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On the compression of a rigid disc by finitely deformed elastic halfspaces☆

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ABSTRACT

The paper examines the problem of a flat rigid circular disc that is compressed between two finitely deformed incompressible elastic halfspace regions with smooth surfaces. This problem yields a unilateral contact problem where the zone of separation needs to be determined. The analysis of the unilateral contact problem is reduced to the solution of a set of triple integral equations associated with the internal indentation of a penny-shaped crack and the internal tensile pressurization of an annular crack. The solutions to these problems can be obtained in an approximate series form in terms of the non-dimensional parameter involving the radius of the rigid disc to the radius of the separation zone. The extent of the separation zone is determined from the vanishing of the contact stress at the point of separation. Specific solutions are developed for the case where the initial finite deformation is for halfspace regions with a strain energy function of the Mooney-Rivlin form.

1. Introduction

The class of problems that deal with the mechanics of incremental deformations superposed on finitely deformed elastic bodies has several engineering applications that relate to both pre-stretched and pre-compressed rubber-like elastic solids used in load transmission. The initial pre-stress is assumed to follow the classical theory of finite deformations of elastic solids and the superposed or incremental deformations are described by an infinitesimal elasticity theory. A number of researchers, including Trefftz (1933), Biot (1939), Neuber (1943) and Green et al (1952), have made seminal contributions to this topic particularly as it relates to the study of stability problems in elastic solids. The theory of small deformations superposed on large proposed by Green et al (1952) has a rigorous development to accommodate initial finite deformations that are applicable to modern developments in the mechanics of rubber-like materials (Rivlin, 1960; Spencer, 1970; Ogden, 1984; Rajagopal, 1995; Barenblatt and Joseph, 1997; Selvadurai, 1977, 1980, 2006, 2011, 2015). Comprehensive expositions of the topic of small deformations superposed on large have also been presented by Truesdell and Noll (1965), Green and Zerna (1968), Green and Adkins (1970), Eringen and Suhubi (1974), Beatty and Usmani (1975) and Hill (1975a,b, 1976, 1977). The theory of small deformations superposed on large is a useful approach for examining hyperelastic materials that are pre-stressed, as opposed to continuously undergoing moderately large deformations similar to that described by the theory of second-order elasticity (Rivlin, 1953; Green and Spratt, 1954; Carlson and Shield, 1965; Selvadurai and Spencer, 1972; Selvadurai, 1974, 1975; Choi and Shield, 1981; Carroll and Rooney, 1984; Lindsay, 1985, 1992; Sabin and Kaloni, 1983). The present paper deals with the compression of a rigid disc inclusion by two identical elastic halfspace regions, with frictionless surfaces that are subjected to a radial finite deformation. The analogous classical elasticity problem of a disc inclusion embedded between initially undeformed elastic halfspace regions was examined by Selvadurai (1994a,b). Of related interest are the classical elasticity problems examined by Barber (1976) dealing with the concave rigid punch problem, the axisymmetric indentation by an annulus and the axisymmetric compression of an oblate spheroidal body solved, respectively, by Gladwell and Gupta (1979) and Gladwell and Hara (1981). The axisymmetric problem of indentation of the single face of a pre-compressed penny-shaped crack was examined by Selvadurai (2000a) and the elegant study by Gladwell (1995), which extends the work of Selvadurai (1994a,b) to include inclusions with an arbitrary planform. Further references to contact problems along these lines are given by Galin (1961), Ufliand (1965), Selvadurai (1979), Gladwell (1980, 2008), Johnson (1985), Curnier (1992), Hills et al. (1993), Selvadurai (2000b), Kachanov et al. (2003), Willner (2003), Selvadurai and Atluri (2010), Aleynikov (2011), Barber (1974, 2002, 2018), Popov et al (2019), Selvadurai and Samea (2020) and Samea and Selvadurai (2020).

The presence of the disc inclusion and axial compression of the finitely deformed halfspaces gives rise to a *unilateral contact problem*

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where the radius of the separation region is an unknown. Unilateral contact problems in elasticity have a well-founded mathematical tradition highlighted by the works of many researchers including Signorini (1933), Prager (1963), Fichera (1963, 1964, 1972), Duvaut and Lions (1976), Villaggio (1977, 1980), Kinderlehrer and Stampaccia (1980), Haslinger and Janovsky (1983), Panagiotopoulos (1989), Moreau et al. (1988), Fremond (1988), Kikuchi and Oden (1988), Kalker (1990), Klarbring (1986,1993), Klarbring et al. (1991), Raous et al. (1995), Selvadurai and Boulon (1995), Selvadurai (2003), Wriggers and Laursen (2007), Hills et al (2017), and many others, who approached the topic incorporating a variety of mechanical actions including friction, slip, adhesion and separation. The classical contact mechanics problems have antecedents commencing with the works of Hertz (1882, 1895), (see also Johnson, 1982), Boussinesq (1885), Love (1928) and Harding and Sneddon (1945), with the development of compact results for indentation problems that have seen a great deal of use and, on some occasions, abuse. These studies also generated advances by a number of elasticians including Mindlin and Deresiewicz (1953), Galin (1961), Lur'e (1964), Dundurs and Stippes (1970), de Pater and Kalker (1975), Goodman (1962, 1974), Johnson et al. (1971) and others that are referred to in the preceding articles. The extension of these classical studies to include frictional contact was also examined by Spence (1968, 1975), Turner (1979), Klarbring et al. (1991), Popov (2010) and more recent studies are given by Zhupanska (2009), Ballard and Jarušek (2011), Ballard (2013) and Selvadurai (2016, 2020). In order to examine the problem of the incremental compression of the disk by finitely deformed elastic halfspaces, we examine two auxiliary problems: the first deals with the frictionless indentation of a penny-shaped crack contained in a finitely deformed elastic infinite space by a rigid disc and the second with the internal tensile traction loading of an annular crack located in a finitely deformed elastic infinite space. In both cases, the Mode I stress intensity factors are evaluated at the outer boundary of the respective defects. As proposed by Barenblatt (1956, 1962) the vanishing of the combined Mode I stress intensity factor obtained from the two auxiliary problems provides the condition for estimating the unknown radius of the separation zone.

2. Governing equations

The fundamental equations governing small elastic deformations of an incompressible isotropic elastic material subjected to an initial finite deformation are given by Green et al. (1952) and in Green and Zerna (1968) and only the salient results required for the formulation of the auxiliary problems are presented for completeness. Also, for ease of reference, the formulation is kept to that presented by Green et al. (1952) and, when necessary, the relevant expressions applicable for the formulation of the axisymmetric problems will be summarized. The material points in the isotropic elastic material are defined by a general curvilinear coordinate system θ_i ($\theta_1 = x$; $\theta_2 = y$; $\theta_3 = z$), which moves with the body as it deforms. The covariant and contravariant metric tensors associated with the undeformed and deformed states are given by g_{ij} , G_{ij} and g^{ij} , G^{ij}, respectively. Although the theory of small deformations superposed on large can be developed for a general form of a strain energy function, the solution of the governing equations in an analytical form is feasible when attention is restricted to a specialized form of a strain energy function. Attention is therefore restricted to incompressible hyperelastic materials with a strain energy function $W(I_1, I_2)$ of the Mooney-Rivlin type, defined by

$$W(I_1, I_2) = C_1(I_1 - 3) + C_2(I_2 - 3) \tag{1}$$

where C_1 and C_2 are constants and I_1 and I_2 are the principal invariants given by

$$I_1 = g^{rs}G_{rs} \quad ; \quad I_2 = g_{rs}G^{rs} \tag{2}$$

and for an incompressible material, $I_3 = 1$. For this class of materials, we can define a contravariant stress tensor σ^{ij}, measured per unit area of the deformed body and referred to θ_i coordinates of the deformed body; the hyperelastic constitutive equation governing the incompressible elastic material is thus given by

$$\sigma^{ij} = \Phi g^{ij} + \Psi B^{ij} + pG^{ij} \tag{3}$$

where p is an isotropic stress to be determined by satisfying the boundary conditions of the problem and

$$\begin{gathered} B^{ij} = I_1 g^{ij} - g^{ir}g^{js}G_{rs} \\ \Phi = 2\frac{\partial W}{\partial I_1} \quad ; \quad \Psi = 2\frac{\partial W}{\partial I_2} \end{gathered} \tag{4}$$

We restrict attention to the special case where the finite deformation in the incompressible elastic material is maintained by the equal biaxial stretch in the x and y directions, which give rise to stretches $\lambda_1 = \lambda_2 = \lambda$ and $\lambda_3 = \mu$. The incompressibility constraint gives $\lambda^2\mu = 1$ and the stress state corresponding to the initial finite deformation is given by the contravariant stress tensor

$$\begin{aligned} \sigma^{11} = \sigma^{22} &= \Phi\lambda^2 + \Psi\lambda^2(\lambda^2 + \mu^2) + p \\ \sigma^{33} &= \Phi\mu^2 + 2\Psi\lambda^2\mu^2 + p \end{aligned} \tag{5}$$

The scalar invariant p is determined from the boundary condition of the initial finite deformation problem. If the elastic medium is subjected to only an equal bi-axial stress field, the zero axial stress requirement gives

$$p = -\mu^2(\Phi + 2\lambda^2\Psi) \tag{6}$$

and the equal bi-axial stress field corresponding to (5) can be obtained from (5) and (6).i.e.

$$\sigma^{11} = \sigma^{22} = \left\{\lambda^2 - \frac{1}{\lambda^4}\right\}(\Phi + \lambda^2\Psi) \tag{7}$$

In the absence of any initial finite stretch, $\lambda = \mu = 1$ and all $\sigma^{ij} \equiv 0$

We superimpose a further *infinitesimal state of deformation* on the finitely deformed elastic region, characterized by the following displacement field:

$$u_x(x,y,z) = u(x,y,z) \; ; \quad u_y(x,y,z) = v(x,y,z) \; ; \quad u_z(x,y,z) = w(x.y,z) \tag{8}$$

where the incremental displacement field satisfies the incompressibility condition

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{9}$$

As has been pointed out by Green et al (1952) and Woo and Shield (1961), the solution of the equations governing superposed deformations is facilitated by the introduction of displacement function techniques, where the functions $\varphi_n(x,y,z)$ are pseudo-Laplacian and satisfy the equations

$$\left\{\tilde{\nabla}^2 + k_n\frac{\partial^2}{\partial z^2}\right\}\varphi_n(x,y,z) = 0 \quad ; \quad (n = 1,2) \tag{10}$$

where

$$\tilde{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{11}$$

In (10), k_1 and k_2 are the roots of the equation

$$k^2 d_{55} + k\{d_{44} + d_{55} - a - c\} + d_{44} = 0 \tag{12}$$

and, since attention is restricted to a strain energy function of the Mooney-Rivlin type (1), we have

$$\begin{aligned} a &= 4\lambda^2(C_1 + \lambda^2C_2) \quad ; \quad c = 4\mu^2(C_1 + \lambda^2C_2) \\ d_{44} &= 2\mu^2(C_1 + \lambda^2C_2) \quad ; \quad d_{55} = 2\lambda^2(C_1 + \lambda^2C_2) \end{aligned} \tag{13}$$

For the case of a strain energy function of the Mooney-Rivlin type

(1), these constitutive parameters and the solution of (12) give the following roots:

$$k_1 = 1 \quad ; \quad k_2 = \lambda^{-6} = \mu^3 \tag{14}$$

The displacements and stresses governing the superposed incremental deformation can be expressed in the terms of the potentials $\varphi_n(x, y, z)$. The expressions relevant to the formulation of the mixed boundary value problems required to solve the unilateral contact problem can be written as follows:

$$u(x,y,z) = \frac{\partial}{\partial x}(\varphi_1 + \varphi_2); \quad v(x,y,z) = \frac{\partial}{\partial y}(\varphi_1 + \varphi_2); \quad w(x,y,z) = \frac{\partial}{\partial z}(k_1\varphi_1 + k_2\varphi_2) \tag{15}$$

and

$$\begin{aligned} \sigma_{xz} &= d_{44}\left\{(1+k_1)\frac{\partial^2\varphi_1}{\partial x\partial z} + (1+k_2)\frac{\partial^2\varphi_2}{\partial x\partial z}\right\} \\ \sigma_{yz} &= d_{44}\left\{(1+k_1)\frac{\partial^2\varphi_1}{\partial y\partial z} + (1+k_2)\frac{\partial^2\varphi_2}{\partial y\partial z}\right\} \\ \sigma_{zz} &= k_1(k_1 d_{55} + d_{44})\frac{\partial^2\varphi_1}{\partial z^2} + k_2(k_2 d_{55} + d_{44})\frac{\partial^2\varphi_2}{\partial z^2} \end{aligned} \tag{16}$$

The results (15) and (16) can also be transformed to generate appropriate expressions to formulate problems with axial symmetry.

3. The unilateral contact problem

We consider the problem of two finitely deformed incompressible elastic halfspace regions, with smooth surfaces, that are subjected a radial stress field σ^{11}. A smooth rigid disc inclusion of radius a and thickness $2h$ is placed at the interface of the halfspace regions and the entire region is subjected to an incremental uniform axial compressive stress field σ_0. This results in contact between the disc inclusion and the finitely deformed halfspace region with a separation region of radius b. Smooth contact between the two finitely deformed halfspace regions is re-established beyond the radius b(Fig. 1).

The objective of the study is to solve the resulting three-part mixed boundary value problem associated with the axisymmetric contact problem and to establish the influence of the initial finitely deformed state of the two halfspace regions, the aspect ratio of the rigid disc and the influence of the constitutive parameters characterizing the Mooney-Rivlin form of the strain energy function (1), on the extent of the separation zone.

The three-part boundary value problem governing the unilateral contact problem can be posed in terms of the following mixed boundary conditions applicable to the surface of the single finitely deformed halfspace region $0 \leqslant z < \infty$: i.e.

$$w(r,0) = h \quad ; \quad 0 \leqslant r \leqslant a \tag{17}$$

$$\sigma'_{zz}(r,0) = 0 \quad ; \quad a < r < b \tag{18}$$

$$w(r,0) = 0 \quad ; \quad b \leqslant r < \infty \tag{19}$$

$$\sigma'_{rz}(r,0) = 0 \quad ; \quad 0 < r < \infty \tag{20}$$

along with the requirement that $\sigma'_{ij} \rightarrow \sigma'_{zz} = \sigma_0$ as $(r,z) \rightarrow \infty$. The contact stress uniformly reduces to zero at the location of separation, satisfying the condition

$$\lim_{r\to b^+}\{2(r-b)\}^{1/2}\sigma'_{zz}(r,0) \rightarrow 0 \tag{21}$$

This constraint provides the Barenblatt (1956,1962) condition for determining the radius of the zone of separation. In order to solve the posed unilateral contact problem it is convenient to adopt the representation of the mixed boundary value problem posed by (17) to (20) as the summation of two *auxiliary problems* related to (i) the internal indentation of a penny-shaped crack by a smooth rigid disc inclusion of radius a and thickness $2h$ and (ii) the internal loading of an annular crack of internal radius a and external radius b by uniform tensile normal

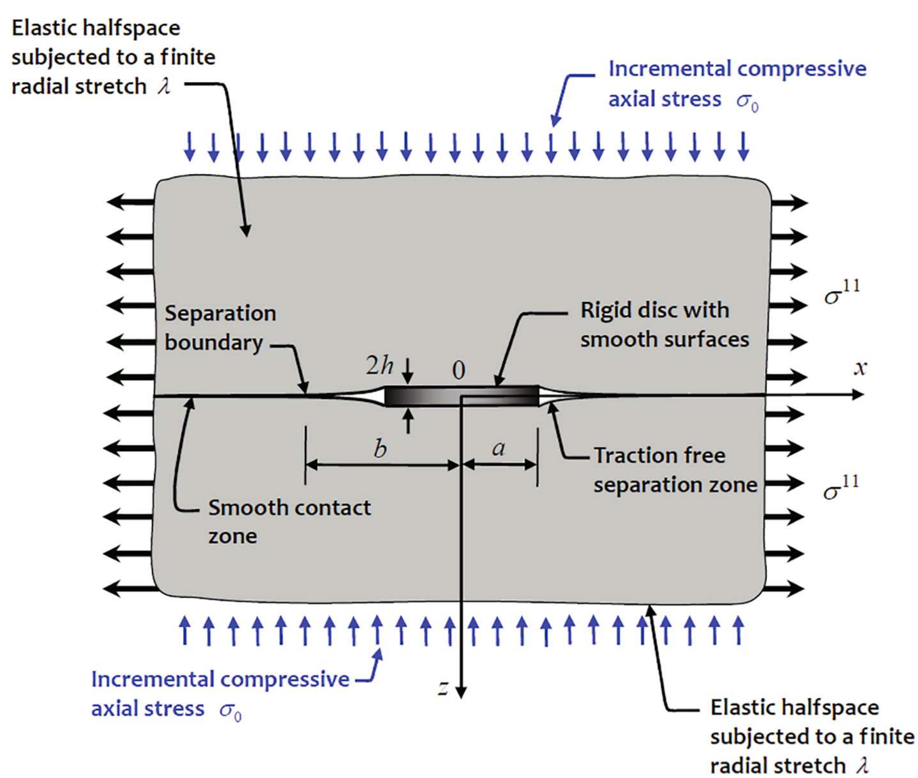

Fig. 1. Axisymmetric compression of a rigid disc by finitely deformed incompressible elastic halfspaces.

tractions σ_0. The combination of these auxiliary problems and the superposition of the far-field compressive axial stress field σ_0 can satisfy the mixed boundary conditions governing the unilateral contact problem. For the solution of the resulting axisymmetric mixed boundary value problems associated with the auxiliary problems, we seek Hankel transform developments (Sneddon, 1951) of the governing partial differential equation (10). For a halfspace region occupying $0 \leqslant z < \infty$, this can be expressed in the form

$$\varphi_n(r,z) = \int_0^\infty \xi A_n(\xi) exp\left(-\frac{\xi z}{\sqrt{k_n}}\right) J_0(\xi r) d\xi \quad ; \quad (n = 1, 2) \tag{22}$$

In (22), $A_n(\xi)$ are arbitrary functions that need to be determined for each auxiliary problem.

3.1. The internal indentation of a penny-shaped crack in a finitely deformed elastic medium

We consider an incompressible elastic infinite space, which contains a penny-shaped crack and, in its finitely deformed configuration, the radius of the penny-shaped crack is b. The penny-shaped crack is symmetrically indented by a smooth rigid disc inclusion of radius a and thickness $2h$ (Fig. 2). Considering the symmetry of the internal indentation of a penny-shaped crack about the plane $z = 0$ we can formulate the first auxiliary problem as a mixed boundary value problem for a single halfspace region $z \geqslant 0$, which can be posed as follows;

$$\sigma'_{rz}(r,0) = 0 \quad ; \quad 0 < r < \infty \tag{23}$$

$$w(r,0) = h \quad ; \quad 0 \leqslant r \leqslant a \tag{24}$$

$$w(r,0) = 0 \quad ; \quad b \leqslant r < \infty \tag{25}$$

$$\sigma'_{zz}(r,0) = 0 \quad ; \quad a < r \leqslant b \tag{26}$$

The boundary condition (23) can be used to obtain a relationship between $A_1(\xi)$ and $A_2(\xi)$. The remaining boundary conditions give rise to a system of triple integral equations for a single unknown function $A(\xi)$, which take the forms

$$\int_0^\infty \xi^2 A(\xi) J_0(\xi r) d\xi = \frac{h((1+k_1)(1+k_2)}{(k_1 - k_2)} \quad ; \quad 0 \leqslant r \leqslant a \tag{27}$$

$$\int_0^\infty \xi^3 A(\xi) J_0(\xi r) d\xi = 0 \quad ; \quad a < r < b \tag{28}$$

$$\int_0^\infty \xi^2 A(\xi) J_0(\xi r) d\xi = 0 \quad ; \quad b < r < \infty \tag{29}$$

Triple integral equations of the type (27) to (29) can be solved in a variety of ways and these are documented by Tranter (1960), Cooke (1963a, b), Williams (1963), Sneddon (1966), Kanwal (1971), Jain and Kanwal (1972), Gladwell (1980). Selvadurai and Singh (1984), Selvadurai (1985) and Barber (2018). Assuming that (28) admits a representation

$$\int_0^\infty \xi^3 A(\xi) J_0(\xi r) d\xi = 0 \quad ; \begin{cases} f_1(r) & ; \quad 0 < r < a \\ f_2(r) & ; \quad b < r < \infty \end{cases} \tag{30}$$

we obtain the following system of coupled integral equations for the unknown functions $f_1(r)$ and $f_2(r)$:

$$f_1(\eta) = \frac{2}{\pi\sqrt{a^2 - \eta^2}} \left(\frac{\eta h (1+k_1)(1+k_2)}{(k_1 - k_2)} - \int_b^\infty \frac{t\sqrt{t^2 - a^2} f_2(t) dt}{(t^2 - \eta^2)} \right) \quad ; \quad 0 \leqslant \eta \leqslant a \tag{31}$$

$$f_2(\eta) = \frac{2}{\pi\sqrt{\eta^2 - b^2}} \left(-\int_0^a \frac{t\sqrt{b^2 - t^2} f_1(t) dt}{(\eta^2 - t^2)} \right) \quad ; \quad b \leqslant \eta < \infty \tag{32}$$

These two coupled integral equations can be reduced to a single Fredholm integral equation of the second kind for an unknown function $\psi(\widetilde{\eta})$ in the form

$$\psi(\widetilde{\eta}) = 1 + \int_0^1 \psi(\widetilde{\xi}) K(\widetilde{\xi}, \widetilde{\eta}) \, d\widetilde{\xi} \quad ; \quad 0 \leqslant \widetilde{\xi} \leqslant 1 \tag{33}$$

where

$$K(\widetilde{\xi}, \widetilde{\eta}) = -\frac{2\widetilde{\xi}}{\pi^2} \left\{ \Phi(\widetilde{\xi}) - \Phi(\widetilde{\eta}) \right\} \sqrt{\frac{1 - c^2 \widetilde{\xi}^2}{1 - \widetilde{\xi}^2}} \tag{34}$$

$$\Phi(\gamma) = \zeta ln\left(\frac{1 + c\zeta}{1 - c\zeta}\right) \quad ; \quad \zeta = \sqrt{\frac{1 - \gamma^2}{1 - c^2\gamma^2}} \quad ; \quad \gamma = \widetilde{\xi} \text{ or } \widetilde{\eta} \tag{35}$$

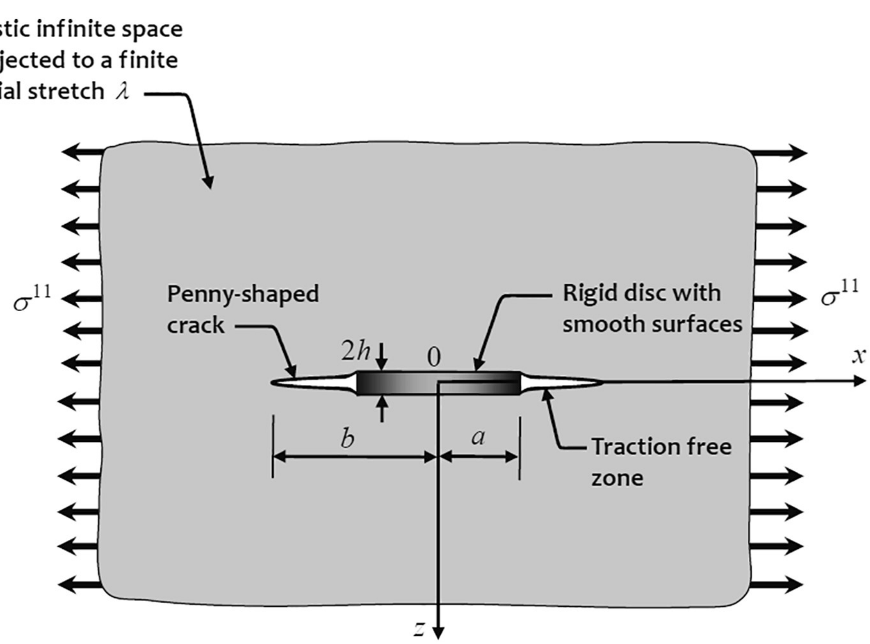

Fig. 2. Axisymmetric smooth indentation of a penny-shaped crack in a finitely deformed elastic solid by a rigid disc.

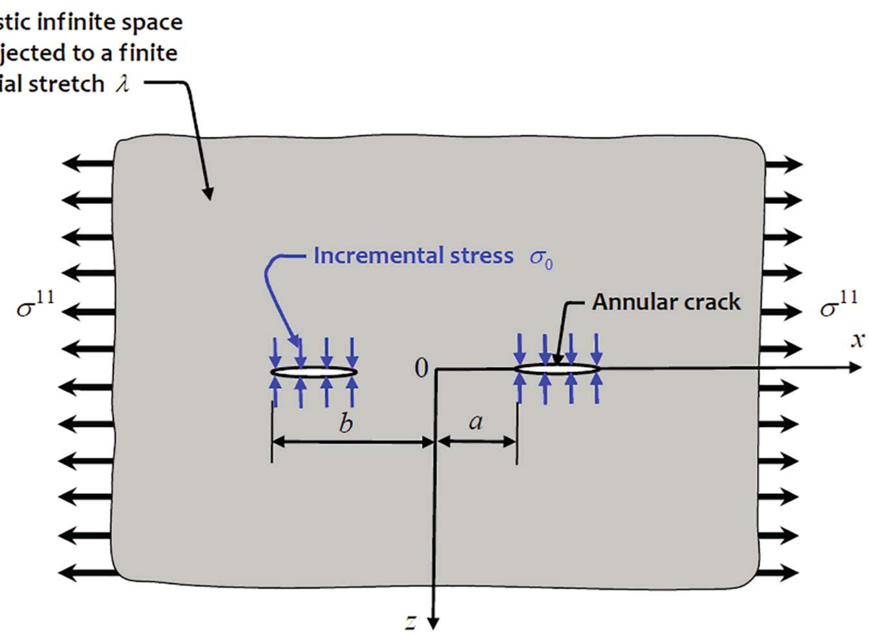

Fig. 3. Action of compressive tractions on the surfaces of an annular crack.

and $c = a/b < 1$. The limit for the expression for $K(\widetilde{\xi},\widetilde{\eta})$, as $\widetilde{\xi}\to\widetilde{\eta}$, can be obtained using L'Hospital's rule, which gives

$$\operatorname*{Lim}_{\widetilde{\xi}\to\widetilde{\eta}} = \frac{(1-c^2)\widetilde{\eta}\,\Omega(\widetilde{\eta})}{\pi^2(1-c^2\widetilde{\eta}^2)(1-\widetilde{\eta}^2)} \tag{36}$$

where

$$\Omega_0 = \left\{ \frac{\sqrt{k_1}\,(k_1 d_{55}+d_{44})(1+k_2)-\sqrt{k_2}\,(k_2 d_{55}+d_{44})(1+k_1)}{(\Phi+\lambda^2\Psi)(k_1-k_2)} \right\} \tag{39}$$

$$F_h(c) = \left(\frac{4}{\pi}c + \frac{16}{\pi^3}c^2 + \left(\frac{4}{3\pi}+\frac{64}{\pi^5}\right)c^3 + \left(\frac{80}{9\pi^3}+\frac{256}{\pi^7}\right)c^4 + \left(\frac{448}{9\pi^5}+\frac{1024}{\pi^9}+\frac{4}{5\pi}\right)c^5 \; + \; \mathrm{O}(c^6) \right) \tag{40}$$

$$\Omega(\widetilde{\eta}) = \frac{2\rho}{(1-\rho^2)} + ln\left(\frac{1+\rho}{1-\rho}\right) \quad ; \quad \rho = c\sqrt{\frac{1-\widetilde{\eta}^2}{1-c^2\widetilde{\eta}^2}} \tag{37}$$

The methods for the solution of Fredholm integral equations of the second-kind are many and varied. The purely numerical solution of such equations is described by Atkinson (1976, 1997) , Baker (1977), Delves and Mohamed (1985), Atkinson and Shampine (2008) and others. Methods applicable to specialized types of kernel functions are also presented by Kanwal (1971), Delves and Mohamed (1985) and Polyanin and Manzhirov (2008). The intention here is to develop an *approximate analytical result* that can be used to estimate the Mode I stress intensity factor at the boundary of the penny-shaped crack. The approach given in Selvadurai and Singh (1984) was used to consider the solution for $\psi(\widetilde{\eta})$ in (33) as a power series in terms of the parameter $c(<1)$. Omitting details, it can be shown that the approximate series expression, in terms of c, for the Mode I stress intensity factor at the boundary of the internally indented crack in a finitely deformed incompressible elastic medium of infinite extent is given by

$$K_I^h = \frac{h(C_1+\lambda^2 C_2)}{\pi\sqrt{b}}\Omega_0\, F_h(c) \tag{38}$$

where

C_1 and C_2 are the elasticity parameters characterizing the Mooney-Rivlin form of the strain energy function (1) with the linear elastic shear modulus of the incompressible elastic material G defined by

$$G = 2C_1(1+\Gamma) \quad ; \quad \Gamma = C_2/C_1 \tag{41}$$

The neo-Hookean form of the strain energy function is obtained by setting $\Gamma\equiv 0$. The accuracy of the series approximation for the estimation of the Mode I stress intensity factor for the internally indented penny-shaped crack for the classical elasticity problem has also been verified through analysis based on boundary integral equation techniques (Tan and Selvadurai, 1986).

3.2. The application of tensile tractions to an annular crack in a finitely deformed incompressible elastic infinite domain

We consider the problem of an annular crack located in an incompressible elastic domain of infinite extent that is subjected to the radial stress field σ^{11}. In the finitely deformed configuration, the plane $z=0$ contains an annular crack of external radius b and internal radius a. The entire domain is subjected to an incremental *compressive axial stress* σ_0, such that the entire plane $z=0$ is under a uniform compressive stress. In the unilateral contact problem, however, the separated zones are subjected to zero traction. The zero shear traction constraint is satisfied by the symmetry about the plane $z=0$, which renders the entire plane free of shear tractions. The surfaces of the annular crack are now subjected to a *uniform tensile traction* σ_0 (Fig. 3). Considering the halfspace region

$z \geqslant 0$, the mixed boundary problem associated with the internal traction loading of the annular crack can be posed as follows:

$$\sigma_{rz}(r,0) = 0 \quad ; \quad 0 < r < \infty \tag{42}$$

$$u_z(r,0) = 0 \quad ; \quad 0 \leqslant r \leqslant a \tag{43}$$

$$u_z(r,0) = 0 \quad ; \quad b \leqslant r < \infty \tag{44}$$

$$\sigma_{zz}(r,0) = \sigma_0 \quad ; \quad a < r \leqslant b \tag{45}$$

The solution of the annular crack problem has been investigated extensively in fracture mechanics literature and references to relevant articles are given in Smetanin (1968), Moss and Kobayashi (1971), Kassir and Sih (1975), Cherepanov (1979), Choi and Shield (1982) and Broberg (1999). An approximate solution to the annular crack was also developed by Selvadurai and Singh (1985), who used the power series expansion technique to solve the triple integral equations resulting from the mixed boundary value problem posed by (42)–(45). The approaches used to estimate the stress intensity factors at the boundaries of the annular crack can give varying estimates; comparisons of the results are given by Choi and Shield (1982) and Selvadurai and Singh (1985). The details are not repeated but the result of interest to the analysis of the unilateral contact problem is the Mode I stress intensity factor (negative) at the outer boundary of the annular crack ($r = b$), which can be expressed in the form

$$K_I^{\sigma_0} = \frac{2\sigma_0\sqrt{b}}{\pi} F_{\sigma_0}(c) \tag{46}$$

where

$$F_{\sigma_0}(c) = 1 - \left(\frac{4}{\pi^2}\right)c - \left(\frac{16}{\pi^4}\right)c^2 - \left(\frac{1}{8} + \frac{64}{\pi^6}\right)c^3 - \left(\frac{16}{3\pi^4} + \frac{4}{\pi^2}\left\{\frac{1}{24} - \frac{8}{9\pi^2} + \frac{64}{\pi^6} + \frac{4}{9\pi^3}\right\}\right)c^4 - \left(\frac{16}{\pi^4}\left\{\frac{1}{24} - \frac{8}{9\pi^3} + \frac{64}{\pi^6} + \frac{8}{9\pi^2}\right\} + \frac{256}{9\pi^6} - \frac{4}{15\pi^2}\right)c^5 + \mathrm{O}(c^6) \tag{47}$$

Implicit in this analysis is the requirement that the crack tip remains open in order to develop a non-zero Mode I stress intensity factor. As is evident, the displacement boundary conditions associated with the internal loading of the annular crack are null boundary conditions. Consequently, the stress intensity factor will be *independent* of the mechanical properties of the incompressible elastic material and the initial finite deformation that is applied to the domain and dependent only on the applied incremental stress σ_0 .

4. The location of the separation zone with unilateral contact

The unilateral contact problem associated with the frictionless compression of the frictionless disc inclusion by the finitely deformed incompressible elastic halfspace regions, under a radial stress σ^{11} and subjected to an incremental uniform compressive loading σ_0, requires that: (i) The plane surfaces of the inclusion region are in contact with the halfspace regions. (ii) In the separation region the surfaces of the halfspace regions are traction free. (iii) The finitely deformed elastic halfspace regions re-establish frictionless contact beyond the separation zone. (iv) The normal stress $\sigma_{zz}(r,0) \to \sigma_0$ as $r \to \infty$. (v) As demonstrated by Barenblatt (1956, 1962), the contact stresses must uniformly reduce to zero at the location of separation $r = b$. All the conditions associated with the unilateral contact problem as indicated by (i) to (iv) will be satisfied by the mixed boundary conditions associated with the auxiliary problems outlined in sections 3.1 and 3.2. The solutions developed in the auxiliary problems are indeterminate to within the radius of the location of the zone of separation $r = b$. Barenblatt's condition requires that, to determine the radius of the separation zone, the combined Mode I stress intensity factors for the two auxiliary problems at the location $r = b$ should reduce to zero. The vanishing of the combined Mode I stress intensity factor gives the following characteristic equation for the evaluation of b/a:

$$\left(\frac{h(\Phi + \lambda^2\Psi)\Omega_0}{2\sigma_0 a}\right) cF_h(c) - F_{\sigma_0}(c) = 0 \tag{48}$$

This equation can be solved to determine the radius of the zone of separation. The *lowest positive root* of (48) gives the radius of the separation zone. It should be remarked that the resulting value for the radius of the separation zone is an analytically derived result that uses a series expansion technique in terms of $c < 1$. The parameter that controls the radius of the normalized separation zone (b/a) is the multiplier

$$\mathrm{N} = \left(\frac{h}{a}\right)\left(\frac{\Phi + \lambda^2\Psi}{2\sigma_0}\right)\Omega_0 \tag{49}$$

Since N is a non-dimensional parameter, (48) can be solved for the separation at the frictionless interface between initially stressed incompressible elastic halfspace regions containing the rigid disc of thickness $2h$ and radius a. Fig. 4 illustrates the variation of (b/a) as a function of the non-dimensional parameter N.

In the limiting case when the halfspace regions are initially unstressed, $\lambda \to 1$, $d_{44} = d_{55} = 2(C_1 + C_2) = 2G$ and (49) reduces to

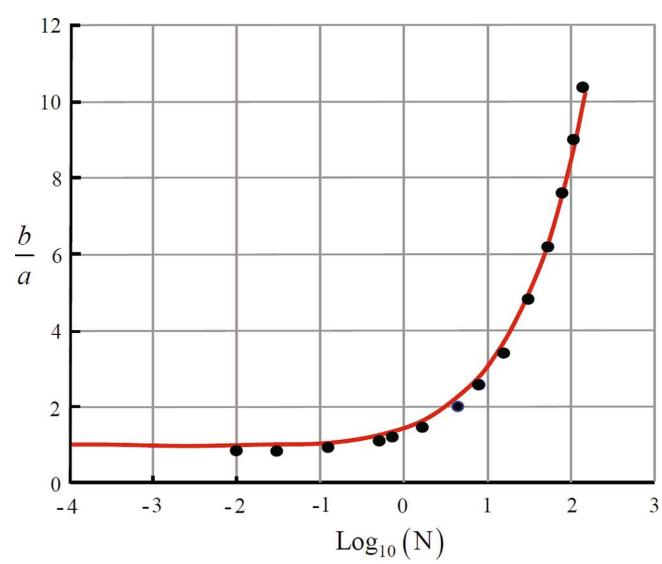

Fig. 4. Variation in the normalized zone of separation as a function of the parameter N [The solid line is the analytical result based on the solution of Eq. (48); The solid circles ● are the results based on FE computations using ABAQUS ™].

$$\mathrm{N} = \lim_{\lambda \to 1} \left\{ \left(\frac{h}{a}\right) \left(\frac{G}{\sigma_0}\right) \left(\frac{\Omega_0}{4}\right) \right\} = \frac{hG}{a\sigma_0} \tag{50}$$

This result is in agreement with the analogous classical elasticity interface separation problem for an incompressible elastic interface that can be deduced from the results given by Selvadurai (1994a,b) and Selvadurai et al. (2018).

It could be visualized that, as the non-dimensional pre-compression $\sigma_0/(\Phi + \lambda^2\Psi)$ increases, the zone of separation will decrease and $c \to 1$. Therefore, the result based on the series expansion-based analytical approach will not be entirely accurate. To assess the limits of applicability of the analytical estimation of the separation zone, the unilateral contact problem was examined using the finite element (FE) scheme available in the general-purpose FE code ABAQUS™. In the computational simulations, the penalty function and augmented Lagrangian techniques were used as a Signorini-constraint enforcement method. The results of the FE computations showed good agreement with the analytical approach. Fig. 4 illustrates the variation of parameter b/a with the non-dimensional parameter N that accounts for the normalized axial stress, the aspect ratio of the rigid disc and the constitutive parameters characterizing the finitely deformed elastic halfspace regions. The solid line in Fig. 4 is derived from the solution of equation (48) and the solid circles indicate the results derived from the FE modelling. There is good correlation between the analytical and computational results for a wide range of the non-dimensional parameter N. As the non-dimensional parameter $(Gh/\sigma_0 a) \to 0$, the non-dimensional parameter $\mathrm{N} \to 0$ and the ratio $(b/a) \to 1$. This conforms to the expected response of the unilateral contact problem.

The developments involving small deformations superposed on large are also viewed as a methodology for estimating stability of elastic media subjected to large deformations, where the incremental deformations are perturbations introduced to assess the stability of the system. In the context of the unilateral contact problem, development of instability can occur when the elastic media are subjected to finite radial compression; instability will occur when $\Omega_0 \to 0$. For the Mooney-Rivlin material with

$$d_{44} = \frac{2\Delta}{\lambda^4} \quad ; \quad d_{55} = 2\Delta\lambda^2 \quad ; \quad k_1 = 1 \quad ; \quad k_2 = \lambda^{-6} \tag{51}$$

we obtain the condition for the development of instability as

$$\lambda^9 + \lambda^6 + 3\lambda^3 - 1 = 0 \tag{52}$$

This gives $\lambda \approx 2/3$, which is *identical* to the result obtained by Green et al. (1952) and others, for the compressive finite strain needed to cause surface instability in a Mooney-Rivlin material. The lowest positive root of (48) is selected to avoid surface instabilities that can load to non-uniqueness.

5. Concluding remarks

The axisymmetric unilateral contact problem resulting from incremental compression of a smooth rigid disc by finitely deformed incompressible elastic halfspace regions, is examined using its reduction to two three-part boundary value problems. Since the equations governing the incremental deformations are linear, the superposition of the two auxiliary problems is permissible. The application of Barenblatt's condition for the vanishing of the combined Mode 1 stress intensity factor enables the determination of the extent of the separation zone. The methodology presented in the paper is such that it can be applied to a wide range of similar axisymmetric problems; these include the estimation of the zone of separation between (i) two identical but differing initially finitely deformed elastic halfspace regions, (ii) two dissimilar finitely deformed halfspace regions, (iii) two transversely isotropic elastic media, (iv) identical or dissimilar inhomogeneous elastic media where the elasticity properties vary in the axial direction, and the methodology can be adopted to estimate the peak anchoring capacity of a disc anchor embedded in either frictional or dilatant contact properties at the interfaces, a topic of interest to geomechanics. The study culminates in a generalized result for the normalized dimension of the zone of separation, where the constitutive influences of cases (i) to (iv) can be accommodated through a single variable encountered in the parameter N.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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