \{c_{13}(2c_{44} + c_{13}) - c_{13}c_{33}\}\nu + c_{33}c_{44} = 0 \tag{2}$$

The cylindrical polar coordinate system (r, θ, z) is chosen such that the z-axis is parallel to the axis of elastic symmetry. The roots ν_i may be real or complex depending upon the elastic constants c_{ij}. The displacement and stress fields in the transversely isotropic elastic medium can be represented in terms of ϕ_i; the results of particular interest to the analysis of the title problem, namely u_r, u_z and σ_{zz}, can be expressed in the form

$$u_r = \frac{\partial}{\partial r}\{\phi_1 + \phi_2\} \qquad u_z = \frac{\partial}{\partial z}\{k_1\phi_1 + k_2\phi_2\} \tag{3}$$

and

$$\sigma_{zz} = \{k_1 c_{33} - \nu_1 c_{13}\}\frac{\partial^2 \phi_1}{\partial z^2} + \{k_2 c_{33} - \nu_2 c_{13}\}\frac{\partial^2 \phi_2}{\partial z^2} \tag{4}$$

respectively, where k_i $(i = 1, 2)$ are given by

$$k_i = \frac{c_{11}\nu_i - c_{44}}{c_{13} + c_{44}} \tag{5}$$

As a prelude to the application of the variational procedure, we consider the problem in which the region corresponding to the bonded flexible disc inhomogeneity (i.e. $r \leq a, z = \pm 0$) is subjected to the displacement field

$$u_r(r, 0\pm) = 0 \qquad u_z(r, 0\pm) = \pm af(\rho) \tag{6}$$

where a is the radius of the inhomogeneity and $\rho = r/a$. The notation $(+)$ or $(-)$ refers to the faces of the disc inhomogeneity in contact with the halfspace regions $z > 0$ and $z < 0$ respectively. We note here that the geometry of the disc inhomogeneity is such that its thickness $h \ll$ a. As such we may assume that $z = \pm h/2$ approximately corresponds to the plane $z = \pm 0$. It is evident that the inhomogeneity problem thus formulated is antisymmetric in the normal stress σ_{zz} and in the radial displacement u_r, in the region $r \geq a$, about the plane $z = 0$. Furthermore, owing to the bonded nature of the inclusion interface (and its inextensibility in the radial direction specified by eqn (6)), we have

$$u_r(r, 0\pm) = 0 \qquad r \geq 0 \tag{7}$$

We may thus restrict the analysis of the inhomogeneity problem to that of a mixed boundary value problem associated with a halfspace region (say $z \geq 0$) subjected to the displacement boundary condition (7) and the mixed boundary conditions

$$u_z(r,0) = af(\rho) \qquad 0 \leq r \leq a \tag{8}$$

$$\sigma_{zz}(r,0) = 0 \qquad a < r < \infty \tag{9}$$

In addition, the displacement and stress fields derived from ϕ_i should reduce to zero as $r, z \to \infty$, in the halfspace region $z > 0$.

Following Sneddon,[5] we introduce the zero order Hankel transform of $\phi_i(r,z)$ as follows:

$$\bar{\phi}_i^0(\xi, z) = H_0\{\phi(r,z):\xi\} = \int_0^\infty r\phi_i(r,z)J_0(\xi r/a)\,\mathrm{d}r \tag{10}$$

We shall also record the appropriate Hankel inversion theorem

$$\phi_i(r,z) = H_0^{-1}\{\bar{\phi}_i^0(\xi,z):r\} = \frac{1}{a^2}\int_0^\infty \xi\bar{\phi}_i^0(\xi,z)J_0(\xi r/a)\,\mathrm{d}\xi \tag{11}$$

The solutions of eqn (1) appropriate to the halfspace region $z \geq 0$ take the form

$$\phi_i(r,z) = \frac{1}{a^2}\int_0^\infty \xi A_i(\xi)\exp(-\lambda_i z)J_0(\xi r/a)\,\mathrm{d}\xi \qquad (i = 1,2) \tag{12}$$

where $A_i(\xi)$ are arbitrary functions and $\lambda_i = \xi/a\sqrt{v_i}$. From eqns (3) and (12) it is evident that, in order to satisfy the boundary condition (7), we require

$$A_1(\xi) = -A_2(\xi)(=A(\xi)) \tag{13}$$

By making use of the above result in the general integral expressions for $u_z(r,z)$ and $\sigma_{zz}(r,z)$ it can be shown that the boundary conditions (8) and (9) are equivalent to the pair of dual integral equations

$$\begin{aligned} &H_0\{\xi^{-1}B(\xi);\rho\} = w(\rho) \qquad 0 \leq \rho \leq 1 \\ &H_0\{B(\xi);\rho\} = 0 \qquad 1 < \rho \leq \infty \end{aligned} \tag{14}$$

where

$$w(\rho) = -\frac{a^4\sqrt{v_1 v_2}f(\rho)}{\{k_1\sqrt{v_2} - k_2\sqrt{v_1}\}} \qquad \rho = \frac{r}{a} \qquad B(\xi) = \xi^2 A(\xi) \tag{15}$$

The solution of the dual system (14) can be readily obtained from the generalised results given by Sneddon.[5,6] The result of particular interest to the ensuing analysis is the contact stress σ_{zz} at the bonded interface $r \leq a$; $z = 0^+$. Briefly, by employing a representation of the form

$$B(\xi) = \int_0^1 \chi(t)\cos(\xi t)\,\mathrm{d}t \tag{16}$$

the second part of the dual system (14) is identically satisfied and the first equation is reduced to the solution of Abel's integral equation, which has a solution

$$\chi(t) = \frac{2}{\pi}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^t \frac{\rho w(\rho)\,\mathrm{d}\rho}{\sqrt{t^2-\rho^2}} \tag{17}$$

The contact stress at the interface $z = 0^+$ can be expressed in terms of $\chi(t)$ as follows:

$$\sigma_{zz}(r, 0^+) = \frac{c_{44}\{k_1 v_2 - k_2 v_1\}}{a^4 v_1 v_2}\left(\frac{\chi(1)}{\sqrt{1-\rho^2}} - \int_\rho^1 \frac{\chi'(t)\,\mathrm{d}t}{\sqrt{t^2-\rho^2}}\right) \tag{18}$$

3. THE FLEXURE OF THE DISC INHOMOGENEITY

We now direct our attention to the variational formulation of the flexure of the embedded disc-shaped inhomogeneity. The flexure of the disc inhomogeneity is induced by an axisymmetric distribution of loads applied to its central plane (Fig. 1). We assume that the flexural deflections of the disc inclusion $w(r) = u_z(r, 0\pm)$ can be approximated by

$$w(r) = a\left[C_0 + C_2\left(\frac{r}{a}\right)^2\right] \tag{19}$$

where C_0 and C_2 are arbitrary constants.

The total potential energy of the inhomogeneity–elastic medium system can be developed by making use of eqn (19). The total potential energy functional (U) is composed of the strain energy of the transversely isotropic elastic medium (U_E), the flexural energy of the disc inhomogeneity (U_I) and the potential energy of the applied loads (U_P) i.e. $U = U_E + U_I + U_P$.

(a) Strain energy of the transversely isotropic elastic medium

The contact stresses at the inhomogeneity–elastic medium interface consistent with the imposed displacement field (eqn (19)) are given by

$$\sigma_{zz}(r, 0\pm) = \pm\frac{2c_{33}\{k_1 v_2 - k_2 v_1\}}{\pi\sqrt{v_1 v_2}\{k_1\sqrt{v_2} - k_2\sqrt{v_1}\}}\left(\frac{C_0 + 2C_2}{\sqrt{1-\rho^2}} - 4C_2\sqrt{1-\rho^2}\right) \tag{20}$$

The strain energy of the transversely isotropic elastic medium can be evaluated by computing the work component of the contact stresses which comprise the interface tractions. Since $u_r(r, 0\pm) = 0$, only the tractions $\sigma_{zz}(r, 0\pm)$ contribute to the strain energy of the infinite medium. Since the displacements and stress components in the infinite medium vanish as $r, z \to \infty$, this energy component is bounded. Avoiding details of calculation it can be shown that

$$U_E = \frac{4c_{33}a^3\{k_1\nu_2 - k_2\nu_1\}}{\sqrt{\nu_1\nu_2}\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}}(C_0^2 + \Omega_1 C_0 C_2 + \Omega_2 C_2^2) \tag{21}$$

where $\Omega_1 = 4/3$; $\Omega_2 = 4/5$.

(b) Flexural energy of the inhomogeneity

The strain energy of the disc inhomogeneity consists of only the flexural energy of the region subjected to the flexural displacement field (eqn (19)). Thus, we obtain

$$U_I = 4\pi D(1 + \nu_i)C_2^2 \tag{22}$$

where

$$D = \frac{E_i h^3}{12(1 - \nu_i^2)} \tag{23}$$

is the flexural rigidity, E_i, ν_i are the elastic constants of the inhomogeneity and $h(\ll a)$ is its thickness.

(c) The potential energy of loads

We assume that the loads are applied over a finite region of the embedded inhomogeneity. When the radius of the loaded region is λa we have

$$U_P = -\pi p_0 \lambda^2 a^3 \left(C_0 + \frac{\lambda^2}{2} C_2\right) \tag{24}$$

The total potential energy functional for the inhomogeneity–elastic medium system can be developed by combining eqns (22)–(24). The constants C_0 and C_2 characterising the displacement field (eqn (19)) can be uniquely determined from the equations generated by minimising U, i.e.

$$\frac{\partial U}{\partial C_0} = \frac{\partial U}{\partial C_2} = 0 \tag{25}$$

The variational estimate for the displacement of the elastic inhomogeneity embedded in a transversely isotropic elastic medium reduces to the form

$$\frac{w(r)}{\pi p_0 \lambda^2 a / 4c_{33}} = \frac{\sqrt{\nu_1\nu_2}\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}}{\{k_1\nu_2 - k_2\nu_1\}}(\tilde{C}_0 + \tilde{C}_2\rho^2) \tag{26}$$

where

$$[\tilde{C}_0; \tilde{C}_2] = \frac{[\{\Omega_1\lambda^2 - 4\Omega_2 - 2R_A\}; 2\{\Omega_1 - \lambda^2\}]}{2\{\Omega_1^2 - 4\Omega_2 - 2R_A\}} \tag{27}$$

and R_A is a relative stiffness parameter defined by

$$R_A = \frac{\pi\sqrt{v_1 v_2}\{k_1\sqrt{v_2} - k_2\sqrt{v_1}\}}{6(1 - v_i)\{k_1 v_2 - k_2 v_1\}} \frac{E_i}{c_{33}} \frac{h^3}{a^3} \tag{28}$$

4. LIMITING CASES

The accuracy of the energy estimate (eqn (26)) for the flexural deflections of an embedded inhomogeneity can be established by comparing with eqn (26) certain known exact solutions.

(a) Infinitely rigid inhomogeneity

In the limiting case as $R_A \to \infty$, we have the problem of an infinitely rigid disc inhomogeneity embedded in bonded contact with a transversely isotropic elastic medium. In this case eqn (26) reduces to

$$[w]^{\text{tr·iso·}}_{\text{rigid}} = \frac{P\sqrt{v_1 v_2}\{k_1\sqrt{v_2} - k_2\sqrt{v_1}\}}{8c_{33}a\{k_1 v_2 - k_2 v_1\}} \tag{29}$$

where $P = p_0\pi\lambda^2 a^2$ is the total load. This result is in agreement with the solution for the displacement of the rigid disc-shaped inhomogeneity embedded in a transversely isotropic elastic medium, obtained by Selvadurai.[7] Also, in the limit as $v_1, v_2 \to 1$, we recover from eqn (29) the corresponding solution for the disc-shaped inhomogeneity embedded in an isotropic elastic medium.

As $v_1, v_2 \to 1$

$$\frac{k_1\sqrt{v_2} - k_2\sqrt{v_1}}{k_1 v_2 - k_2 v_1} = \frac{c_{11} + c_{44}}{2c_{44}} \tag{30a}$$

where

$$c_{11} = c_{33} = \lambda + 2\mu \qquad c_{44} = \mu \tag{30b}$$

and λ, μ are Lame's constants for the isotropic elastic solid. Making use of these results in eqn (29) we obtain

$$[w]^{\text{iso·}}_{\text{rigid}} = \frac{P(3 - 4v)}{32\mu a(1 - v)} \tag{31}$$

The above result is in agreement with solutions obtained by Collins,[8] Kanwal and Sharma[9] and Selvadurai[10] for the penny-shaped, rigid inclusion problem by making use of complex potential function techniques, singularity methods and direct spheroidal harmonic function techniques, respectively.

(b) Infinitely flexible inhomogeneity

In the particular case when the stiffness of the inhomogeneity reduces to zero (i.e. $E_i \to 0$) the applied load is directly transmitted to the interior of the transversely isotropic elastic medium. To enable comparison of the energy estimate with certain known solutions, we set $\lambda = 1$. In this case the inhomogeneity is subjected to a uniformly distributed load of intensity p_0 and radius a. The exact solution to the problem of a transversely isotropic elastic medium subjected to a concentrated axial load (perpendicular to the plane of transverse isotropy) is given by Elliott,[1] Kroner[11] and, more recently, by Pan and Chou.[12] A comparison of solutions yields the following: the deflection at the centre of the loaded area $w(0)$ is given by

$$[\{w(0)\}_{\text{energy}} : \{w(0)\}_{\text{exact}}] = \frac{\pi p_0 a\sqrt{\nu_1\nu_2}\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}}{8c_{33}\{k_1\nu_2 - k_2\nu_1\}}\left[\frac{21}{16}; \frac{4}{\pi}\right] \tag{32a}$$

Similarly, the deflection at the edge of the circular area $w(a)$ is given by

$$[\{w(a)\}_{\text{energy}} : \{w(a)\}_{\text{exact}}] = \frac{\pi p_0 a\sqrt{\nu_1\nu_2}\{k_1\sqrt{\nu_2} - k_2\sqrt{\nu_1}\}}{8c_{33}\{k_1\nu_2 - k_2\nu_1\}}\left[\frac{27}{32}; \frac{8}{\pi^2}\right] \tag{32b}$$

Also, in the limit as $\nu_i \to 1$, eqns (32) reduce to their counterparts for the isotropic elastic medium.

5. CONCLUSIONS

This paper presents an approximate method, based on a variational approach, which is used to determine the flexural displacements of a disc-shaped inhomogeneity embedded in bonded contact with a transversely isotropic elastic medium of infinite extent. The approximate solution is valid for situations in which the inhomogeneity is loaded over a finite region. The estimate for the flexural deflection of the inhomogeneity compares accurately with known exact solutions for inhomogeneities of infinite rigidity or flexibility. The second order approximation for the deflection of the inhomogeneity is, of course, unsuitable for the analysis of highly flexible inhomogeneities subjected to localised loads. In this instance an improved displacement field which satisfies the possible singular character of the stress distribution has to be used in the variational estimate.

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